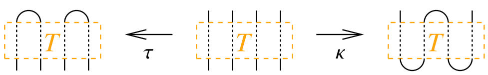
Work in Progress!

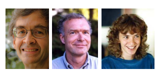
Gauss-Gassner Invariants, What?

Abstract. In a "degree d Gauss diagram formula" one produces a number by summing over all possibilities of paying very close attention to d crossings in some n-crossing knot diagram while observing the rest of the diagram only very loosely, minding only its skeleton. The result is always poly-time computable as only $\binom{n}{d}$ states need to be considered. An under-explained paper by Goussarov, Polyak, and Viro [GPV] shows that every type d knot invariant has a formula of this kind. Yet only finitely many integer invariants can be computed in this manner within any specific polynomial time bound.

I suggest to do the same as [GPV], except replacing "the skeleton" with "the Gassner invariant", which is still poly-time. One poly-time invariant that arises in this way is the Alexander polynomial (in itself it is infinitely many numerical invariants) and I believe (and have evidence to support my belief) that there are more.

The QUILT Target. QUick Invariants of Large Tangles, for little had been found since Alexander (and if they're there, how can we not know all about them?), and for {ribbon} ≠ {slice}:

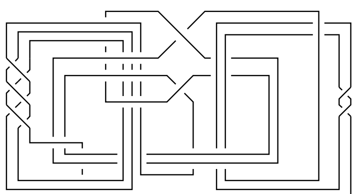




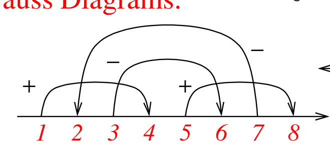
Gompf, Scharlemann, Thompson [GST]

$$U \in \mathcal{T}_n \qquad 1 \in \mathcal{A}_n$$
$$\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n} \quad (\tau \nearrow,\ \kappa \searrow)$$
$$\text{ribbon } K \in \mathcal{T}_1 \qquad z(K) \in \mathcal{A}_1$$

Faster is better, leaner is meaner!



Gauss Diagrams. (just QUILK, today)



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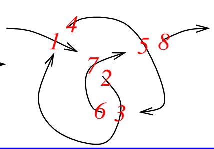
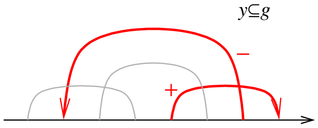
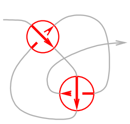


Gauss

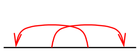
Gauss Diagram Formulas [PV, GPV]. If g is a Gauss diagram and F an unsigned Gauss diagram, $\langle F, g\rangle_{PV} := \sum_{y\subseteq g} (-1)^y \delta(F, \bar{y})$:

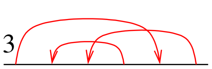
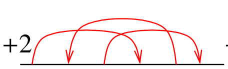
Goussarov-Polyak-Viro





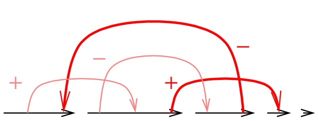
Under-Explain Theorem [GPV]. Every finite type invariant arises in this way.

$F_2 =$  $\Rightarrow \langle F_2, K\rangle = v_2(K)$

$F_3 =$ 3  +2  + rotations

$$\Rightarrow \langle F_3, K\rangle = 6v_3(K)$$

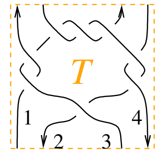
Gauss-Gassner Invariants. Want more? Increase your environmental awareness! Instead of nearly-forgetting y^c, compute its Burau/Gassner invariant (note that y^c is a tangle in a Swiss cheese; more easily, a virtual tangle):

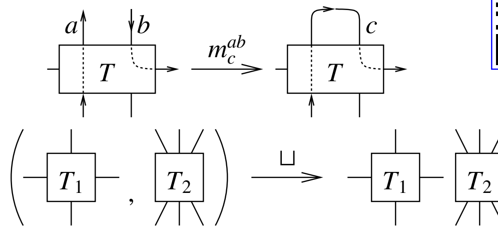


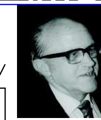
$$GG_{k,F}(g) = \sum_{y\subseteq g,\ |y|\le k} \bar{F}\left(y, z(y^c)\right) = \sum_{y\subseteq g,\ |y|\le k} F\left(y, z(g \text{ cut near } y)\right),$$

where k is fixed and $F(y,\gamma)$ is a function of a list of arrows y and a square matrix γ of side $|y|+1 \le k+1$.

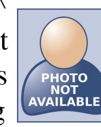
The (Burau-)Gassner Invariant.







Burau



Gassner

Theorem 1. ∃! an invariant z: {pure framed S-component tangles} → $\Gamma(S) := M_{S\times S}(R_S)$, where $R_S = \mathbb{Z}((T_a)_{a\in S})$ is the ring of rational functions in S variables, intertwining

$$\left(\begin{array}{c|c} & S_1 \\ \hline S_1 & A_1 \end{array}, \begin{array}{c|c} & S_2 \\ \hline S_2 & A_2 \end{array} \right) \xrightarrow{\sqcup} \begin{array}{c|cc} & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array},$$

$$\begin{array}{c|ccc} & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a, T_b \to T_c,\ \mu := 1-\beta]{m_c^{ab}} \left(\begin{array}{c|cc} & c & S \\ \hline c & \gamma + \alpha\delta/\mu & \epsilon + \delta\theta/\mu \\ S & \phi + \alpha\psi/\mu & \Xi + \psi\theta/\mu \end{array} \right),$$

and satisfying $\left(|_a;\ {}_a\!\nearrow\!\!\nwarrow_b,\ {}_b\!\nearrow\!\!\nwarrow_a\right) \xrightarrow{z} \left(\begin{array}{c|c} & a \\ \hline a & 1 \end{array};\ \begin{array}{c|cc} & a & b \\ \hline a & 1 & 1-T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array} \right).$

See also [LD, KLW, CT, BNS].

Theorem 2. With $k=1$ and F_A defined by

$$F_A(\xrightarrow{s}, \gamma) = s\frac{\gamma_{22}\gamma_{33} - \gamma_{23}\gamma_{32}}{\gamma_{33} + \gamma_{13}\gamma_{32} - \gamma_{12}\gamma_{33}}\Bigg|_{T_a\to T},$$

$$F_A(\xleftarrow{s}, \gamma) = s\frac{\gamma_{13}\gamma_{32} - \gamma_{12}\gamma_{33}}{\gamma_{32} - \gamma_{23}\gamma_{32} + \gamma_{22}\gamma_{33}}\Bigg|_{T_a\to T},$$

$GG_{1,F_A}(K)$ is a regular isotopy invariant. Unfortunately, for every knot K, $GG_{1,F_A}(K) - T\frac{d}{dT}\log A(K)(T) \in \mathbb{Z}$, where $A(K)$ is the Alexander polynomial of K.

Expectation. Higher Gauss-Gassner invariants exist . . .
(though right now I can reach for them only wearing my exoskeleton)

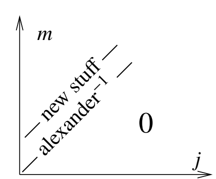
Jones, Melvin, Morton, Rozansky, Garoufalidis

. . . and they are the "higher diagonals" in the MMR expansion of the coloured Jones polynomial J_λ.

