Pensieve header: A talk and a program about Archibald- and Γ-calculus and the Halacheva map between them. Continues pensieve://2021-03/

Title. I Still don’t Understand the Alexander Polynomial

Abstract. As an algebraic knot theorist, I still don’t understand the Alexander polynomial. There are two conventions as for how to present tangle theory in algebra: one may name the strands of a tangle, or one may name their ends. The distinction might seem too minor to matter, yet it leads to a completely different view of the set of tangles as an algebraic structure. There are lovely formulas for the Alexander polynomial as viewed from either perspective, and they even agree where they meet. But the “strands” formulas know about strand doubling while the “ends” ones don’t, and the “ends” formulas know about skein relations while the “strands” ones don’t. There ought to be a common generalization, but I don’t know what it is.

𝒜-Calculus

In[∘]:=
```
WP[Wedge[u___], Wedge[v___]] := Signature[{u, v}] * Wedge @@ Sort[{u, v}];
WP[0, _] = WP[_, 0] = 0;
WP[A_, B_] :=
  Expand[Distribute[A ** B] /. (a_. * u_Wedge) ** (b_. * v_Wedge) :> a b WP[u, v]];
```

In[∘]:=
```
WExp[A_] := Module[{s = Wedge[], t = Wedge[], k = 0},
  While[t =!= 0, s += (t = Expand[WP[t, A] / (++k)])]; s]
```

In[∘]:=
```
c_{x_,y_}[w_Wedge] := Module[{i, j},
   {i} = FirstPosition[w, x, {0}]; {j} = FirstPosition[w, y, {0}];
   { w                                               (i == 0) ∧ (j == 0)
   { (-1)^(i+j+If[i>j,1,0]) Delete[w, {{i}, {j}}]   (i > 0) ∧ (j > 0)
  ];
c_{x_,y_}[ℰ_] := ℰ /. w_Wedge :> c_{x,y}[w]
```

In[∘]:=
```
𝒜[Γ[ω_, _, λ_]] := Expand[ω WExp[Expand[λ] /. t_a_ h_b_ :> ξ_a ∧ x_b]]
```

In[∘]:=
```
𝒜 /: 𝒜[is1_, os1_, cs1_, w1_] 𝒜[is2_, os2_, cs2_, w2_] :=
 𝒜[is1 ⋃ is2, os1 ⋃ os2, Join[cs1, cs2], WP[w1, w2]]
```

In[∘]:=
```
c_{h_,t_}@𝒜[is_, os_, cs_, w_] := 𝒜[
    DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {x_h, ξ_t}], c_{x_h,ξ_t}[w]
   ] /. If[cs[ξ_t][[1]] === T, cs[ξ_t] → cs[x_h], cs[x_h] → cs[ξ_t]];
c@𝒜[is_, os_, cs_, w_] := Fold[c_{#2,#2}[#1] &, 𝒜[is, os, cs, w], is ⋂ os]
```

In[∘]:=
```
𝒜[X_{i_,j_,k_,l_}[u_, o_]] := 𝒜[{i, l}, {j, k}, ⟨|ξ_i → u, x_j → o, x_k → u, ξ_l → o|⟩,
    Expand[o^{-1/2} WExp[Expand[-{ξ_l, ξ_i}.(1 1-o; 0 o).{x_j, x_k}] /. ξ_a_ x_b_ :→ ξ_a ∧ x_b]]];
𝒜[X̄_{i_,j_,k_,l_}[u_, o_]] := 𝒜[{i, j}, {k, l}, ⟨|ξ_i → u, ξ_j → o, x_k → u, x_l → o|⟩,
    Expand[o^{1/2} WExp[Expand[-{ξ_j, ξ_i}.(1 1-o^{-1}; 0 o^{-1}).{x_l, x_k}] /. ξ_a_ x_b_ :→ ξ_a ∧ x_b]]];
𝒜[X_{i_,j_,k_,l_}] := 𝒜[X_{i,j,k,l}[T_i, T_l]];
𝒜[X̄_{i_,j_,k_,l_}] := 𝒜[X̄_{i,j,k,l}[T_i, T_j]];
```

In[∘]:= `WP[Wedge[x], Wedge[y]]`

Out[∘]= x ∧ y

In[∘]:= `WP[Wedge[y], Wedge[x]]`

Out[∘]= − (x ∧ y)

In[∘]:= `A = Sum[a_i (x_i ∧ y_i), {i, 2}]`

Out[∘]= $a_1\, x_1 \wedge y_1 + a_2\, x_2 \wedge y_2$

In[∘]:= `WExp[Sum[a_i (x_i ∧ y_i), {i, 4}]]`

Out[∘]= $\mathrm{Wedge}[\,] + a_1\, x_1 \wedge y_1 + a_2\, x_2 \wedge y_2 + a_3\, x_3 \wedge y_3 + a_4\, x_4 \wedge y_4 - a_1 a_2\, x_1 \wedge x_2 \wedge y_1 \wedge y_2 - a_1 a_3\, x_1 \wedge x_3 \wedge y_1 \wedge y_3 - a_1 a_4\, x_1 \wedge x_4 \wedge y_1 \wedge y_4 - a_2 a_3\, x_2 \wedge x_3 \wedge y_2 \wedge y_3 - a_2 a_4\, x_2 \wedge x_4 \wedge y_2 \wedge y_4 - a_3 a_4\, x_3 \wedge x_4 \wedge y_3 \wedge y_4 - a_1 a_2 a_3\, x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3 - a_1 a_2 a_4\, x_1 \wedge x_2 \wedge x_4 \wedge y_1 \wedge y_2 \wedge y_4 - a_1 a_3 a_4\, x_1 \wedge x_3 \wedge x_4 \wedge y_1 \wedge y_3 \wedge y_4 - a_2 a_3 a_4\, x_2 \wedge x_3 \wedge x_4 \wedge y_2 \wedge y_3 \wedge y_4 + a_1 a_2 a_3 a_4\, x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge y_1 \wedge y_2 \wedge y_3 \wedge y_4$

