Loading KnotTheory`

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<< KnotTheory`
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Loading KnotTheory` version
  of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

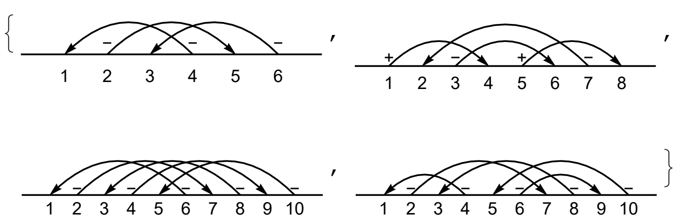
Gauss Diagram Utilities

```
GD[g_GD] := g;
GD[L_] := GD @@ PD[L] /.
   X[i_, j_, k_, l_] :> If[PositiveQ@X[i, j, k, l],
    Ap_{l,i}, Am_{j,i}];
Draw[g_GD] := Module[{n = Max@Cases[g, _Integer, ∞]},
  Graphics[{
    Line[{{0, 0}, {n + 1, 0}}],
    List @@ g /. (ah_)_{i_,j_} :> {
       Arrow[BezierCurve[{{i, 0}, {i + j, Abs[j - i]} / 2,
          {j, 0}}]],
       Text[ah /. {Ap → "+", Am → "-"}, {i, 0.3}]},
    Table[Text[i, {i, -0.5}], {i, n}]}]]
```

Some Gauss Diagrams

```
Draw /@ GD /@ AllKnots@{3, 5}
```

KnotTheory::loading : Loading precomputed data in PD4Knots`.



Some Gauss Diagrams, 2

```
GD /@ AllKnots@{3, 5}
```

```
{GD[Am_{4,1}, Am_{6,3}, Am_{2,5}], GD[Ap_{1,4}, Ap_{5,8}, Am_{3,6}, Am_{7,2}],
 GD[Am_{6,1}, Am_{8,3}, Am_{10,5}, Am_{2,7}, Am_{4,9}],
 GD[Am_{4,1}, Am_{8,3}, Am_{10,5}, Am_{6,9}, Am_{2,7}]}
```

V_2 Definition

```
CF[g_GD] := Sort[
   g /. Thread[Sort@Cases[g, _Integer, ∞] →
      Range[2 Length[g]]]];
PV[F_GD, g_GD] /; Length[F] > Length[g] := 0;
PV[F_GD, g_GD] /; Length[F] < Length[g] := Sum[
   PV[F, y], {y, Subsets[g, {Length[F]}]}];
PV[F_GD, g_GD] /; Length[F] == Length[g] := If[
   CF[F] === CF[g /. Ap | Am → A], (-1)^Count[g, Am__], 0];
V2[g_] := V2[g] = PV[GD[A_{3,1}, A_{2,4}], GD[g]];
```

Computing V_2

```
Format[Knot[n_, k_]] := n_k;
Table[K → V2[K], {K, AllKnots@{3, 7}}]
```

$\{3_1\to 1,\ 4_1\to -1,\ 5_1\to 3,\ 5_2\to 2,\ 6_1\to -2,\ 6_2\to -1,\ 6_3\to 1,\ 7_1\to 6,\ 7_2\to 3,\ 7_3\to 5,\ 7_4\to 4,\ 7_5\to 4,\ 7_6\to 1,\ 7_7\to -1\}$

V_3 Definition

```
PV[F1_ + F2_, g_] := PV[F1, g] + PV[F2, g];
PV[c_ * F_GD, g_] := c PV[F, g];
ρk_[g_] := g /. i_Integer :> Mod[i - k, 2 Length@g, 1];
F3 = Sum_{k=0}^{5} (3 ρk@GD[A_{1,5}, A_{4,2}, A_{6,3}] + 2 ρk@GD[A_{1,4}, A_{5,2}, A_{3,6}]);
V3[K_] := V3[K] = PV[F3, GD@K] / 6;
```

