I Still Don't Understand the Alexander Polynomial

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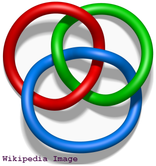
Abstract. As an algebraic knot theorist, I still don't understand the Alexander polynomial. There are two conventions as for how to present tangle theory in algebra: one may name the strands of a tangle, or one may name their ends. The distinction might seem too minor to matter, yet it leads to a completely different view of the set of tangles as an algebraic structure. There are lovely formulas for the Alexander polynomial as viewed from either perspective, and they even agree where they meet. But the "strands" formulas know about strand doubling while the "ends" ones don't, and the "ends" formulas know about skein relations while the "strands" ones don't. There ought to be a common generalization, but I don't know what it is.

Thanks for inviting me to Moscow! As most of you have never seen it, here's a picture of the lecture room:

I use talks to self-motivate; so often I choose a topic and write an abstract when I know I can do it, yet when I haven't done it yet. This time it turns out my abstract was wrong — I'm still uncomfortable with the Alexander polynomial, but in slightly different ways than advertised two slides before.

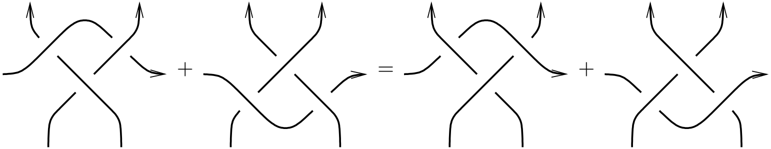
My discomfort.

- I can compute the multivariable Alexander polynomial real fast:



$$\longrightarrow (uvw)^{-1/2}(u-1)(v-1)(w-1).$$

- But I can only prove "skein relations" real slow:



This talk is to a large extent an elucidation of the Ph.D. theses of my former students Jana Archibald and Iva Halacheva. See [Ar, Ha1, Ha2].

Also thanks to Roland van der Veen for comments.

A technicality. There's supposed to be fire alarm testing in my building today. Don't panic!

1. Virtual Skein Theory Heaven

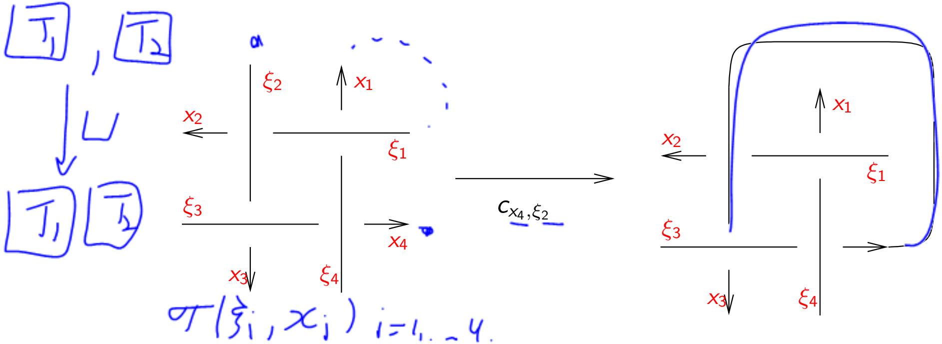
Definition. A "Contraction Algebra" assigns a set $\mathcal{T}(\mathcal{X}, X)$ to any pair of finite sets $\mathcal{X} = \{\xi \dots\}$ and $X = \{x, \dots\}$ provided $|\mathcal{X}| = |X|$, and has operations

- "Disjoint union" $\sqcup\colon \mathcal{T}(\mathcal{X}, X) \times \mathcal{T}(\mathcal{Y}, Y) \to \mathcal{T}(\mathcal{X} \sqcup \mathcal{Y}, X \sqcup Y)$, provided $\mathcal{X} \cap \mathcal{Y} = X \cap Y = \emptyset$.
- "Contractions" $c_{x,\xi}\colon \mathcal{T}(\mathcal{X}, X) \to \mathcal{T}(\mathcal{X} \setminus \xi, X \setminus x)$, provided $x \in X$ and $\xi \in \mathcal{X}$.
- Renaming operations $\sigma^\xi_\eta\colon \mathcal{T}(\mathcal{X} \sqcup \{\xi\}, X) \to \mathcal{T}(\mathcal{X} \sqcup \{\eta\}, X)$ and $\sigma^x_y\colon \mathcal{T}(\mathcal{X}, X \sqcup \{x\}) \to \mathcal{T}(\mathcal{X}, X \sqcup \{y\})$.

Subject to axioms that will be specified right after the two examples in the next three slides.

If R is a ring, a contraction algebra is said to be "R-linear" if all the $\mathcal{T}(\mathcal{X}, X)$'s are R-modules, if the disjoint union operations are R-bilinear, and if the contractions $c_{x,\xi}$ and the renamings $\sigma^\cdot_\cdot$ are R-linear.