In[∘]:= `c_{x_6,x_5}[x_1 ∧ x_2 ∧ x_3 ∧ x_4]`

Out[∘]= $x_1 \wedge x_2 \wedge x_3 \wedge x_4$

In[∘]:= `lhs = c_{x_4,y_4}[WExp[Sum[a_i (x_i ∧ y_i), {i, 4}]]]`

Out[∘]= $\mathrm{Wedge}[\,] - a_4\,\mathrm{Wedge}[\,] + a_1\, x_1 \wedge y_1 - a_1 a_4\, x_1 \wedge y_1 + a_2\, x_2 \wedge y_2 - a_2 a_4\, x_2 \wedge y_2 + a_3\, x_3 \wedge y_3 - a_3 a_4\, x_3 \wedge y_3 - a_1 a_2\, x_1 \wedge x_2 \wedge y_1 \wedge y_2 + a_1 a_2 a_4\, x_1 \wedge x_2 \wedge y_1 \wedge y_2 - a_1 a_3\, x_1 \wedge x_3 \wedge y_1 \wedge y_3 + a_1 a_3 a_4\, x_1 \wedge x_3 \wedge y_1 \wedge y_3 - a_2 a_3\, x_2 \wedge x_3 \wedge y_2 \wedge y_3 + a_2 a_3 a_4\, x_2 \wedge x_3 \wedge y_2 \wedge y_3 - a_1 a_2 a_3\, x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3 + a_1 a_2 a_3 a_4\, x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3$

In[∘]:= `rhs = Expand[(1 - a_4) WExp[Sum[a_i (x_i ∧ y_i), {i, 3}]]]`

Out[∘]= $\mathrm{Wedge}[\,] - a_4\,\mathrm{Wedge}[\,] + a_1\, x_1 \wedge y_1 - a_1 a_4\, x_1 \wedge y_1 + a_2\, x_2 \wedge y_2 - a_2 a_4\, x_2 \wedge y_2 + a_3\, x_3 \wedge y_3 - a_3 a_4\, x_3 \wedge y_3 - a_1 a_2\, x_1 \wedge x_2 \wedge y_1 \wedge y_2 + a_1 a_2 a_4\, x_1 \wedge x_2 \wedge y_1 \wedge y_2 - a_1 a_3\, x_1 \wedge x_3 \wedge y_1 \wedge y_3 + a_1 a_3 a_4\, x_1 \wedge x_3 \wedge y_1 \wedge y_3 - a_2 a_3\, x_2 \wedge x_3 \wedge y_2 \wedge y_3 + a_2 a_3 a_4\, x_2 \wedge x_3 \wedge y_2 \wedge y_3 - a_1 a_2 a_3\, x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3 + a_1 a_2 a_3 a_4\, x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3$

In[∘]:= `lhs == rhs`

Out[∘]= True

In[∘]:= $\mathbf{lhs} = c_{x_3,y_3}\left[c_{x_4,y_4}[\text{WExp}[\text{Sum}[a_i\ (x_i \wedge y_i),\ \{i, 4\}]]]\right]$

Out[∘]= $\text{Wedge}[\,] - a_3\,\text{Wedge}[\,] - a_4\,\text{Wedge}[\,] + a_3\,a_4\,\text{Wedge}[\,] + a_1\,x_1 \wedge y_1 - a_1\,a_3\,x_1 \wedge y_1 - a_1\,a_4\,x_1 \wedge y_1 + a_1\,a_3\,a_4\,x_1 \wedge y_1 + a_2\,x_2 \wedge y_2 - a_2\,a_3\,x_2 \wedge y_2 - a_2\,a_4\,x_2 \wedge y_2 + a_2\,a_3\,a_4\,x_2 \wedge y_2 - a_1\,a_2\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 + a_1\,a_2\,a_3\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 + a_1\,a_2\,a_4\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 - a_1\,a_2\,a_3\,a_4\,x_1 \wedge x_2 \wedge y_1 \wedge y_2$

In[∘]:= $\mathbf{n = 4;\ \gamma 0} = \Gamma\left[\omega,\ \sum_{a=1}^{n} h_a\,\sigma_a,\ \sum_{a=1}^{n}\sum_{b=1}^{n} t_a\,h_b\,\alpha_{10\,a+b}\right]$

$\gamma 0$ // tr[2]

Out[∘]= $\Gamma[\omega,\ h_1\,\sigma_1 + h_2\,\sigma_2 + h_3\,\sigma_3 + h_4\,\sigma_4,\ h_1\,\alpha_{11}\,\text{Wedge}[\,]_1 + h_2\,\alpha_{12}\,\text{Wedge}[\,]_1 + h_3\,\alpha_{13}\,\text{Wedge}[\,]_1 + h_4\,\alpha_{14}\,\text{Wedge}[\,]_1 + h_1\,\alpha_{21}\,\text{Wedge}[\,]_2 + h_2\,\alpha_{22}\,\text{Wedge}[\,]_2 + h_3\,\alpha_{23}\,\text{Wedge}[\,]_2 + h_4\,\alpha_{24}\,\text{Wedge}[\,]_2 + h_1\,\alpha_{31}\,\text{Wedge}[\,]_3 + h_2\,\alpha_{32}\,\text{Wedge}[\,]_3 + h_3\,\alpha_{33}\,\text{Wedge}[\,]_3 + h_4\,\alpha_{34}\,\text{Wedge}[\,]_3 + h_1\,\alpha_{41}\,\text{Wedge}[\,]_4 + h_2\,\alpha_{42}\,\text{Wedge}[\,]_4 + h_3\,\alpha_{43}\,\text{Wedge}[\,]_4 + h_4\,\alpha_{44}\,\text{Wedge}[\,]_4]$

Out[∘]= $\text{tr}[2][\Gamma[\omega,\ h_1\,\sigma_1 + h_2\,\sigma_2 + h_3\,\sigma_3 + h_4\,\sigma_4,\ h_1\,\alpha_{11}\,\text{Wedge}[\,]_1 + h_2\,\alpha_{12}\,\text{Wedge}[\,]_1 + h_3\,\alpha_{13}\,\text{Wedge}[\,]_1 + h_4\,\alpha_{14}\,\text{Wedge}[\,]_1 + h_1\,\alpha_{21}\,\text{Wedge}[\,]_2 + h_2\,\alpha_{22}\,\text{Wedge}[\,]_2 + h_3\,\alpha_{23}\,\text{Wedge}[\,]_2 + h_4\,\alpha_{24}\,\text{Wedge}[\,]_2 + h_1\,\alpha_{31}\,\text{Wedge}[\,]_3 + h_2\,\alpha_{32}\,\text{Wedge}[\,]_3 + h_3\,\alpha_{33}\,\text{Wedge}[\,]_3 + h_4\,\alpha_{34}\,\text{Wedge}[\,]_3 + h_1\,\alpha_{41}\,\text{Wedge}[\,]_4 + h_2\,\alpha_{42}\,\text{Wedge}[\,]_4 + h_3\,\alpha_{43}\,\text{Wedge}[\,]_4 + h_4\,\alpha_{44}\,\text{Wedge}[\,]_4]]$