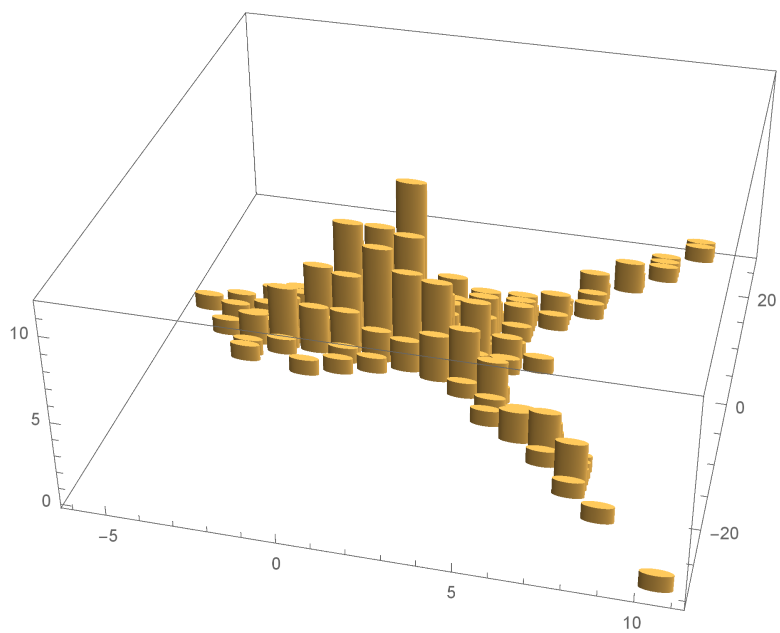
Computing V_3

```
Table[K → V3[K], {K, AllKnots@{3, 7}}]
```

$\{3_1\to -1,\ 4_1\to 0,\ 5_1\to -5,\ 5_2\to -3,\ 6_1\to 1,\ 6_2\to 1,\ 6_3\to 0,\ 7_1\to -14,\ 7_2\to -6,\ 7_3\to 11,\ 7_4\to 8,\ 7_5\to -8,\ 7_6\to -2,\ 7_7\to -1\}$

Willerton's Fish

```
Histogram3D[
 Table[{V2[K], V3[K]}, {K, AllKnots@{3, 10}}],
 {1}]
```



Gassner Utilities

```
G[λ_]_{a_,b_} := ∂_{t_a,h_b} λ;
G /: Factor[G[λ_]] :=
  G[Collect[λ, h_, Collect[#, t_, Factor] &]];
Format@γ_G := Module[{S = Union@Cases[γ, (h | t)_a_ :> a, ∞]},
   Table[γ_{a,b}, {a, S}, {b, S}] // MatrixForm] ;
```

The Gassner Program

```
G /: G[λ1_] G[λ2_] := G[λ1 + λ2];
m_{a_,b_→c_}[G[λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
```

$$\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix} = \begin{pmatrix}\partial_{t_a,h_a}\lambda & \partial_{t_a,h_b}\lambda & \partial_{t_a}\lambda\\ \partial_{t_b,h_a}\lambda & \partial_{t_b,h_b}\lambda & \partial_{t_b}\lambda\\ \partial_{h_a}\lambda & \partial_{h_b}\lambda & \lambda\end{pmatrix} \ /.\ (t\mid h)_{a|b}\to 0;$$

$\mu = 1 - \beta;$

$$G\left[\mathrm{Tr}\left[\begin{pmatrix}t_c\\1\end{pmatrix}^{T}.\begin{pmatrix}\gamma+\alpha\,\delta/\mu & \epsilon+\delta\,\theta/\mu\\ \phi+\alpha\,\psi/\mu & \Xi+\psi\,\theta/\mu\end{pmatrix}.\begin{pmatrix}h_c\\1\end{pmatrix}\right]\right] \ /.\ T_{a|b}\to T_c \ //\ \mathrm{Factor}\Big];$$

$$Rp_{a\_,b\_} := G\left[\mathrm{Tr}\left[\begin{pmatrix}t_a\\t_b\end{pmatrix}^{T}.\begin{pmatrix}1 & 1-T_a\\ 0 & T_a\end{pmatrix}.\begin{pmatrix}h_a\\h_b\end{pmatrix}\right]\right];$$

```
Rm_{a_,b_} := Rp_{a,b} /. T_a → 1 / T_a;
```