Theorem ([BNG], conjectured [MM], elucidated [Ro]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of $sl(2)$. Writing

$$\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}}\Bigg|_{q=e^\hbar} = \sum_{j,m\ge 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot A(K)(e^\hbar) = 1$.

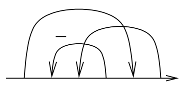


Help Needed!

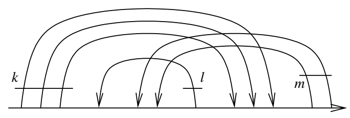
I'm slow and feeble-minded.

Warning. Conventions on this page change randomly from line to line.

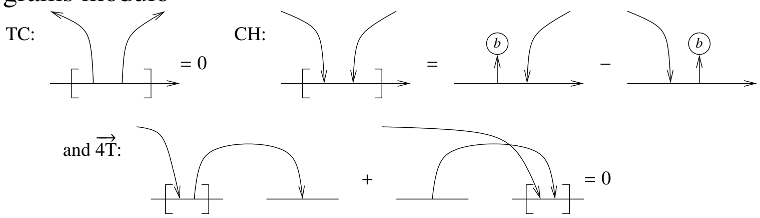
$Z^{w/2}$. The GGA story is about $Z^{w/2}\colon \mathcal{K} \to \mathcal{A}^{w/2}$, defined on arrows a by $\pm a \mapsto \exp(\pm a)$:



$$\to \sum_{k,l,m\geq 0} \frac{(+)^k(-)^l(+)^m}{k!l!m!}$$



Where the target space $\mathcal{A}^{w/2}$ is the space of unsigned arrow diagrams modulo

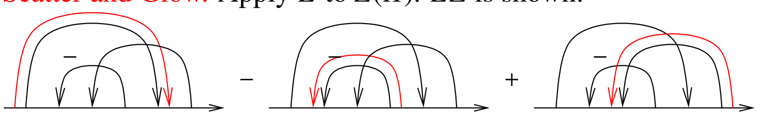


($Z^{w/2}$ is a reduction of the much-studied Z^w [BND, BN]).

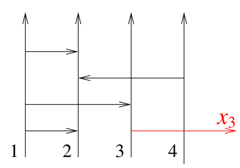
The Euler Trick. How best do non-commutative algebra with exponentials? Logarithms are from hell as $e^fe^g = e^{\text{bch}(f,g)}$, but Euler's from heaven: Let E be the derivation $Ef := (\deg f)f$ $(= xf'$, in $\mathbb{Q}[\![x]\!])$ and let $\tilde{E}Z := Z^{-1}EZ$ $(= x(\log Z)'$ in same). If $\deg x = 1$ then $\tilde{E}e^x = x$ and if $F = e^f$ and $G = e^g$, then $\tilde{E}(FG)$ is

$$(FG)^{-1}((EF)G + F(EG)) = G^{-1}(\tilde{E}F)G + \tilde{E}G = e^{-\operatorname{ad} g}(\tilde{E}F) + \tilde{E}G.$$

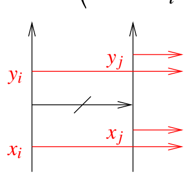
Scatter and Glow. Apply $\tilde{E}$ to $Z(K)$. EZ is shown:



Tail scattering. The algebra $\mathbb{Q}[\![b_i]\!]\langle a_{ij}\rangle$ modulo $[a_{ij}, a_{kl}] = 0$ (loc), $[a_{ij}, a_{ik}] = 0$ (TC), and $[a_{ik}, a_{jk}] = -[a_{ij}, a_{jk}] = b_ja_{ik} - b_ia_{jk}$ (CH and $\overrightarrow{4T}$), acts on $V = \mathbb{Q}[\![b_i]\!]\langle x_i = a_{i\infty}\rangle$ by $[a_{ij}, x_i] = 0$, $[a_{ij}, x_j] = b_ix_j - b_jx_i$. Hence $e^{\operatorname{ad} a_{ij}}x_i = x_i$, $e^{\operatorname{ad} a_{ij}}x_j = e^{b_i}x_j + \frac{b_j}{b_i}(1 - e^{b_i})x_i$. Renaming $\bar{x}_i = x_i/b_i$, $T_i = e^{b_i}$, get $[e^{\operatorname{ad} a_{ij}}]_{\bar{x}_i,\bar{x}_j} = \begin{pmatrix} 1 & 1-T_i \\ 0 & T_i \end{pmatrix}$. Alternatively,



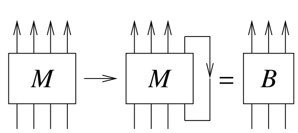
$a_{12}a_{13}a_{42}a_{12}$



$$\bar{y}_i = \bar{x}_i$$
$$\bar{y}_j = (1 - T_i)\bar{x}_i + T_i\bar{x}_j$$

Linear Control Theory.

If $\begin{pmatrix} y \\ y_n \end{pmatrix} = \begin{pmatrix} \Xi & \phi \\ \theta & \alpha \end{pmatrix}\begin{pmatrix} x \\ x_n \end{pmatrix}$, and we further impose $x_n = y_n$, then $y = Bx$ where $B = \Xi + \dfrac{\phi\theta}{1-\alpha}$. This fully explains the Gassner formulas and the GGA formula!



All that remains now is to replace TC by something more interesting: with $\epsilon^2 = 0$,

$$[a_{ij}, a_{ik}] = \epsilon(c_ja_{ik} - c_ka_{ij}).$$

Many further changes are also necessary, and the algebra is a lot more complicated and revolves around "quantization of Lie bialgebras" [EK, En]. But the spirit is right.

References.

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[PV] M. Polyak and O. Viro, *Gauss Diagram Formulas for Vassiliev Invariants,* Inter. Math. Res. Notices **11** (1994) 445–453.

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Dror Bar-Natan: Talks: NCSU-1604: Gauss-Gassner-Alexander Demo

ωεβ:=www.math.toronto.edu/~drorbn/Talks/NCSU-1604/
source notebook: ωεβ/GGAD.

Loading KnotTheory`

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<< KnotTheory`
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Loading KnotTheory` version
  of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

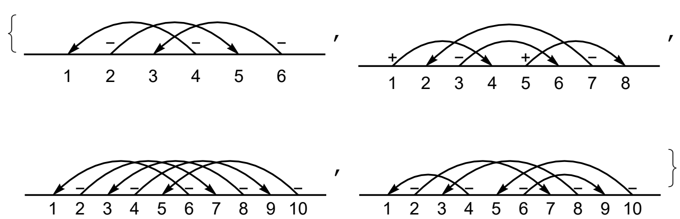
Gauss Diagram Utilities

```
GD[g_GD] := g;
GD[L_] := GD @@ PD[L] /.
   X[i_, j_, k_, l_] :→ If[PositiveQ@X[i, j, k, l],
    Ap_{l,i}, Am_{j,i}];
Draw[g_GD] := Module[{n = Max@Cases[g, _Integer, ∞]},
  Graphics[{
    Line[{{0, 0}, {n + 1, 0}}],
    List @@ g /. (ah_)_{i_,j_} :→ {
       Arrow[BezierCurve[{{i, 0}, {i + j, Abs[j - i]} / 2,
          {j, 0}}]],
       Text[ah /. {Ap → "+", Am → "-"}, {i, 0.3}]},
    Table[Text[i, {i, -0.5}], {i, n}]}]]
```

Some Gauss Diagrams

```
Draw /@ GD /@ AllKnots@{3, 5}
```

KnotTheory::loading : Loading precomputed data in PD4Knots`.