In[∘]:= $\mathbf{lhs} = \mathcal{A}[\gamma 0\ //\ \text{tr}[1]]\ //\ \text{Simplify}$

Out[∘]= $\mathcal{A}[\text{tr}[1][\Gamma[\omega,\ h_1\,\sigma_1 + h_2\,\sigma_2 + h_3\,\sigma_3 + h_4\,\sigma_4,\ h_3\,\alpha_{13}\,\text{Wedge}[\,]_1 + h_4\,\alpha_{14}\,\text{Wedge}[\,]_1 + h_3\,\alpha_{23}\,\text{Wedge}[\,]_2 + h_4\,\alpha_{24}\,\text{Wedge}[\,]_2 + h_3\,\alpha_{33}\,\text{Wedge}[\,]_3 + h_4\,\alpha_{34}\,\text{Wedge}[\,]_3 + h_3\,\alpha_{43}\,\text{Wedge}[\,]_4 + h_4\,\alpha_{44}\,\text{Wedge}[\,]_4 + h_1\,(\alpha_{11}\,\text{Wedge}[\,]_1 + \alpha_{21}\,\text{Wedge}[\,]_2 + \alpha_{31}\,\text{Wedge}[\,]_3 + \alpha_{41}\,\text{Wedge}[\,]_4) + h_2\,(\alpha_{12}\,\text{Wedge}[\,]_1 + \alpha_{22}\,\text{Wedge}[\,]_2 + \alpha_{32}\,\text{Wedge}[\,]_3 + \alpha_{42}\,\text{Wedge}[\,]_4)]]]$

In[ ]:= **rhs = c_{ξ_1,x_1}[𝒜[γ0]] // Simplify**

Out[ ]= $-\omega\ (-\text{Wedge}[\,] + \alpha_{22}\, x_2 \wedge \xi_2 + \alpha_{32}\, x_2 \wedge \xi_3 + \alpha_{42}\, x_2 \wedge \xi_4 +$

$\alpha_{13}\,\alpha_{21}\, x_3 \wedge \xi_2 + \alpha_{23}\, x_3 \wedge \xi_2 + \alpha_{13}\,\alpha_{31}\, x_3 \wedge \xi_3 + \alpha_{33}\, x_3 \wedge \xi_3 + \alpha_{13}\,\alpha_{41}\, x_3 \wedge \xi_4 + \alpha_{43}\, x_3 \wedge \xi_4 +$

$\alpha_{14}\,\alpha_{21}\, x_4 \wedge \xi_2 + \alpha_{24}\, x_4 \wedge \xi_2 + \alpha_{14}\,\alpha_{31}\, x_4 \wedge \xi_3 + \alpha_{34}\, x_4 \wedge \xi_3 + \alpha_{14}\,\alpha_{41}\, x_4 \wedge \xi_4 + \alpha_{44}\, x_4 \wedge \xi_4 +$

$\alpha_{13}\,\alpha_{22}\,\alpha_{31}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 - \alpha_{13}\,\alpha_{21}\,\alpha_{32}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 - \alpha_{23}\,\alpha_{32}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 +$

$\alpha_{22}\,\alpha_{33}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 + \alpha_{13}\,\alpha_{22}\,\alpha_{41}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 - \alpha_{13}\,\alpha_{21}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 -$

$\alpha_{23}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{22}\,\alpha_{43}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{13}\,\alpha_{32}\,\alpha_{41}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{13}\,\alpha_{31}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 - \alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 + \alpha_{32}\,\alpha_{43}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 +$

$\alpha_{14}\,\alpha_{22}\,\alpha_{31}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{14}\,\alpha_{21}\,\alpha_{32}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{24}\,\alpha_{32}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 +$

$\alpha_{22}\,\alpha_{34}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{14}\,\alpha_{22}\,\alpha_{41}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 - \alpha_{14}\,\alpha_{21}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 -$

$\alpha_{24}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{22}\,\alpha_{44}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{14}\,\alpha_{32}\,\alpha_{41}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{14}\,\alpha_{31}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 - \alpha_{34}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 + \alpha_{32}\,\alpha_{44}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 +$

$\alpha_{14}\,\alpha_{23}\,\alpha_{31}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{13}\,\alpha_{24}\,\alpha_{31}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{14}\,\alpha_{21}\,\alpha_{33}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 -$

$\alpha_{24}\,\alpha_{33}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{13}\,\alpha_{21}\,\alpha_{34}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{23}\,\alpha_{34}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 +$

$\alpha_{14}\,\alpha_{23}\,\alpha_{41}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 - \alpha_{13}\,\alpha_{24}\,\alpha_{41}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 - \alpha_{14}\,\alpha_{21}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 -$

$\alpha_{24}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{13}\,\alpha_{21}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{23}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 +$

$\alpha_{14}\,\alpha_{33}\,\alpha_{41}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 - \alpha_{13}\,\alpha_{34}\,\alpha_{41}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 - \alpha_{14}\,\alpha_{31}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{34}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 + \alpha_{13}\,\alpha_{31}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 + \alpha_{33}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 +$

$\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 +$

$\alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{24}\,\alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{13}\,\alpha_{21}\,\alpha_{34}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{23}\,\alpha_{34}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 +$

$\alpha_{14}\,\alpha_{22}\,\alpha_{31}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{14}\,\alpha_{21}\,\alpha_{32}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{24}\,\alpha_{32}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{22}\,\alpha_{34}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{13}\,\alpha_{22}\,\alpha_{31}\,\alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{13}\,\alpha_{21}\,\alpha_{32}\,\alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 +$

$\alpha_{23}\,\alpha_{32}\,\alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{22}\,\alpha_{33}\,\alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 +$

$\alpha_{12}\ (\alpha_{41}\ (x_2 \wedge \xi_4 - \alpha_{33}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 - \alpha_{24}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 - \alpha_{34}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 +$