The Gauss-Gassner-Program

```
GG[g_GD, k_, F_, BB_] :=
  Module[{n = 2 Length@g + Length@BB, y, cuts, rr, γ0, γ},
   γ0 = G[t_{n+1} h_{n+1}] Times @@ g /. {Ap → Rp, Am → Rm};
   γ0 *= G[Sum[β_{a,b} t_a h_b, {a, BB}, {b, BB}]];
   Sum[ γ = γ0;
    cuts = Cases[y, _Integer, ∞] ⋃ {n + 1};
    rr = Thread[cuts → Range[Length@cuts]];
    Do[If[! MemberQ[cuts, j], γ = γ // m_{j,j+1→j+1}], {j, n}];
    F[y /. rr, γ /. (v_)_a_ :> v_{a/.rr}],
    (*over*) {y, Subsets[List @@ g, k]}]];
GG[g_GD, k_, F_] := GG[g, k, F, {}];
```

```
GG[GD@Knot[4, 1], {1}, F]
```

Example: 4_1

$$F\left[\{Am_{1,2}\}, \begin{pmatrix} -\frac{-1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1 T_3} & \frac{(-1+T_1)(1-T_2+T_1 T_2)(-1+T_3)}{T_1 T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_1} \\ -\frac{(-1+T_2)(-1+T_3)}{T_1 T_3} & \frac{-1+T_1+T_2-T_1 T_2+T_3-T_2 T_3+T_1 T_2 T_3}{T_1 T_3} & -\frac{-1+T_2}{T_1} \\ \frac{T_2(-1+T_3)}{T_3} & -\frac{(-1+T_1) T_2 (-1+T_3)}{T_3} & T_2 \end{pmatrix}\right] +$$

$$F\left[\{Am_{2,1}\}, \begin{pmatrix} \frac{1}{T_2} & \frac{-1+T_1}{-T_1-T_2+T_1 T_2} & -\frac{(-1+T_1)(-1+T_2)^2}{T_2(-T_1-T_2+T_1 T_2)} \\ \frac{-1+T_2}{T_2} & \frac{1-2 T_1-T_2+T_1 T_2}{-T_1-T_2+T_1 T_2} & -\frac{(-1+T_2)\left(-1+T_1+T_2-2 T_1 T_2-T_2^2+T_1 T_2^2\right)}{T_2(-T_1-T_2+T_1 T_2)} \\ 0 & 0 & T_2 \end{pmatrix}\right] +$$

$$F\left[\{Ap_{1,2}\}, \begin{pmatrix} -\frac{1-2 T_1-T_2+T_1 T_2}{-1+T_1+T_2} & \frac{(-1+T_1)^2(-1+T_2)}{-1+T_1+T_2} & 0 \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_1(1-T_1-2 T_2+T_1 T_2)}{-1+T_1+T_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] + F\left[\{Ap_{1,2}\}, \begin{pmatrix} 1 & \frac{(-1+T_1)(1-2 T_2-T_3+T_2 T_3)}{-1+T_2+T_3} & -\frac{(-1+T_1)(-1+T_2)}{-1+T_2+T_3} \\ 0 & -\frac{T_1(1-2 T_2-T_3+T_2 T_3)}{-1+T_2+T_3} & \frac{T_1(-1+T_2)}{-1+T_2+T_3} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}\right]$$

The Alexander Functional

$$FA[\{x\_\}, \gamma\_] := \text{Simplify}\Big[\text{Switch}[x, Ap\_\_, 1, Am\_\_, -1] * \text{Switch}\Big[x, \_{}_{1,2}, \frac{\gamma_{2,2}\gamma_{3,3}-\gamma_{2,3}\gamma_{3,2}}{\gamma_{3,3}+\gamma_{1,3}\gamma_{3,2}-\gamma_{1,2}\gamma_{3,3}}, \_{}_{2,1}, \frac{\gamma_{1,3}\gamma_{3,2}-\gamma_{1,2}\gamma_{3,3}}{\gamma_{3,2}-\gamma_{2,3}\gamma_{3,2}+\gamma_{2,2}\gamma_{3,3}}\Big] /. T\_ \to T\Big];$$

```
GGA[K_, bb___] := GG[GD@K, {1}, FA, bb];
```

Example: 4_1

```
Simplify@With[{K = Knot[4, 1]},
  {GGA[K], Alexander[K][T], T ∂_T Log[Alexander[K][T]]}]
```

$$\left\{\frac{T(-3+2 T)}{1-3 T+T^2}, 3-\frac{1}{T}-T, \frac{-1+T^2}{1-3 T+T^2}\right\}$$