Some Gauss Diagrams, 2

```
GD /@ AllKnots@{3, 5}
```

```
{GD[Am_{4,1}, Am_{6,3}, Am_{2,5}], GD[Ap_{1,4}, Ap_{5,8}, Am_{3,6}, Am_{7,2}],
 GD[Am_{6,1}, Am_{8,3}, Am_{10,5}, Am_{2,7}, Am_{4,9}],
 GD[Am_{4,1}, Am_{8,3}, Am_{10,5}, Am_{6,9}, Am_{2,7}]}
```

V_2 Definition

```
CF[g_GD] := Sort[
   g /. Thread[Sort@Cases[g, _Integer, ∞] →
      Range[2 Length[g]]]];
PV[F_GD, g_GD] /; Length[F] > Length[g] := 0;
PV[F_GD, g_GD] /; Length[F] < Length[g] := Sum[
   PV[F, y], {y, Subsets[g, {Length[F]}]}];
PV[F_GD, g_GD] /; Length[F] == Length[g] := If[
   CF[F] === CF[g /. Ap | Am → A], (-1)^Count[g, Am__], 0];
V_2[g_] := V_2[g] = PV[GD[A_{3,1}, A_{2,4}], GD[g]];
```

Computing V_2

```
Format[Knot[n_, k_]] := n_k;
Table[K → V_2[K], {K, AllKnots@{3, 7}}]
```

```
{3_1 → 1, 4_1 → -1, 5_1 → 3, 5_2 → 2, 6_1 → -2, 6_2 → -1, 6_3 → 1,
 7_1 → 6, 7_2 → 3, 7_3 → 5, 7_4 → 4, 7_5 → 4, 7_6 → 1, 7_7 → -1}
```

V_3 Definition

```
PV[F1_ + F2_, g_] := PV[F1, g] + PV[F2, g];
PV[c_ * F_GD, g_] := c PV[F, g];
ρ_k_[g_] := g /. i_Integer :→ Mod[i - k, 2 Length@g, 1];
F_3 = Sum_{k=0}^{5} (3 ρ_k@GD[A_{1,5}, A_{4,2}, A_{6,3}] + 2 ρ_k@GD[A_{1,4}, A_{5,2}, A_{3,6}]);
V_3[K_] := V_3[K] = PV[F_3, GD@K] / 6;
```

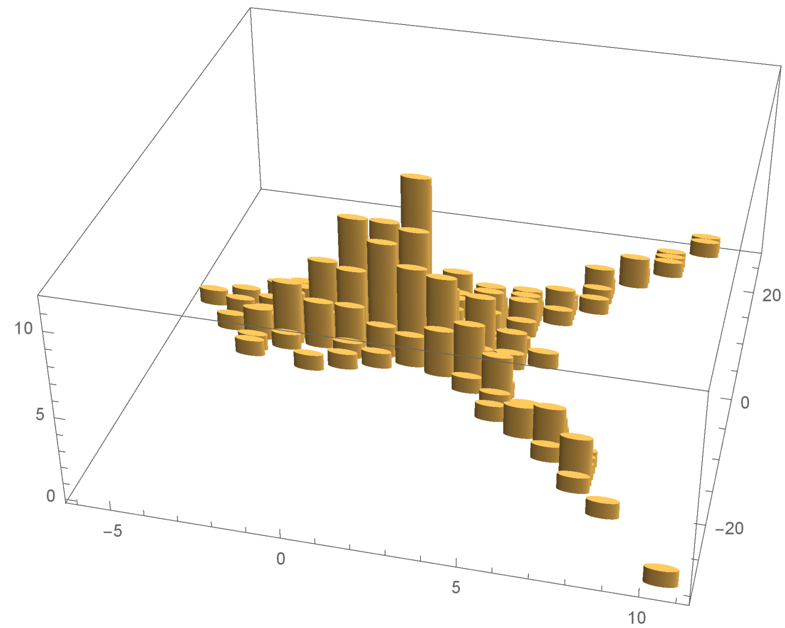
Computing V_3

```
Table[K → V_3[K], {K, AllKnots@{3, 7}}]
```

```
{3_1 → -1, 4_1 → 0, 5_1 → -5, 5_2 → -3, 6_1 → 1, 6_2 → 1, 6_3 → 0,
 7_1 → -14, 7_2 → -6, 7_3 → 11, 7_4 → 8, 7_5 → -8, 7_6 → -2, 7_7 → -1}
```

Willerton's Fish

```
Histogram3D[
 Table[{V_2[K], V_3[K]}, {K, AllKnots@{3, 10}}],
 {1}]
```



Gassner Utilities

```
G[λ_]_{a_,b_} := ∂_{t_a,h_b} λ;
G /: Factor[G[λ_]] :=
  G[Collect[λ, h_, Collect[#, t_, Factor] &]];
Format@γ_G := Module[{S = Union@Cases[γ, (h | t)_a_ :→ a, ∞]},
   Table[γ_{a,b}, {a, S}, {b, S}] // MatrixForm] ;
```

The Gassner Program

```
G /: G[λ1_] G[λ2_] := G[λ1 + λ2];
m_{a_,b_→c_}[G[λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
   ( α  β  θ )   ( ∂_{t_a,h_a} λ   ∂_{t_a,h_b} λ   ∂_{t_a} λ )
   ( γ  δ  ϵ ) = ( ∂_{t_b,h_a} λ   ∂_{t_b,h_b} λ   ∂_{t_b} λ ) /. (t | h)_{a|b} → 0;
   ( ϕ  ψ  Ξ )   ( ∂_{h_a} λ       ∂_{h_b} λ       λ         )
   μ = 1 - β;
   G[Tr[(t_c, 1)^T . ( γ + α δ / μ    ϵ + δ θ / μ ) . (h_c, 1)]] /. T_{a|b} → T_c //
                     ( ϕ + α ψ / μ    Ξ + ψ θ / μ )
    Factor];
Rp_{a_,b_} := G[Tr[(t_a, t_b)^T . ( 1  1 - T_a ) . (h_a, h_b)]];
                                  ( 0  T_a     )
Rm_{a_,b_} := Rp_{a,b} /. T_a → 1 / T_a;
```

The Gauss-Gassner-Program

```
GG[g_GD, k_, F_, BB_] :=
  Module[{n = 2 Length@g + Length@BB, y, cuts, rr, γ0, γ},
   γ0 = G[t_{n+1} h_{n+1}] Times @@ g /. {Ap → Rp, Am → Rm};
   γ0 *= G[Sum[β_{a,b} t_a h_b, {a, BB}, {b, BB}]];
   Sum[ γ = γ0;
    cuts = Cases[y, _Integer, ∞] ⋃ {n + 1};
    rr = Thread[cuts → Range[Length@cuts]];
    Do[If[! MemberQ[cuts, j], γ = γ // m_{j,j+1→j+1}], {j, n}];
    F[y /. rr, γ /. (v_)_a_ :→ v_{a/.rr}],
    (*over*) {y, Subsets[List @@ g, k]}]];
GG[g_GD, k_, F_] := GG[g, k, F, {}];
```

```
GG[GD@Knot[4, 1], {1}, F]
```