$\alpha_{24}\,\alpha_{33}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{23}\ (x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{34}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4)) +$

$\alpha_{21}\ (x_2 \wedge \xi_2 + \alpha_{43}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{34}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{44}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 +$

$\alpha_{34}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{33}\ (x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 - \alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4)) +$

$\alpha_{31}\ (x_2 \wedge \xi_3 + \alpha_{43}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 - \alpha_{24}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{44}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{24}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{23}\ (-(x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3) + \alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4))) +$

$\alpha_{11}\ (\text{Wedge}[\,] - \alpha_{42}\, x_2 \wedge \xi_4 - \alpha_{23}\, x_3 \wedge \xi_2 - \alpha_{33}\, x_3 \wedge \xi_3 - \alpha_{43}\, x_3 \wedge \xi_4 - \alpha_{24}\, x_4 \wedge \xi_2 - \alpha_{34}\, x_4 \wedge \xi_3 -$

$\alpha_{44}\, x_4 \wedge \xi_4 + \alpha_{23}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 + \alpha_{24}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 +$

$\alpha_{34}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 + \alpha_{24}\,\alpha_{33}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{23}\,\alpha_{34}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{24}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 -$

$\alpha_{23}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{34}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 - \alpha_{33}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{24}\,\alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{23}\,\alpha_{34}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{22}\ (x_2 \wedge \xi_2 + \alpha_{43}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{34}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{44}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 +$

$\alpha_{34}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{33}\ (x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 - \alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4)) -$

$\alpha_{32}\ (x_2 \wedge \xi_3 + \alpha_{43}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 - \alpha_{24}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{44}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$

$\alpha_{24}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{23}\ (-(x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3) + \alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4))))$