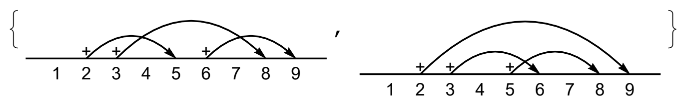
Testing for up to 7 crossings

```
Table[
 K → Simplify[GGA[K] - T ∂_T Log[Alexander[K][T]]],
 {K, AllKnots@{3, 7}}]
```

$\{3_1 \to -1, 4_1 \to 1, 5_1 \to -2, 5_2 \to -2, 6_1 \to 0, 6_2 \to 0, 6_3 \to 0, 7_1 \to -3, 7_2 \to -3, 7_3 \to 4, 7_4 \to 4, 7_5 \to -3, 7_6 \to -1, 7_7 \to 2\}$

Invariance

$\text{Draw} /@ \{R3L = GD[Ap_{2,5}, Ap_{3,8}, Ap_{6,9}],$
$R3R = GD[Ap_{5,8}, Ap_{2,9}, Ap_{3,6}]\}$



$\text{Simplify}[$
$GGA[R3L, \{1, 4, 7, 10\}] == GGA[R3R, \{1, 4, 7, 10\}] /.$
$\beta_{10,b\_} :\to 1 - \beta_{1,b} - \beta_{4,b} - \beta_{7,b}]$

True

Example: Degree 2 Gauss-Gassner for 4_1

```
GG[GD@Knot[4, 1], {1, 2}, F] /. F[y_List, γ_G] :> F[Column@y, γ]
```

$$F\left[Am_{1,2}, \begin{pmatrix} -\frac{-1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1 T_3} & \frac{(-1+T_1)(1-T_2+T_1 T_2)(-1+T_3)}{T_1 T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_1} \\ -\frac{(-1+T_2)(-1+T_3)}{T_1 T_3} & \frac{-1+T_1+T_2-T_1 T_2+T_3-T_2 T_3+T_1 T_2 T_3}{T_1 T_3} & -\frac{-1+T_2}{T_1} \\ \frac{T_2(-1+T_3)}{T_3} & -\frac{(-1+T_1) T_2 (-1+T_3)}{T_3} & T_2 \end{pmatrix}\right] +$$

$$F\left[Am_{2,1}, \begin{pmatrix} \frac{1}{T_2} & \frac{-1+T_1}{-T_1-T_2+T_1 T_2} & -\frac{(-1+T_1)(-1+T_2)^2}{T_2(-T_1-T_2+T_1 T_2)} \\ \frac{-1+T_2}{T_2} & \frac{1-2 T_1-T_2+T_1 T_2}{-T_1-T_2+T_1 T_2} & -\frac{(-1+T_2)\left(-1+T_1+T_2-2 T_1 T_2-T_2^2+T_1 T_2^2\right)}{T_2(-T_1-T_2+T_1 T_2)} \\ 0 & 0 & T_2 \end{pmatrix}\right] + F\left[Ap_{1,2}, \begin{pmatrix} -\frac{1-2 T_1-T_2+T_1 T_2}{-1+T_1+T_2} & \frac{(-1+T_1)^2(-1+T_2)}{-1+T_1+T_2} & 0 \\ \frac{T_1(-1+T_2)}{-1+T_1+T_2} & -\frac{T_1(1-T_1-2 T_2+T_1 T_2)}{-1+T_1+T_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] +$$