Example: 4_1

$$F\left[\{Am_{1,2}\}, \begin{pmatrix} -\frac{-1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1 T_3} & \frac{(-1+T_1)(1-T_2+T_1 T_2)(-1+T_3)}{T_1 T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_1} \\ -\frac{(-1+T_2)(-1+T_3)}{T_1 T_3} & \frac{-1+T_1+T_2-T_1 T_2+T_3-T_2 T_3+T_1 T_2 T_3}{T_1 T_3} & -\frac{-1+T_2}{T_1} \\ \frac{T_2(-1+T_3)}{T_3} & -\frac{(-1+T_1) T_2 (-1+T_3)}{T_3} & T_2 \end{pmatrix}\right] +$$

$$F\left[\{Am_{2,1}\}, \begin{pmatrix} \frac{1}{T_2} & \frac{-1+T_1}{-T_1-T_2+T_1 T_2} & -\frac{(-1+T_1)(-1+T_2)^2}{T_2(-T_1-T_2+T_1 T_2)} \\ \frac{-1+T_2}{T_2} & \frac{1-2 T_1-T_2+T_1 T_2}{-T_1-T_2+T_1 T_2} & -\frac{(-1+T_2)(-1+T_1+T_2-2 T_1 T_2-T_2^2+T_1 T_2^2)}{T_2(-T_1-T_2+T_1 T_2)} \\ 0 & 0 & T_2 \end{pmatrix}\right] +$$

$$F\left[\{Ap_{1,2}\}, \begin{pmatrix} -\frac{1-2 T_1-T_2+T_1 T_2}{-1+T_1+T_2} & \frac{(-1+T_1)^2(-1+T_2)}{-1+T_1+T_2} & 0 \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_1(1-T_1-2 T_2+T_1 T_2)}{-1+T_1+T_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] + F\left[\{Ap_{1,2}\}, \begin{pmatrix} 1 & \frac{(-1+T_1)(1-2 T_2-T_3+T_2 T_3)}{-1+T_2+T_3} & -\frac{(-1+T_1)(-1+T_2)}{-1+T_2+T_3} \\ 0 & -\frac{T_1(1-2 T_2-T_3+T_2 T_3)}{-1+T_2+T_3} & \frac{T_1(-1+T_2)}{-1+T_2+T_3} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}\right]$$

The Alexander Functional

```
FA[{x_}, γ_] := Simplify[
  Switch[x, Ap__, 1, Am__, -1] *
   Switch[x, _1,2, (γ2,2 γ3,3 - γ2,3 γ3,2)/(γ3,3 + γ1,3 γ3,2 - γ1,2 γ3,3),
    _2,1, (γ1,3 γ3,2 - γ1,2 γ3,3)/(γ3,2 - γ2,3 γ3,2 + γ2,2 γ3,3)] /. T_ → T];
GGA[K_, bb___] := GG[GD@K, {1}, FA, bb];
```

Example: 4_1

```
Simplify@With[{K = Knot[4, 1]},
  {GGA[K], Alexander[K][T], T ∂T Log[Alexander[K][T]]}]
```

$$\left\{\frac{T(-3+2T)}{1-3T+T^2}, 3-\frac{1}{T}-T, \frac{-1+T^2}{1-3T+T^2}\right\}$$

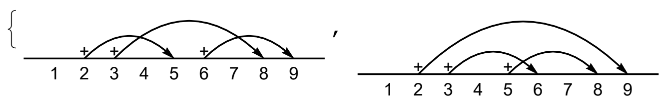
Testing for up to 7 crossings

```
Table[
 K → Simplify[GGA[K] - T ∂T Log[Alexander[K][T]]],
 {K, AllKnots@{3, 7}}]
```

$\{3_1 \to -1, 4_1 \to 1, 5_1 \to -2, 5_2 \to -2, 6_1 \to 0, 6_2 \to 0, 6_3 \to 0, 7_1 \to -3, 7_2 \to -3, 7_3 \to 4, 7_4 \to 4, 7_5 \to -3, 7_6 \to -1, 7_7 \to 2\}$

Invariance

```
Draw /@ {R3L = GD[Ap2,5, Ap3,8, Ap6,9],
  R3R = GD[Ap5,8, Ap2,9, Ap3,6]}
```




```
Simplify[
 GGA[R3L, {1, 4, 7, 10}] == GGA[R3R, {1, 4, 7, 10}] /.
  β10,b_ :> 1 - β1,b - β4,b - β7,b]
```

True

Example: Degree 2 Gauss-Gassner for 4_1

```
GG[GD@Knot[4, 1], {1, 2}, F] /. F[y_List, γ_G] :> F[Column@y, γ]
```

$$F\left[Am_{1,2}, \begin{pmatrix} -\frac{-1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1 T_3} & \frac{(-1+T_1)(1-T_2+T_1 T_2)(-1+T_3)}{T_1 T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_1} \\ -\frac{(-1+T_2)(-1+T_3)}{T_1 T_3} & \frac{-1+T_1+T_2-T_1 T_2+T_3-T_2 T_3+T_1 T_2 T_3}{T_1 T_3} & -\frac{-1+T_2}{T_1} \\ \frac{T_2(-1+T_3)}{T_3} & -\frac{(-1+T_1) T_2 (-1+T_3)}{T_3} & T_2 \end{pmatrix}\right] +$$

$$F\left[Am_{2,1}, \begin{pmatrix} \frac{1}{T_2} & \frac{-1+T_1}{-T_1-T_2+T_1 T_2} & -\frac{(-1+T_1)(-1+T_2)^2}{T_2(-T_1-T_2+T_1 T_2)} \\ \frac{-1+T_2}{T_2} & \frac{1-2 T_1-T_2+T_1 T_2}{-T_1-T_2+T_1 T_2} & -\frac{(-1+T_2)(-1+T_1+T_2-2 T_1 T_2-T_2^2+T_1 T_2^2)}{T_2(-T_1-T_2+T_1 T_2)} \\ 0 & 0 & T_2 \end{pmatrix}\right] + F\left[Ap_{1,2}, \begin{pmatrix} -\frac{1-2 T_1-T_2+T_1 T_2}{-1+T_1+T_2} & \frac{(-1+T_1)^2(-1+T_2)}{-1+T_1+T_2} & 0 \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_1(1-T_1-2 T_2+T_1 T_2)}{-1+T_1+T_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] +$$

$$F\left[Ap_{1,2}, \begin{pmatrix} 1 & \frac{(-1+T_1)(1-2 T_2-T_3+T_2 T_3)}{-1+T_2+T_3} & -\frac{(-1+T_1)(-1+T_2)}{-1+T_2+T_3} \\ 0 & -\frac{T_1(1-2 T_2-T_3+T_2 T_3)}{-1+T_2+T_3} & \frac{T_1(-1+T_2)}{-1+T_2+T_3} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}\right] + F\left[\begin{matrix} Am_{2,3} \\ Am_{4,1} \end{matrix}, \begin{pmatrix} \frac{1}{T_4} & 0 & -\frac{-1+T_1}{T_4} & 0 & 0 \\ 0 & 1 & \frac{T_1(-1+T_2)}{T_2} & 0 & -\frac{(-1+T_2)(-1+T_3)}{T_2} \\ 0 & 0 & \frac{T_1}{T_2} & 0 & -\frac{-1+T_3}{T_2} \\ \frac{-1+T_4}{T_4} & 0 & -\frac{(-1+T_1)(-1+T_4)}{T_4} & 1 & 0 \\ 0 & 0 & 0 & 0 & T_3 \end{pmatrix}\right] + F\left[\begin{matrix} Ap_{1,2} \\ Ap_{3,4} \end{matrix}, \begin{pmatrix} 1 & -\frac{-1+T_1}{T_4} & 0 & -\frac{(-1+T_1)(-1+T_2)}{T_2} & 0 \\ 0 & \frac{T_1}{T_4} & 0 & \frac{T_1(-1+T_2)}{T_2} & 0 \\ 0 & -\frac{(-1+T_3)(-1+T_4)}{T_4} & 1 & -\frac{-1+T_3}{T_2} & 0 \\ 0 & \frac{T_3(-1+T_4)}{T_4} & 0 & \frac{T_3}{T_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right] +$$