In[ ]:=
```
Simplify[lhs == rhs]
```

Out[ ]=

$$\mathcal{A}[\mathrm{tr}[1][\Gamma[\omega,\ h_1\,\sigma_1 + h_2\,\sigma_2 + h_3\,\sigma_3 + h_4\,\sigma_4,\ h_3\,\alpha_{13}\,\mathrm{Wedge}[\,]_1 + h_4\,\alpha_{14}\,\mathrm{Wedge}[\,]_1 +$$
$$h_3\,\alpha_{23}\,\mathrm{Wedge}[\,]_2 + h_4\,\alpha_{24}\,\mathrm{Wedge}[\,]_2 + h_3\,\alpha_{33}\,\mathrm{Wedge}[\,]_3 + h_4\,\alpha_{34}\,\mathrm{Wedge}[\,]_3 + h_3\,\alpha_{43}\,\mathrm{Wedge}[\,]_4 +$$
$$h_4\,\alpha_{44}\,\mathrm{Wedge}[\,]_4 + h_1\,(\alpha_{11}\,\mathrm{Wedge}[\,]_1 + \alpha_{21}\,\mathrm{Wedge}[\,]_2 + \alpha_{31}\,\mathrm{Wedge}[\,]_3 + \alpha_{41}\,\mathrm{Wedge}[\,]_4) +$$
$$h_2\,(\alpha_{12}\,\mathrm{Wedge}[\,]_1 + \alpha_{22}\,\mathrm{Wedge}[\,]_2 + \alpha_{32}\,\mathrm{Wedge}[\,]_3 + \alpha_{42}\,\mathrm{Wedge}[\,]_4)]]] ==$$
$$-\omega\,(-\mathrm{Wedge}[\,] + \alpha_{22}\,x_2\wedge\xi_2 + \alpha_{32}\,x_2\wedge\xi_3 + \alpha_{42}\,x_2\wedge\xi_4 + \alpha_{13}\,\alpha_{21}\,x_3\wedge\xi_2 + \alpha_{23}\,x_3\wedge\xi_2 +$$
$$\alpha_{13}\,\alpha_{31}\,x_3\wedge\xi_3 + \alpha_{33}\,x_3\wedge\xi_3 + \alpha_{13}\,\alpha_{41}\,x_3\wedge\xi_4 + \alpha_{43}\,x_3\wedge\xi_4 + \alpha_{14}\,\alpha_{21}\,x_4\wedge\xi_2 +$$
$$\alpha_{24}\,x_4\wedge\xi_2 + \alpha_{14}\,\alpha_{31}\,x_4\wedge\xi_3 + \alpha_{34}\,x_4\wedge\xi_3 + \alpha_{14}\,\alpha_{41}\,x_4\wedge\xi_4 + \alpha_{44}\,x_4\wedge\xi_4 +$$
$$\alpha_{13}\,\alpha_{22}\,\alpha_{31}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_3 - \alpha_{13}\,\alpha_{21}\,\alpha_{32}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_3 - \alpha_{23}\,\alpha_{32}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_3 +$$
$$\alpha_{22}\,\alpha_{33}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_3 + \alpha_{13}\,\alpha_{22}\,\alpha_{41}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4 - \alpha_{13}\,\alpha_{21}\,\alpha_{42}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4 -$$
$$\alpha_{23}\,\alpha_{42}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4 + \alpha_{22}\,\alpha_{43}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4 + \alpha_{13}\,\alpha_{32}\,\alpha_{41}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{13}\,\alpha_{31}\,\alpha_{42}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4 - \alpha_{33}\,\alpha_{42}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4 + \alpha_{32}\,\alpha_{43}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4 +$$
$$\alpha_{14}\,\alpha_{22}\,\alpha_{31}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3 - \alpha_{14}\,\alpha_{21}\,\alpha_{32}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3 - \alpha_{24}\,\alpha_{32}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3 +$$
$$\alpha_{22}\,\alpha_{34}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3 + \alpha_{14}\,\alpha_{22}\,\alpha_{41}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4 - \alpha_{14}\,\alpha_{21}\,\alpha_{42}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4 -$$
$$\alpha_{24}\,\alpha_{42}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4 + \alpha_{22}\,\alpha_{44}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4 + \alpha_{14}\,\alpha_{32}\,\alpha_{41}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{14}\,\alpha_{31}\,\alpha_{42}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4 - \alpha_{34}\,\alpha_{42}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4 + \alpha_{32}\,\alpha_{44}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4 +$$
$$\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3 - \alpha_{13}\,\alpha_{24}\,\alpha_{31}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3 - \alpha_{14}\,\alpha_{21}\,\alpha_{33}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3 -$$
$$\alpha_{24}\,\alpha_{33}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3 + \alpha_{13}\,\alpha_{21}\,\alpha_{34}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3 + \alpha_{23}\,\alpha_{34}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3 +$$
$$\alpha_{14}\,\alpha_{23}\,\alpha_{41}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4 - \alpha_{13}\,\alpha_{24}\,\alpha_{41}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4 - \alpha_{14}\,\alpha_{21}\,\alpha_{43}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4 -$$
$$\alpha_{24}\,\alpha_{43}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4 + \alpha_{13}\,\alpha_{21}\,\alpha_{44}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4 + \alpha_{23}\,\alpha_{44}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4 +$$
$$\alpha_{14}\,\alpha_{33}\,\alpha_{41}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4 - \alpha_{13}\,\alpha_{34}\,\alpha_{41}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4 - \alpha_{14}\,\alpha_{31}\,\alpha_{43}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{34}\,\alpha_{43}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4 + \alpha_{13}\,\alpha_{31}\,\alpha_{44}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4 + \alpha_{33}\,\alpha_{44}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4 +$$
$$\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 - \alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 +$$
$$\alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{24}\,\alpha_{33}\,\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{13}\,\alpha_{21}\,\alpha_{34}\,\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 - \alpha_{23}\,\alpha_{34}\,\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 +$$
$$\alpha_{14}\,\alpha_{22}\,\alpha_{31}\,\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 - \alpha_{14}\,\alpha_{21}\,\alpha_{32}\,\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{24}\,\alpha_{32}\,\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{22}\,\alpha_{34}\,\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{13}\,\alpha_{22}\,\alpha_{31}\,\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{13}\,\alpha_{21}\,\alpha_{32}\,\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 +$$
$$\alpha_{23}\,\alpha_{32}\,\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 - \alpha_{22}\,\alpha_{33}\,\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 +$$
$$\alpha_{12}\,(\alpha_{41}\,(x_2\wedge\xi_4 - \alpha_{33}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4 - \alpha_{24}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4 - \alpha_{34}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4 +$$
$$\alpha_{24}\,\alpha_{33}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 - \alpha_{23}\,(x_2\wedge x_3\wedge\xi_2\wedge\xi_4 + \alpha_{34}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4)) +$$
$$\alpha_{21}\,(x_2\wedge\xi_2 + \alpha_{43}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4 + \alpha_{34}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3 + \alpha_{44}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4 +$$
$$\alpha_{34}\,\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{33}\,(x_2\wedge x_3\wedge\xi_2\wedge\xi_3 - \alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4)) +$$
$$\alpha_{31}\,(x_2\wedge\xi_3 + \alpha_{43}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4 - \alpha_{24}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3 + \alpha_{44}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{24}\,\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{23}\,(-(x_2\wedge x_3\wedge\xi_2\wedge\xi_3) + \alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4))) +$$
$$\alpha_{11}\,(\mathrm{Wedge}[\,] - \alpha_{42}\,x_2\wedge\xi_4 - \alpha_{23}\,x_3\wedge\xi_2 - \alpha_{33}\,x_3\wedge\xi_3 - \alpha_{43}\,x_3\wedge\xi_4 - \alpha_{24}\,x_4\wedge\xi_2 - \alpha_{34}\,x_4\wedge\xi_3 -$$
$$\alpha_{44}\,x_4\wedge\xi_4 + \alpha_{23}\,\alpha_{42}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4 + \alpha_{33}\,\alpha_{42}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4 + \alpha_{24}\,\alpha_{42}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4 +$$
$$\alpha_{34}\,\alpha_{42}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4 + \alpha_{24}\,\alpha_{33}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3 - \alpha_{23}\,\alpha_{34}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3 +$$
$$\alpha_{24}\,\alpha_{43}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4 - \alpha_{23}\,\alpha_{44}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4 + \alpha_{34}\,\alpha_{43}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4 - \alpha_{33}\,\alpha_{44}$$
$$x_3\wedge x_4\wedge\xi_3\wedge\xi_4 - \alpha_{24}\,\alpha_{33}\,\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{23}\,\alpha_{34}\,\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{22}\,(x_2\wedge\xi_2 + \alpha_{43}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4 + \alpha_{34}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3 + \alpha_{44}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4 +$$
$$\alpha_{34}\,\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{33}\,(x_2\wedge x_3\wedge\xi_2\wedge\xi_3 - \alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4)) -$$
$$\alpha_{32}\,(x_2\wedge\xi_3 + \alpha_{43}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4 - \alpha_{24}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3 + \alpha_{44}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4 -$$
$$\alpha_{24}\,\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4 + \alpha_{23}\,(-(x_2\wedge x_3\wedge\xi_2\wedge\xi_3) + \alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4))))$$

In[∘]:= $\mathbf{Expand}\left[\left(Xp_{a,b}\ //\ \Gamma\right)[\![3]\!]\right]$

Part: Part 3 of $\Gamma[Xp_{a,b}]$ does not exist.

Out[∘]= $\Gamma\left[Xp_{a,b}\right][\![3]\!]$

In[∘]:= $\mathbf{Expand}\left[\left(Xp_{a,b}\ //\ \Gamma\right)[\![3]\!]\right]\ /.\ t_{a\_}\ h_{b\_} :\to -(\xi_a \wedge x_b)\ /.\ \{\xi_a \to \xi_l,\ \xi_b \to \xi_i,\ x_a \to x_j,\ x_b \to x_k,\ T_a \to o\}$

Part: Part 3 of $\Gamma[Xp_{a,b}]$ does not exist.

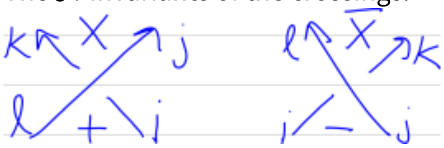
Out[∘]= $\Gamma\left[Xp_{a,b}\right][\![3]\!]$

In[∘]:= $\mathbf{Expand}\left[\left(Xm_{a,b}\ //\ \Gamma\right)[\![3]\!]\right]\ /.\ t_{a\_}\ h_{b\_} :\to -(\xi_a \wedge x_b)\ /.\ \{\xi_a \to \xi_j,\ \xi_b \to \xi_i,\ x_a \to x_l,\ x_b \to x_k,\ T_a \to o\}$

Part: Part 3 of $\Gamma[Xm_{a,b}]$ does not exist.