$$F\left[Ap_{1,2}, \begin{pmatrix} 1 & \frac{(-1+T_1)(1-2 T_2-T_3+T_2 T_3)}{-1+T_2+T_3} & -\frac{(-1+T_1)(-1+T_2)}{-1+T_2+T_3} \\ 0 & -\frac{T_1(1-2 T_2-T_3+T_2 T_3)}{-1+T_2+T_3} & \frac{T_1(-1+T_2)}{-1+T_2+T_3} \\ 0 & \frac{T_2(-1+T_3)}{-1+T_2+T_3} & \frac{T_3}{-1+T_2+T_3} \end{pmatrix}\right] + F\left[\begin{matrix} Am_{2,3} \\ Am_{4,1} \end{matrix}, \begin{pmatrix} \frac{1}{T_4} & 0 & -\frac{-1+T_1}{T_4} & 0 & 0 \\ 0 & 1 & \frac{T_1(-1+T_2)}{T_2} & 0 & -\frac{(-1+T_2)(-1+T_3)}{T_2} \\ 0 & 0 & \frac{T_1}{T_2} & 0 & -\frac{-1+T_3}{T_2} \\ \frac{-1+T_4}{T_4} & 0 & -\frac{(-1+T_1)(-1+T_4)}{T_4} & 1 & 0 \\ 0 & 0 & 0 & 0 & T_3 \end{pmatrix}\right] + F\left[\begin{matrix} Ap_{1,2} \\ Ap_{3,4} \end{matrix}, \begin{pmatrix} 1 & -\frac{-1+T_1}{T_4} & 0 & -\frac{(-1+T_1)(-1+T_2)}{T_2} & 0 \\ 0 & \frac{T_1}{T_4} & 0 & \frac{T_1(-1+T_2)}{T_2} & 0 \\ 0 & -\frac{(-1+T_3)(-1+T_4)}{T_4} & 1 & -\frac{-1+T_3}{T_2} & 0 \\ 0 & \frac{T_3(-1+T_4)}{T_4} & 0 & \frac{T_3}{T_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right] +$$

$$F\left[\begin{matrix} Ap_{1,3} \\ Am_{2,4} \end{matrix}, \begin{pmatrix} 1 & 0 & 1-T_1 & 0 & 0 \\ 0 & -\frac{-1+T_4-T_2 T_4+T_5-T_2 T_5-T_4 T_5+T_2 T_4 T_5}{T_2 T_5} & 0 & \frac{-1+T_2}{T_2} & -\frac{(-1+T_2)(-1+T_4)}{T_2} \\ 0 & 0 & T_1 & 0 & 0 \\ 0 & -\frac{(-1+T_4)(-1+T_5)}{T_2 T_5} & 0 & \frac{1}{T_2} & -\frac{-1+T_4}{T_2} \\ 0 & \frac{T_4(-1+T_5)}{T_5} & 0 & 0 & T_4 \end{pmatrix}\right] + F\left[\begin{matrix} Ap_{1,3} \\ Am_{4,2} \end{matrix}, \begin{pmatrix} 1 & 0 & 1-T_1 & -\frac{(-1+T_1)(-1+T_3)}{T_3} & \frac{(-1+T_1)(-1+T_3)(-1+T_4)}{T_3} \\ 0 & \frac{1}{T_4} & 0 & 0 & 0 \\ 0 & 0 & T_1 & \frac{T_1(-1+T_3)}{T_3} & -\frac{T_1(-1+T_3)(-1+T_4)}{T_3} \\ 0 & \frac{-1+T_4}{T_4} & 0 & \frac{1}{T_3} & -\frac{-1+T_4}{T_3} \\ 0 & 0 & 0 & 0 & T_4 \end{pmatrix}\right] +$$

$$F\left[\begin{matrix} Ap_{2,4} \\ Am_{1,3} \end{matrix}, \begin{pmatrix} \frac{1}{T_4} & -\frac{-1+T_1}{T_4} & \frac{-1+T_1}{T_1} & 0 & 0 \\ -\frac{(-1+T_2)(-1+T_4)}{T_4} & \frac{-1+T_1+T_2-T_1 T_2+T_4-T_2 T_4+T_1 T_2 T_4}{T_4} & 0 & 1-T_2 & 0 \\ 0 & 0 & \frac{1}{T_1} & 0 & 0 \\ \frac{T_2(-1+T_4)}{T_4} & -\frac{(-1+T_1) T_2 (-1+T_4)}{T_4} & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right] + F\left[\begin{matrix} Ap_{2,4} \\ Am_{3,1} \end{matrix}, \begin{pmatrix} \frac{1}{T_3} & -\frac{-1+T_1}{T_3} & -\frac{(-1+T_1)(-1+T_2)}{T_2 T_3} & 0 & 0 \\ 0 & T_1 & \frac{T_1(-1+T_2)}{T_2} & 1-T_2 & 0 \\ \frac{-1+T_3}{T_3} & -\frac{(-1+T_1)(-1+T_3)}{T_3} & -\frac{-1+T_1+T_2-T_1 T_2-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_2 T_3} & 0 & 0 \\ 0 & 0 & 0 & T_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]$$