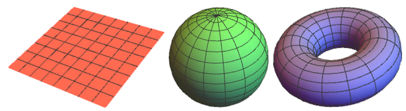
$$F\left[\begin{matrix} Ap_{1,3} \\ Am_{2,4} \end{matrix}, \begin{pmatrix} 1 & 0 & 1-T_1 & 0 & 0 \\ 0 & -\frac{-1+T_4-T_2 T_4+T_5-T_2 T_5-T_4 T_5+T_2 T_4 T_5}{T_2 T_5} & 0 & \frac{-1+T_2}{T_2} & -\frac{(-1+T_2)(-1+T_4)}{T_2} \\ 0 & 0 & T_1 & 0 & 0 \\ 0 & -\frac{(-1+T_4)(-1+T_5)}{T_2 T_5} & 0 & \frac{1}{T_2} & -\frac{-1+T_4}{T_2} \\ 0 & \frac{T_4(-1+T_5)}{T_5} & 0 & 0 & T_4 \end{pmatrix}\right] + F\left[\begin{matrix} Ap_{1,3} \\ Am_{4,2} \end{matrix}, \begin{pmatrix} 1 & 0 & 1-T_1 & -\frac{(-1+T_1)(-1+T_3)}{T_3} & \frac{(-1+T_1)(-1+T_3)(-1+T_4)}{T_3} \\ 0 & \frac{1}{T_4} & 0 & 0 & 0 \\ 0 & 0 & T_1 & \frac{T_1(-1+T_3)}{T_3} & -\frac{T_1(-1+T_3)(-1+T_4)}{T_3} \\ 0 & \frac{-1+T_4}{T_4} & 0 & \frac{1}{T_3} & -\frac{-1+T_4}{T_3} \\ 0 & 0 & 0 & 0 & T_4 \end{pmatrix}\right] +$$

$$F\left[\begin{matrix} Ap_{2,4} \\ Am_{1,3} \end{matrix}, \begin{pmatrix} \frac{1}{T_4} & -\frac{-1+T_1}{T_4} & \frac{-1+T_1}{T_1} & 0 & 0 \\ -\frac{(-1+T_2)(-1+T_4)}{T_4} & \frac{-1+T_1+T_2-T_1 T_2+T_4-T_2 T_4+T_1 T_2 T_4}{T_4} & 0 & 1-T_2 & 0 \\ 0 & 0 & \frac{1}{T_1} & 0 & 0 \\ \frac{T_2(-1+T_4)}{T_4} & -\frac{(-1+T_1) T_2 (-1+T_4)}{T_4} & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right] + F\left[\begin{matrix} Ap_{2,4} \\ Am_{3,1} \end{matrix}, \begin{pmatrix} \frac{1}{T_3} & -\frac{-1+T_1}{T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_2 T_3} & 0 & 0 \\ 0 & T_1 & \frac{T_1(-1+T_2)}{T_2} & 1-T_2 & 0 \\ \frac{-1+T_3}{T_3} & -\frac{(-1+T_1)(-1+T_3)}{T_3} & -\frac{-1+T_1+T_2-T_1 T_2-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_2 T_3} & 0 & 0 \\ 0 & 0 & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

A research proposal I am now writing - I really mean what I write, so I may as well advertise it. Comments are welcome!

Polynomial Time Knot Polynomials — Summary of Project

Construct, implement, and document poly-time computable polynomial invariants of knots and tangles



Topology is the study of spaces in themselves. What are all the spaces we might be living in? If we were living in 2 dimensions, the answer might have been the plane, or the sphere, or the torus, shown on the right. The list of possibilities is in fact somewhat longer, and topologists understand it quite well. But we live in 4 dimensions (or perhaps more, according to modern physics), and we understand 3- and 4- and higher-dimensional topology much less well. Much is known, but little can be explained in simple terms and on just one page, for even the meaning of the phrase "4-dimensional space" requires some reading.

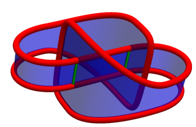
Just one part of topology is easily visible for all: the study of placements of 1-dimensional objects inside 3-dimensional objects, or **Knot Theory**. By luck or by divine justice, the study of knots is both interesting in itself and relevant to the study of higher dimensional topology. My project primarily takes place in knot theory and in closely related subjects in algebra.

Knots are often studied via their *invariants*. Invariants are quantities that can be computed from a picture of a knot and that yet depend only on the intrinsic topological properties of the knot and not on the specifics of how the picture was taken. We know a good number of very useful invariants. Unfortunately many of them are very difficult to compute and hence can be computed only for knots of relatively small size. Technically speaking, most knot invariants are "exponential time" or worse, whereas we tend to think that truly computable quantities are "polynomial time". It would be extremely desirable, perhaps revolutionary, if we had a few further poly-time computable knot invariants.

The only poly-time invariants we now know are the Alexander polynomial and finite type invariants. The Alexander polynomial can be computed in about $O(n^4)$ operations (where n is any reasonable measure of the complexity of the knot). Each coefficient of the Alexander polynomial is a numerical invariant, and hence there are infinitely many numerical invariants that can be computed in about $O(n^4)$ time. Finite type invariants can also be computed in poly-time, but there are only finitely many of them below any time bound $O(n^d)$. So Alexander stands out as an anomaly or a miracle: an infinite garden within a scarcely planted desert.

This informal picture, of an Alexander miracle in a great desert, is so unusual it must either be proven[1] or refuted. I plan to do the latter by constructing a few new poly-time computable polynomial-valued knot invariants.

I know how to do it (using near-cocommutative 2-dimensional reductions of the Etingof-Kazhdan-Enriquez quantization of classical Yang-Baxter operators and related matters; see "detailed project description"), and I know it will work. I also know it will be hard for the details are daunting (at least for me), though I hope and expect that once the mountains are first climbed, easier paths up will be found. I've been struggling and making progress with these ideas for years now, and I have a unique and uniquely powerful combination of theoretical and computational tools at my disposal. And now is the most critical time within this project: the goal is within reach, yet reaching it would still take a couple years of serious concentration.



I should add that the polynomial invariants I intend to construct extend to tangles and hence stand a fair chance of detecting counterexamples to the "slice equal ribbon" conjecture of 4-dimensional topology (a "ribbon knot", on the right). See my "detailed project description".

[1] Namely, it must be shown that there are no further poly-time invariants.

Dror Bar-Natan: Academic Pensieve: Projects: Killam-2017: ωεβ:=http://drorbn.net/K17/

Polynomial Time Knot Polynomials — Detailed Project Description

The primary goal of my project is ridiculously easy to state:

Construct, implement, and document poly-time computable polynomial invariants of knots and tangles

The following questions should immediately come to mind, and are answered below:

Q1. Why is this important? **Q2. Is it possible?** **Q3. If it is possible, why isn't it done already?**

Q1. Why is this important? I wrote a few words on the importance of knot theory in itself in the "Summary of Project" section of this proposal, so I'll only answer "why is this important within knot theory?".