Out[∘]= $\Gamma\left[Xm_{a,b}\right][\![3]\!]$

The $\mathcal{A}$-invariants of the crossings.



In[∘]:= $\mathcal{A}\left[X_{i,j,k,l}\right]$

Out[∘]= $\mathcal{A}\Big[\{i, l\},\ \{j, k\},\ \langle|\ \xi_i \to T_i,\ x_j \to T_l,\ x_k \to T_i,\ \xi_l \to T_l\ |\rangle,$
$\quad \frac{\text{Wedge}[\,]}{\sqrt{T_l}} + \frac{x_j \wedge \xi_l}{\sqrt{T_l}} + \sqrt{T_l}\ x_k \wedge \xi_i + \frac{x_k \wedge \xi_l}{\sqrt{T_l}} - \sqrt{T_l}\ x_k \wedge \xi_l + \sqrt{T_l}\ x_j \wedge x_k \wedge \xi_i \wedge \xi_l\Big]$

In[∘]:= $\mathcal{A}\left[\overline{X}_{i,j,k,l}\right]$

Out[∘]= $\mathcal{A}\Big[\{i, j\},\ \{k, l\},\ \langle|\ \xi_i \to T_i,\ \xi_j \to T_j,\ x_k \to T_i,\ x_l \to T_j\ |\rangle,$
$\quad \sqrt{T_j}\ \text{Wedge}[\,] + \frac{x_k \wedge \xi_i}{\sqrt{T_j}} - \frac{x_k \wedge \xi_j}{\sqrt{T_j}} + \sqrt{T_j}\ x_k \wedge \xi_j + \sqrt{T_j}\ x_l \wedge \xi_j - \frac{x_k \wedge x_l \wedge \xi_i \wedge \xi_j}{\sqrt{T_j}}\Big]$

In[∘]:= $\mathcal{A}[X_{2,5,4,1}]\ \mathcal{A}[X_{3,7,6,5}]\ \mathcal{A}[X_{6,9,8,4}]\ //\ \mathbf{Short}$

Out[∘]//Short= $\mathcal{A}\Big[\{1, 2, 3, 4, 5, 6\},\ \{4, \ll 4 \gg, 9\},\ \ll 1 \gg,$
$\quad \frac{\text{Wedge}[\,]}{\sqrt{T_1}\ \sqrt{T_4}\ \sqrt{T_5}} + \ll 322 \gg + \sqrt{T_1}\ \sqrt{T_4}\ \sqrt{T_5}\ x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 \wedge \xi_5 \wedge \xi_6\Big]$

In[∘]:= $\mathcal{A}[X_{2,5,4,1}]\ \mathcal{A}[X_{3,7,6,5}]$

Out[∘]=

$$\mathcal{A}\Big[\{1,2,3,5\},\{4,5,6,7\},\ \langle\big|\ \xi_2\to T_2,\ x_5\to T_1,\ x_4\to T_2,\ \xi_1\to T_1,\ \xi_3\to T_3,\ x_7\to T_5,\ x_6\to T_3,\ \xi_5\to T_5\ \big|\rangle,$$

$$\frac{\text{Wedge}[\,]}{\sqrt{T_1}\,\sqrt{T_5}}+\frac{x_4\wedge\xi_1}{\sqrt{T_1}\,\sqrt{T_5}}-\frac{\sqrt{T_1}\,x_4\wedge\xi_1}{\sqrt{T_5}}+\frac{\sqrt{T_1}\,x_4\wedge\xi_2}{\sqrt{T_5}}+\frac{x_5\wedge\xi_1}{\sqrt{T_1}\,\sqrt{T_5}}+\frac{\sqrt{T_5}\,x_6\wedge\xi_3}{\sqrt{T_1}}+$$

$$\frac{x_6\wedge\xi_5}{\sqrt{T_1}\,\sqrt{T_5}}-\frac{\sqrt{T_5}\,x_6\wedge\xi_5}{\sqrt{T_1}}+\frac{x_7\wedge\xi_5}{\sqrt{T_1}\,\sqrt{T_5}}+\frac{\sqrt{T_1}\,x_4\wedge x_5\wedge\xi_1\wedge\xi_2}{\sqrt{T_5}}-\frac{\sqrt{T_5}\,x_4\wedge x_6\wedge\xi_1\wedge\xi_3}{\sqrt{T_1}}+$$

$$\sqrt{T_1}\,\sqrt{T_5}\,x_4\wedge x_6\wedge\xi_1\wedge\xi_3-\frac{x_4\wedge x_6\wedge\xi_1\wedge\xi_5}{\sqrt{T_1}\,\sqrt{T_5}}+\frac{\sqrt{T_1}\,x_4\wedge x_6\wedge\xi_1\wedge\xi_5}{\sqrt{T_5}}+\frac{\sqrt{T_5}\,x_4\wedge x_6\wedge\xi_1\wedge\xi_5}{\sqrt{T_1}}-$$

$$\sqrt{T_1}\,\sqrt{T_5}\,x_4\wedge x_6\wedge\xi_1\wedge\xi_5-\sqrt{T_1}\,\sqrt{T_5}\,x_4\wedge x_6\wedge\xi_2\wedge\xi_3-\frac{\sqrt{T_1}\,x_4\wedge x_6\wedge\xi_2\wedge\xi_5}{\sqrt{T_5}}+$$

$$\sqrt{T_1}\,\sqrt{T_5}\,x_4\wedge x_6\wedge\xi_2\wedge\xi_5-\frac{x_4\wedge x_7\wedge\xi_1\wedge\xi_5}{\sqrt{T_1}\,\sqrt{T_5}}+\frac{\sqrt{T_1}\,x_4\wedge x_7\wedge\xi_1\wedge\xi_5}{\sqrt{T_5}}-\frac{\sqrt{T_1}\,x_4\wedge x_7\wedge\xi_2\wedge\xi_5}{\sqrt{T_5}}-$$

$$\frac{\sqrt{T_5}\,x_5\wedge x_6\wedge\xi_1\wedge\xi_3}{\sqrt{T_1}}-\frac{x_5\wedge x_6\wedge\xi_1\wedge\xi_5}{\sqrt{T_1}\,\sqrt{T_5}}+\frac{\sqrt{T_5}\,x_5\wedge x_6\wedge\xi_1\wedge\xi_5}{\sqrt{T_1}}-\frac{x_5\wedge x_7\wedge\xi_1\wedge\xi_5}{\sqrt{T_1}\,\sqrt{T_5}}-$$