First Answer. The first and primary answer is that the importance of the project is self evident (and so the more important questions are Q2 and Q3). Indeed the value of an invariant is inversely correlated with its computational complexity, for the more complicated invariants can be computed on such a small number of knots that they may have very little value beyond the purely theoretical. A few "old" knot invariants are poly-time (meaning, easy to compute and hence potentially more valuable): the Alexander polynomial, some signatures, and perhaps a little more. These invariants are "classical" — they've been known for many years (nearly a century [Al], in some cases), and they have been studied to exhaustion. The somewhat newer "finite type invariants" [Va, Go, BN1] are also poly-time [BN3], but there are only finitely many scalar-valued finite type invariants below any specific polynomial complexity bound $O(n^d)$ (where n is some measure of the complexity of a knot), and there are theorems that limit the usefulness of individual finite type invariants [Ng].

Practically all the "new" knot invariants which emerged from the work of Jones [Jo], Witten [Wi], and Khovanov [Kh] are exponential time or worse, and hence can be computed only on relatively small or very small knots, severely limiting their practical usefulness (and in fact, also limiting our ability to understand them in theory, for theories are hard to come upon when only few examples are present). Possible exceptions are the "original" Jones and HOMFLY-PT [HOMFLY, PT] polynomials and the "original" Khovanov homology which can be computed using exponential yet surprisingly efficient algorithms (the efficient Khovanov homology algorithm is due to myself [BN3]). And indeed those three invariants are better studied and better-used than the rest.

The introduction of new poly-time knot polynomials may prove at least as valuable to knot theory as the Jones and HOMFLY-PT polynomials, and as Khovanov homology.

Second Answer. The second answer has to do with "Algebraic Knot Theory", so let me start with that. Somewhat informally, a "tangle" is a piece of a knot, or a "knot with endpoints" (an example is on the right). Knots can be assembled by stitching together the strands of several tangles, or the different strands of a single tangle. Some interesting classes of knots can be defined algebraically using tangles and these stitching operations. Here is the most interesting example:

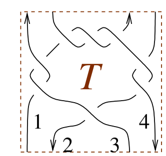


Figure 1. A tangle.

Definition 1. A "ribbon knot" is a knot K that can be presented as the boundary of a disk D which is allowed to have "ribbon singularities" but not "clasp singularities". See Figure 2.

Definition 2. Let $\mathcal{T}_{2n}$ denote the set of all tangles T with $2n$ components that connect $2n$ points along a "top end" with $2n$ points along a "bottom end" inducing the identity permutation of ends (an example is the tangle in Figure 1).

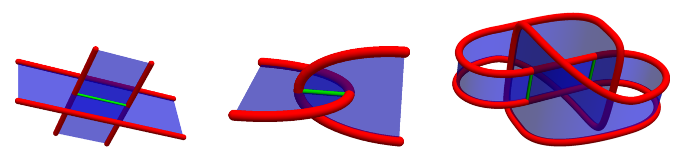
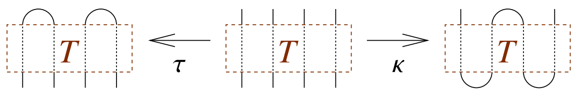


Figure 2. A ribbon singularity, a clasp singularity, and an example of a ribbon knot.

A research proposal I am now writing - I really mean what I write, so I may as well advertise it. Comments are welcome!



gives $\mathcal{T}_n \xleftarrow{\tau} \mathcal{T}_{2n} \xrightarrow{\kappa} \mathcal{T}_1$

Figure 3. The definitions of τ and κ.

Given $T \in \mathcal{T}_{2n}$, let $\tau(T)$ be the result of stitching its components at the top in pairs as in Figure 3 — it is an n-component tangle all of whose ends are at the bottom, and we (somewhat loosely) denote the set of all such by $\mathcal{T}_n$, so $\tau\colon \mathcal{T}_{2n} \to \mathcal{T}_n$. Likewise let $\kappa(T)$ be the result of stitching T both at the top and at the bottom, also as in Figure 3. So $\kappa(T)$ is a 1-component tangle, which is the same as a knot, and $\kappa\colon \mathcal{T}_{2n} \to \mathcal{T}_1$.

Theorem 1 (I have not seen this theorem in the literature, yet it is not difficult to prove). The set of ribbon knots is the set of all knots K that can be written as $K = \kappa(T)$ for some tangle T for which $\tau(T)$ is the untangled (crossingless) tangle U:

$$\{\text{ribbon knots}\} = \{\kappa(T)\colon T \in \mathcal{T}_{2n} \text{ and } \tau(T) = U \in \mathcal{T}_n\}.$$

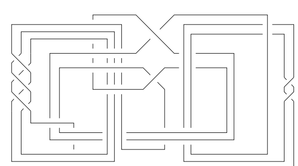
Now suppose we have an invariant $Z\colon \mathcal{T}_k \to A_k$ of tangles, which takes values in some spaces A_k. Suppose also we have operations $\tau_A\colon A_{2n} \to A_n$ and $\kappa_A\colon A_{2n} \to A_1$ such that the diagram below is commutative:

$$\begin{array}{ccccc} \mathcal{T}_n & \xleftarrow{\tau} & \mathcal{T}_{2n} & \xrightarrow{\kappa} & \mathcal{T}_1 \\ \downarrow Z & & \downarrow Z & & \downarrow Z \\ A_n & \xleftarrow{\tau_A} & A_{2n} & \xrightarrow{\kappa_A} & A_1 \end{array} \qquad (1)$$

Then

$$Z(\{\text{ribbon knots}\}) \subseteq \mathcal{R}_A := \{\kappa_A(\zeta)\colon \zeta \in A_{2n} \text{ and } \tau_A(\zeta) = 1_A \in A_n\} \subset \mathcal{A}_1, \qquad (2)$$

where $1_A := Z(U) \in A_n$. If the target spaces A_k are algebraic (polynomials, matrices, matrices of polynomials, etc.) and the operations τ_A and κ_A are algebraic maps between them (at this stage, meaning just "have simple algebraic formulas"), then $\mathcal{R}_A$ is an algebraically defined set. Hence we potentially have an algebraic way to detect non-ribbon knots: if $Z(K) \notin \mathcal{R}_A$, then K is not ribbon.



As it turns out, it is very interesting to detect non-ribbon knots. Indeed the Slice-Ribbon Conjecture (Fox, 1960s) asserts that every slice knot (a knot in S^3 that can be presented as the boundary of a disk embedded in B^4) is ribbon. Gompf, Scharlemann, and Thompson [GST] describe a family of slice knots which they conjecture are not ribbon (the simplest of those is on the right). With the algebraic technology described above it may be possible to show that the [GST] knots are indeed non-ribbon, thus disproving the Slice-Ribbon Conjecture.

What would it take?

C1. An invariant Z which makes sense on tangles and for which diagram (1) commutes.
C2. Z cannot be a simple extension of the Alexander polynomial to tangles, for by Fox-Milnor [FM] the Alexander polynomial does not detect non-ribbon slice knots.
C3. Z cannot be computable from finitely many finite type invariants, for this would contradict the results of Ng [Ng].[1]
C4. Z must be computable on at least the simplest [GST] knot, which has 48 crossings.
C5. It is better if in some meaningful sense the size of the spaces A_k grows slowly in k. Indeed in (2), if A_{2n} is much bigger than A_n and A_1 then at least generically $\mathcal{R}_A$ will be the full set A_1 and our condition will be empty.

[1]A slight subtlety arises: There is no taking limits here, and C3 does not preclude the possibility that Z is computable from infinitely many finite type invariants. The Fox-Milnor condition on the Alexander polynomial of ribbon knots, for example, is expressible in terms of the full Alexander polynomial, yet not in terms of any finite type reduction thereof. Unfortunately by C2 it cannot be used here.

No invariant that I know now meets these criteria. Alexander and Vassiliev fail C2 and C3, respectively. Almost all quantum invariants and knot homologies pass C1-C3, but fail C4. Jones, HOMFLY-PT and Khovanov potentially pass C4, yet fail C5. We must come up with something new.

Q2. Is it possible? For the last 10 years or so I knew that the answer was *yes*, in theory, but *too hard*, in practice. More recently the *too hard* became *hard, but within reach*.

Old Technology. The Kontsevich integral [Ko, BN1] is a knot invariant Z_K with values in some messy graded space $\mathcal{A}$ of diagrams modulo some relations. Using "Drinfel'd Associators" [Dr, LM, BN2] Z_K can be extended to tangles (with some small print regarding parenthesizations). The extended Z_K is way too big to be useful, but $\mathcal{A}$ has many "ideals" $\mathcal{I}$ such that the quotients $\mathcal{A}/\mathcal{I}$ and compositions $\mathcal{T}_k \to \mathcal{A} \to \mathcal{A}/\mathcal{I}$ are almost manageable. But unfortunately, only "almost". A crucial ingredient in the computation of Z_K is a Drinfel'd associator Φ, and hard as I tried, I could not find a quotient $\mathcal{A}/\mathcal{I}$ which is complex enough to carry information beyond finite degree and beyond Alexander and which is simple enough so that Φ would be computable. In short, *too hard*.

New Technology, Executive Summary. Following [EK, Ha, En, Se], imitate the "R-matrix" approach rather than the "Kontsevich/associators" approach to construct an invariant of knots/tangles in a richer space of diagrams $\mathcal{A}^{vc}$, in which the "chords" are directed. The space $\mathcal{A}^{vc}$ is less manageable and less understood than the space $\mathcal{A}$ of the "old technology", but it has many more interesting quotients of the form $\mathcal{A}^{vc}/\mathcal{I}$.

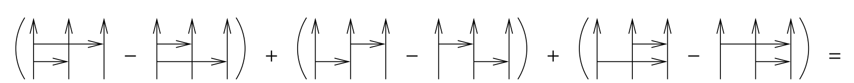
One such quotient, $\mathcal{A}^w := \mathcal{A}^{vc}/TC$, I have studied in great detail along with Z. Dancso [BND1, BND2, BND3, BN4, BN5]; it turned out to be a the key to a deep link between 4D topology and the Kashiwara-Vergne problem of Lie theory [KV, AT]. A further quotient of $\mathcal{A}^w$, call it $\mathcal{A}^{w/2D}$, arises as the "radical" of the pairing of $\mathcal{A}^w$ with co-commutative Lie bialgebras that are at most 2-dimensional. It turns out [BN4, BNS] that $\mathcal{A}^{w/2D}$ contains the Alexander family of invariants (Alexander, Multi-Variable Alexander, Burau, Gassner) in a single neat package.

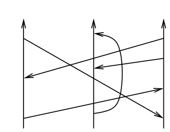
I propose to relax the key relation TC just a bit, so as to get more than the Alexander family at a bearable cost to complexity. Specifically, mod out by TC^2 instead of by TC. In the language of Lie bialgebras, TC means "the co-bracket vanishes". Similarly, TC^2 means "the co-bracket comes with coefficient ϵ such that $\epsilon^2 = 0$". The latter does not make sense in the original R-matrix context but does make sense in the diagrammatic world of $\mathcal{A}^{vc}$. Hence there is an invariant $Z^{2,2}$ valued in the resulting quotient $\mathcal{A}^{2,2} := \mathcal{A}^v/TC^2/2D$.

The invariant $Z^{2,2}$ satisfies C1–C3 and respects C5. The only obstruction to C4 appears to be the human complexity of $Z^{2,2}$: various formulas are "large", even if "simple" in the computational complexity sense. Finding the framework within which these "large" formulas will appear natural is one of the challenges I will be facing (much was done, much still needs to be done). The other major challenge will be the analysis of $\mathcal{R}_{\mathcal{A}^{2,2}}$ of Equation (2).

New Technology, Some (but not all) Details. A celebrated deformation quantization theorem of Etingof and Kazhdan [EK] (also [En, Se]) asserts (with some details suppressed) that every solution r of the Classical Yang-Baxter Equation (CYBE) $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$ can be quantized to a solution of the Quantum Yang-Baxter Equation (QYBE) $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$, which is the key to knot invariants.

This theorem was originally phrased in representation-theoretic language but it was noted by the original authors and further elucidated in [Ha] that it can also be formulated in a "universal", diagrammatic, language. Namely, r can be interpreted as a directed chord connecting two strands ("arrow", below). With this interpretation the CYBE becomes the $6T$ relation:





The $6T$ relation is written in a space $\mathcal{A}^v_k$ of diagrams made of k vertical strands and arrows connecting them ($k = 3$ here, yet whenever possible, we suppress k from the notation); one example of such diagram is displayed on the right. Hence there is a solution of the QYBE in $\mathcal{A}$, and hence there is a knot/tangle invariant Z^v with values in $\mathcal{A}^v$.[2]

[2]For simplicity here and below we tell a few lies and suppress issues having to do with cups and caps and related issues having to do with antipodes and with "cyclic Reidemeister moves". The same issues arise in the representation-theoretic version of this story, and they can be resolved here as they are resolved there. We also suppress the fact that Z^v is an expansion, or a "universal finite type invariant" for a certain class of virtual knots/tangles. That's a lovely perspective which puts Z^v in a larger context, but we don't need it here.

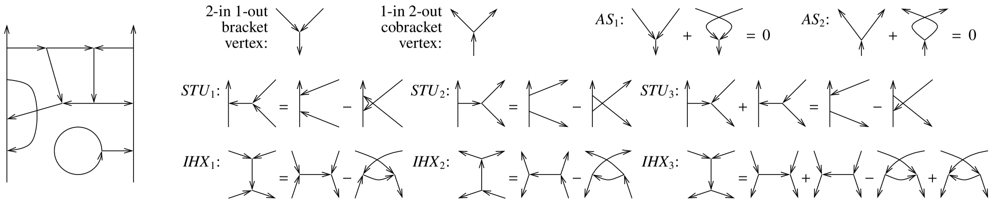


Figure 4. $\mathcal{A}^{vc}$ in some detail: a typical diagram, then the two types of internal trivalent vertices, then the AS, STU, and IHX relations. $\mathcal{A}^{vc}$ is $\mathbb{Q}\langle$such diagrams$\rangle/(AS, STU, IHX)$.