$$\frac{\sqrt{T_5}\,x_6\wedge x_7\wedge\xi_3\wedge\xi_5}{\sqrt{T_1}}+\sqrt{T_1}\,\sqrt{T_5}\,x_4\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_2\wedge\xi_3+\frac{\sqrt{T_1}\,x_4\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_2\wedge\xi_5}{\sqrt{T_5}}-$$

$$\sqrt{T_1}\,\sqrt{T_5}\,x_4\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_2\wedge\xi_5+\frac{\sqrt{T_1}\,x_4\wedge x_5\wedge x_7\wedge\xi_1\wedge\xi_2\wedge\xi_5}{\sqrt{T_5}}-\frac{\sqrt{T_5}\,x_4\wedge x_6\wedge x_7\wedge\xi_1\wedge\xi_3\wedge\xi_5}{\sqrt{T_1}}+$$

$$\sqrt{T_1}\,\sqrt{T_5}\,x_4\wedge x_6\wedge x_7\wedge\xi_1\wedge\xi_3\wedge\xi_5-\sqrt{T_1}\,\sqrt{T_5}\,x_4\wedge x_6\wedge x_7\wedge\xi_2\wedge\xi_3\wedge\xi_5-$$

$$\frac{\sqrt{T_5}\,x_5\wedge x_6\wedge x_7\wedge\xi_1\wedge\xi_3\wedge\xi_5}{\sqrt{T_1}}-\sqrt{T_1}\,\sqrt{T_5}\,x_4\wedge x_5\wedge x_6\wedge x_7\wedge\xi_1\wedge\xi_2\wedge\xi_3\wedge\xi_5\Big]$$

In[∘]:= $c_{4,1}[\mathcal{A}[X_{2,5,4,1}]\ \mathcal{A}[X_{3,7,6,5}]]$

Out[∘]=

$$\mathcal{A}\Big[\{2,3,5\},\{5,6,7\},\ \langle\big|\ \xi_2\to T_2,\ x_5\to T_2,\ \xi_3\to T_3,\ x_7\to T_5,\ x_6\to T_3,\ \xi_5\to T_5\ \big|\rangle,$$

$$\frac{\sqrt{T_2}\,\text{Wedge}[\,]}{\sqrt{T_5}}+\frac{\sqrt{T_2}\,x_5\wedge\xi_2}{\sqrt{T_5}}+\sqrt{T_2}\,\sqrt{T_5}\,x_6\wedge\xi_3+\frac{\sqrt{T_2}\,x_6\wedge\xi_5}{\sqrt{T_5}}-\sqrt{T_2}\,\sqrt{T_5}\,x_6\wedge\xi_5+$$

$$\frac{\sqrt{T_2}\,x_7\wedge\xi_5}{\sqrt{T_5}}-\sqrt{T_2}\,\sqrt{T_5}\,x_5\wedge x_6\wedge\xi_2\wedge\xi_3-\frac{\sqrt{T_2}\,x_5\wedge x_6\wedge\xi_2\wedge\xi_5}{\sqrt{T_5}}+\sqrt{T_2}\,\sqrt{T_5}\,x_5\wedge x_6\wedge\xi_2\wedge\xi_5-$$

$$\frac{\sqrt{T_2}\,x_5\wedge x_7\wedge\xi_2\wedge\xi_5}{\sqrt{T_5}}-\sqrt{T_2}\,\sqrt{T_5}\,x_6\wedge x_7\wedge\xi_3\wedge\xi_5-\sqrt{T_2}\,\sqrt{T_5}\,x_5\wedge x_6\wedge x_7\wedge\xi_2\wedge\xi_3\wedge\xi_5\Big]$$

In[∘]:= $c[\mathcal{A}[X_{2,5,4,1}]\,\mathcal{A}[X_{3,7,6,5}]]$

Out[∘]=
$$\mathcal{A}\Big[\{1,2,3\},\{4,6,7\},\langle|\xi_2\to T_2,\ x_4\to T_2,\ \xi_1\to T_1,\ \xi_3\to T_3,\ x_7\to T_1,\ x_6\to T_3|\rangle,\ \frac{\text{Wedge}[\,]}{T_1}-x_4\wedge\xi_1+\frac{x_4\wedge\xi_1}{T_1}+x_4\wedge\xi_2-x_6\wedge\xi_1+\frac{x_6\wedge\xi_1}{T_1}+x_6\wedge\xi_3+\frac{x_7\wedge\xi_1}{T_1}+x_4\wedge x_6\wedge\xi_1\wedge\xi_2-T_1\,x_4\wedge x_6\wedge\xi_1\wedge\xi_2-x_4\wedge x_6\wedge\xi_1\wedge\xi_3+T_1\,x_4\wedge x_6\wedge\xi_1\wedge\xi_3-T_1\,x_4\wedge x_6\wedge\xi_2\wedge\xi_3+x_4\wedge x_7\wedge\xi_1\wedge\xi_2+x_6\wedge x_7\wedge\xi_1\wedge\xi_3-T_1\,x_4\wedge x_6\wedge x_7\wedge\xi_1\wedge\xi_2\wedge\xi_3\Big]$$

In[∘]:= $\mathbf{lhs}=c[\mathcal{A}[X_{2,5,4,1}]\,\mathcal{A}[X_{3,7,6,5}]\,\mathcal{A}[X_{6,9,8,4}]]$