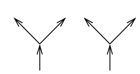
By a standard technique [BN1, Ha, BND1], internal trivalent vertices can be introduced into the diagrams and the $6T$ relation can be replaced by AS, STU, and IHX relations, making a new space of diagrams $\mathcal{A}^{vc}$ along with a map $\iota\colon \mathcal{A}^v \to \mathcal{A}^{vc}$. Figure 4 describes $\mathcal{A}^{vc}$ in some further detail. The composition $Z^{vc} := \iota \circ Z^v$ is an invariant of knots/tangles, and a few notes are in order:

N1. The space $\mathcal{A}^{vc}$ is graded and in each degree Z^{vc} is finite type.
N2. It is not known if every finite type invariant factors through Z^{vc}.
N3. All quantum invariants of knots factor through Z^{vc}: Given any semi-simple Lie algebra $\mathfrak{g}$ and representation ρ thereof, there is a $\mathbb{Q}[\![\hbar]\!]$-valued linear functional $W_{\mathfrak{g},\rho}$ on $\mathcal{A}^{vc}$ such that $W_{\mathfrak{g},\rho} \circ Z^{vc}$ is the quantum invariant associated with $(\mathfrak{g}, \rho)$ subject to the substitution $q \mapsto e^{\hbar}$.
N4. More generally $\mathcal{A}^{vc}_k$ plays the role of a "universal version of the kth tensor power of the universal enveloping algebra of the double of a finite dimensional Lie bialgebra". The details are not important here. It is enough to mention that the 2-in 1-out vertex plays bracket, the 1-in 2-out vertex plays cobracket, and IHX_{1-3} play Jacobi, co-Jacobi, and the 1-cocycle condition, respectively.

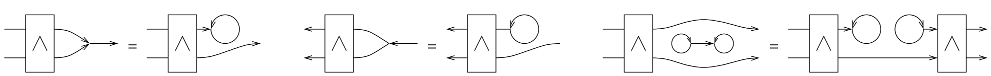
The space $\mathcal{A}^{vc}$ is way too big and hard to work with directly. We hope to construct poly-time knot/tangle polynomials by studying quotients of $\mathcal{A}^{vc}$ by appropriate "ideals" $\mathcal{I}$.[3] So $\mathcal{I}$ should be as big as possible, so as to make $\mathcal{A}^{vc}/\mathcal{I}$ small and manageable, and yet not too big, so as to allow $\mathcal{A}^{vc}$ to contain some new information.

What came next is long a process of elimination which should not be reproduced here. What matters is that I know how to construct suitable ideals, and the simplest of those is $\mathcal{I}^{2,2}$, which leads to the quotient $\mathcal{A}^{2,2} := \mathcal{A}^{vc}/\mathcal{I}^{2,2}$ and which is composed of 2 sets of relations relations revolving around the number 2:

The TC^2 or 2co Relation is shown on the right; it means that whenever a diagram in $\mathcal{A}^{vc}$ contains two "co" 1-in 2-out vertices (not necessarily adjacent), it is set to zero in $\mathcal{A}^{2,2}/\mathcal{I}^{2,2}$. (By the STU_2 relation a "co" vertex is a commutator of arrow tails, or a Tail Commutator TC, explaining the naming of this relation.) It was already noted in the "executive summary" that this relation has a stronger version (hence leading to a weaker theory), the TC or 1co or w relation, which had been studied extensively and for good reasons.



The $2D$ relations are shown next: (a box with a $\wedge$ is a standard "antisymmetrization box")



In the language of Lie bialgebras (N4), these relations correspond to restricting attention to 2-dimensional Lie bialgebras. Indeed, if $\mathfrak{a}$ is $2D$ then any two tensors in $\bigwedge^2(\mathfrak{a})$ are proportional to each other, and our $2D$ relations mean just that. The quantum invariants that come from $2D$ Lie bialgebras include the Alexander polynomial and the coloured Jones polynomial. In agreement with that, we expect $Z^{2,2}$ to be readable from the coloured Jones polynomial: In Melvin-Morton-Rozansky language [BNG], we expect it to be a *workable version* of the diagonal

[3] An "ideal", for our purpose, is a collection of relations that can be imposed without breaking the various algebraic properties of $\mathcal{A}^{vc}$ that are used in the computation of Z^{vc}. For the initiated it means "use only internal relations". All the relations we will write below are such.

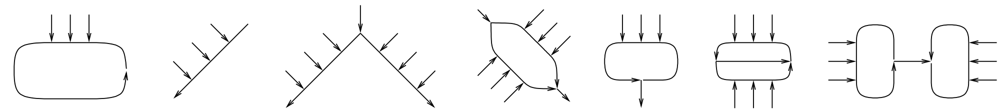


Figure 5. Primitives in $\mathcal{A}^{vc}_k$ with at most one "co" vertex, ignoring some IHX-reducibles. To complete these pictures, all loose ends in these diagrams must be connected to the k skeleton lines $\uparrow\uparrow\ldots\uparrow$ in an arbitrary manner.

above the Alexander diagonal. In the language of the Kontsevich integral, it should be an effectively computable reduction of the 2-loop part of Z_K.

Why should $Z^{2,2}$ be effectively computable? The space $\mathcal{A}^{vc}$ is in itself a Hopf algebra, and Z^{vc} is valued in its group-like elements, or in the exponentials of primitives. So in order to understand Z^{vc} it is enough to understand the primitives $\mathcal{A}^{vc}_{\text{prim}}$ in $\mathcal{A}^{vc}$. Our chosen ideal $\mathcal{I}^{2,2}$ is not a Hopf ideal and the word "primitive" does not make sense in $\mathcal{A}^{2,2}$, yet as a reduction modulo $\mathcal{I}^{2,2}$, our $Z^{2,2}$ takes values in the exponentials of projections of primitives under $\pi\colon \mathcal{A}^{vc} \to \mathcal{A}^{2,2}$. So we need to understand $\mathcal{P}^{2,2} := \pi\left(\mathcal{A}^{vc}_{\text{prim}}\right)$.

In Figure 5 we display all the connected diagrams (that is, primitives) in $\mathcal{A}^{vc}_k$ that involve at most one "co" vertex (as the rest vanish mod 2co). It is not difficult to analyze the reduction of these diagrams modulo the $2D$ relations and find that

$$\mathcal{P}^{2,2}_k \cong 2R_k \oplus V_k \oplus 2V_k^{\otimes 2} \oplus V_k^{\otimes 3} \oplus (S^2V_k)^{\otimes 2},$$

where R_k is a certain commutative ring of polynomials and V_k is a free module of rank k over R_k. What matters here is that the rank of $\mathcal{P}^{2,2}_k$ grows as a polynomial in k (as opposed to spaces of the form $V^{\otimes k}$ that normally arise in quantum algebra, whose growth rate is exponential). This suggests that computations in $\mathcal{A}^{vc}_k$ may be carried out in poly-time (and the details, not shown here, agree).

It remains to actually carry out said computations. The easier ones involve multiplying exponentials of primitives, hence they involve (polynomial reductions of) the Baker-Campbell-Hausdorff (BCH) formula. The harder ones involve the variants of BCH that occur in "stitching" two strands of a tangle to each other. These are difficult, but I have seen their likes [BN4, BN5] and I know what do. Quite a lot is already done [BN6].

Just as von Neumann algebras are no longer necessary in order to understand some of the specific formulas for the Jones polynomial, I expect that the machinery of the last 3 pages will ultimately not be needed in order to understand the formulas for the poly-time polynomial invariants it will produce. Yet I don't know how to discover such formulas by other means.

Q3. If it is possible, why isn't it done already? Because it's hard and requires a concentrated effort. I'll make good use of the Killam Research Fellowship, if I get it.

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