Out[∘]=
$$\mathcal{A}\Big[\{1,2,3\},\{7,8,9\},\langle|x_9\to T_2,\ x_8\to T_3,\ \xi_2\to T_2,\ \xi_1\to T_1,\ \xi_3\to T_3,\ x_7\to T_1|\rangle,\ \frac{\text{Wedge}[\,]}{T_1\sqrt{T_2}}+\frac{x_7\wedge\xi_1}{T_1\sqrt{T_2}}-\frac{x_8\wedge\xi_1}{\sqrt{T_2}}+\frac{x_8\wedge\xi_1}{T_1\sqrt{T_2}}+\frac{x_8\wedge\xi_2}{\sqrt{T_2}}-\sqrt{T_2}\,x_8\wedge\xi_2+\sqrt{T_2}\,x_8\wedge\xi_3-\frac{x_9\wedge\xi_1}{\sqrt{T_2}}+\frac{x_9\wedge\xi_1}{T_1\sqrt{T_2}}+\frac{x_9\wedge\xi_2}{\sqrt{T_2}}-\frac{x_7\wedge x_8\wedge\xi_1\wedge\xi_2}{\sqrt{T_2}}+\sqrt{T_2}\,x_7\wedge x_8\wedge\xi_1\wedge\xi_2-\sqrt{T_2}\,x_7\wedge x_8\wedge\xi_1\wedge\xi_3-\frac{x_7\wedge x_9\wedge\xi_1\wedge\xi_2}{\sqrt{T_2}}-\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_1\wedge\xi_2+T_1\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_1\wedge\xi_2+\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_1\wedge\xi_3-T_1\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_1\wedge\xi_3+T_1\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_2\wedge\xi_3+T_1\sqrt{T_2}\,x_7\wedge x_8\wedge x_9\wedge\xi_1\wedge\xi_2\wedge\xi_3\Big]$$

In[∘]:= $\mathbf{rhs}=c[\mathcal{A}[X_{3,5,4,2}]\,\mathcal{A}[X_{4,6,8,1}]\,\mathcal{A}[X_{5,7,9,6}]]$

Out[∘]=
$$\mathcal{A}\Big[\{1,2,3\},\{7,8,9\},\langle|x_7\to T_1,\ x_9\to T_2,\ \xi_3\to T_3,\ \xi_2\to T_2,\ x_8\to T_3,\ \xi_1\to T_1|\rangle,\ \frac{\text{Wedge}[\,]}{T_1\sqrt{T_2}}+\frac{x_7\wedge\xi_1}{T_1\sqrt{T_2}}-\frac{x_8\wedge\xi_1}{\sqrt{T_2}}+\frac{x_8\wedge\xi_1}{T_1\sqrt{T_2}}+\frac{x_8\wedge\xi_2}{\sqrt{T_2}}-\sqrt{T_2}\,x_8\wedge\xi_2+\sqrt{T_2}\,x_8\wedge\xi_3-\frac{x_9\wedge\xi_1}{\sqrt{T_2}}+\frac{x_9\wedge\xi_1}{T_1\sqrt{T_2}}+\frac{x_9\wedge\xi_2}{\sqrt{T_2}}-\frac{x_7\wedge x_8\wedge\xi_1\wedge\xi_2}{\sqrt{T_2}}+\sqrt{T_2}\,x_7\wedge x_8\wedge\xi_1\wedge\xi_2-\sqrt{T_2}\,x_7\wedge x_8\wedge\xi_1\wedge\xi_3-\frac{x_7\wedge x_9\wedge\xi_1\wedge\xi_2}{\sqrt{T_2}}-\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_1\wedge\xi_2+T_1\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_1\wedge\xi_2+\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_1\wedge\xi_3-T_1\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_1\wedge\xi_3+T_1\sqrt{T_2}\,x_8\wedge x_9\wedge\xi_2\wedge\xi_3+T_1\sqrt{T_2}\,x_7\wedge x_8\wedge x_9\wedge\xi_1\wedge\xi_2\wedge\xi_3\Big]$$

In[∘]:= lhs[[4]] == rhs[[4]]

Out[∘]= True

In[∘]:= $c\big[\mathcal{A}[X_{2,4,3,1}]\,\mathcal{A}[\overline{X}_{3,4,6,5}]\big]$

Out[∘]=
$$\mathcal{A}\Big[\{1,2\},\{5,6\},\langle|\xi_2\to T_2,\ \xi_1\to T_1,\ x_6\to T_2,\ x_5\to T_1|\rangle,\ \text{Wedge}[\,]+x_5\wedge\xi_1+x_6\wedge\xi_2-x_5\wedge x_6\wedge\xi_1\wedge\xi_2\Big]$$

In[∘]:= $\{c[\mathcal{A}[X_{1,1,2,3}]],\ c[\mathcal{A}[X_{1,2,3,3}]],\ c[\mathcal{A}[\overline{X}_{1,2,2,3}]],\ c[\mathcal{A}[\overline{X}_{1,2,3,1}]]\}$

Out[∘]= $\left\{\mathcal{A}\left[\{3\}, \{2\}, \langle| x_2 \to \underset{\cdot}{T}_3, \xi_3 \to \underset{\cdot}{T}_3 |\rangle, \frac{\text{Wedge}[]}{\sqrt{\underset{\cdot}{T}_3}} + \frac{x_2 \wedge \xi_3}{\sqrt{\underset{\cdot}{T}_3}}\right],\right.$

$\mathcal{A}\left[\{1\}, \{2\}, \langle| \xi_1 \to \underset{\cdot}{T}_1, x_2 \to \underset{\cdot}{T}_1 |\rangle, \sqrt{\underset{\cdot}{T}_1}\ \text{Wedge}[] + \sqrt{\underset{\cdot}{T}_1}\ x_2 \wedge \xi_1\right],$

$\mathcal{A}\left[\{1\}, \{3\}, \langle| \xi_1 \to \underset{\cdot}{T}_1, x_3 \to \underset{\cdot}{T}_1 |\rangle, \frac{\text{Wedge}[]}{\sqrt{\underset{\cdot}{T}_1}} + \frac{x_3 \wedge \xi_1}{\sqrt{\underset{\cdot}{T}_1}}\right],$

$\left.\mathcal{A}\left[\{2\}, \{3\}, \langle| \xi_2 \to \underset{\cdot}{T}_2, x_3 \to \underset{\cdot}{T}_2 |\rangle, \sqrt{\underset{\cdot}{T}_2}\ \text{Wedge}[] + \sqrt{\underset{\cdot}{T}_2}\ x_3 \wedge \xi_2\right]\right\}$

In[∘]:= $c[\mathcal{A}[X_{1,1,2,3}]]\ //\ c_{2,3}$

Out[∘]= $\mathcal{A}[\{\}, \{\}, \langle||\rangle, 0]$

In[∘]:= $(c[\mathcal{A}[X_{1,1,2,3}]]\ c[\mathcal{A}[X_{1,1,4,5}]])\ //\ c_{2,3}$

Out[∘]= $\mathcal{A}\left[\{5\}, \{4\}, \langle| x_4 \to \underset{\cdot}{T}_5, \xi_5 \to \underset{\cdot}{T}_5 |\rangle, 0\right]$