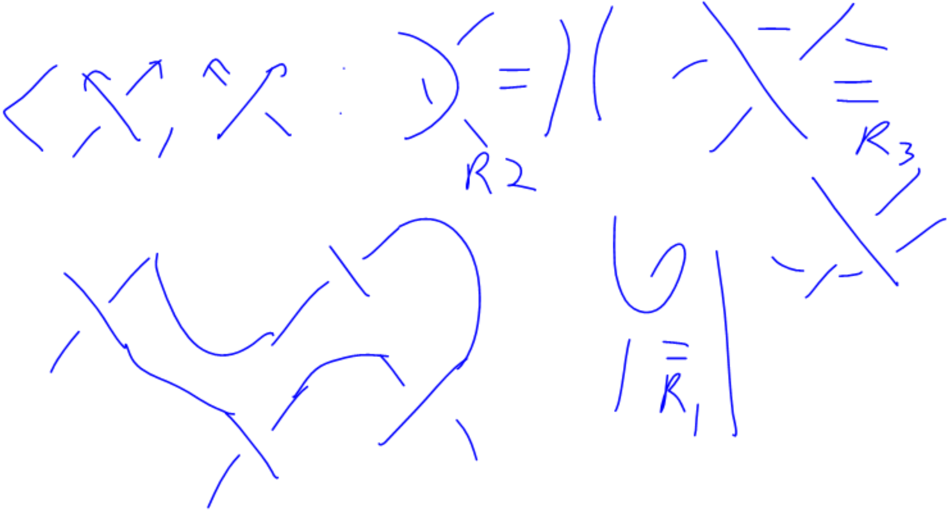
(Contraction algebras with some further "unit" properties are called "wheeled props" in [MMS, DHR])



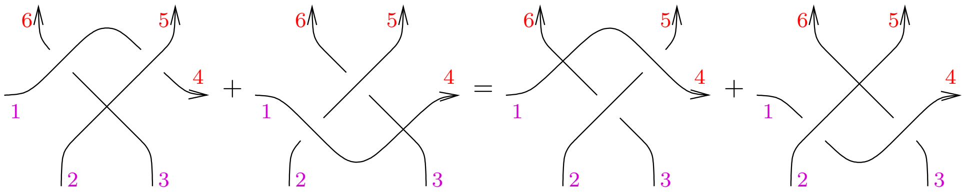
Example 1. Let $\mathcal{T}(\mathcal{X}, X)$ be the set of virtual tangles with incoming ends ("tails") labeled by $\mathcal{X}$ and outgoing ends ("heads") labeled by X, with $\sqcup$ and σ: the obvious disjoint union and end-renaming operations, and with $c_{x,\xi}$ the operation of attaching a head x to a tail ξ while introducing no new crossings.

Note 1. $\mathcal{T}$ can be made linear by allowing formal linear combinations.

Note 2. $\mathcal{T}$ is finitely presented, with generators the positive and negative crossings, and with relations the Reidemeister moves! (If you want, you can take this to be the definition of "virtual tangles").


R2
R3
R1

Note 2. A contraction algebra morphism out of $\mathcal{T}$ is an invariant of virtual tangles (and hence of virtual knots and links) and would be an ideal tool to prove Skein Relations:



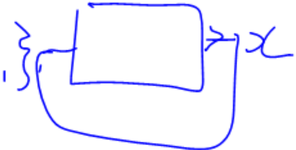
Example 2. Let V be a finite dimensional vector space and set $\mathcal{V}(\mathcal{X}, \mathcal{X}) := (V^*)^{\otimes \mathcal{X}} \otimes V^{\otimes X}$, with $\sqcup = \otimes$, with σ: the operation of renaming a factor, and with $c_{x,\xi}$ the operation of contraction: the evaluation of tensor factor ξ (which is a V^*) on tensor factor x (which is a V).

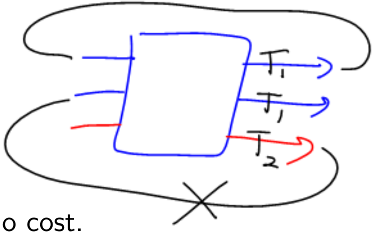
Axioms. One axiom is primary and interesting,

- Contractions commute! Namely, $c_{x,\xi}/\!\!/c_{y,\eta} = c_{y,\eta}/\!\!/c_{x,\xi}$ (or in old-speak, $c_{y,\eta} \circ c_{x,\xi} = c_{x,\xi} \circ c_{y,\eta}$).

And the rest are just what you'd expect:

- $\sqcup$ is commutative and associative, and it commutes with $c_{\cdot,\cdot}$ and with $\sigma_{\cdot}^{\cdot}$ whenever that makes sense.
- $c_{\cdot,\cdot}$ is "natural" relative to renaming: $c_{x,\xi} = \sigma_y^x/\!\!/\sigma_\eta^\xi/\!\!/c_{y,\eta}$.
- $\sigma_\xi^\xi = \sigma_x^x = Id$, $\sigma_\eta^\xi/\!\!/\sigma_\zeta^\eta = \sigma_\zeta^\xi$, $\sigma_y^x/\!\!/\sigma_z^y = \sigma_z^x$, and renaming operations commute where it makes sense.





Comments.

- We can relax $|\mathcal{X}| = |X|$ at no cost.
- We can lose the distinction between $\mathcal{X}$ and X and get "circuit algebras".
- There is a "coloured version", where $\mathcal{T}(\mathcal{X}, X)$ is replaced with $\mathcal{T}(\mathcal{X}, X, \lambda, l)$ where $\lambda\colon \mathcal{X} \to C$ and $l\colon X \to C$ are "colour functions" into some set C of "colours", and contractions $c_{x,\xi}$ are alllowed only if x and ξ are of the same colour, $l(x) = \lambda(\xi)$. In the world of tangles, this is "coloured tangles".

2. Heaven is a Place on Earth

(A version of the main results of Archibald's thesis, [Ar]).

Let us work over the base ring $\mathcal{R} = \mathbb{Q}[\{T^{\pm 1/2} \colon T \in C\}]$. Set

$$\mathcal{A}(\mathcal{X}, X) := \{w \in \Lambda(\mathcal{X} \sqcup X) \colon \deg_{\mathcal{X}} w = \deg_X w\}$$

(so in particular the elements of $\mathcal{A}(\mathcal{X}, X)$ are all of even degree). The union operation is the wedge product, the renaming operations are changes of variables, and $c_{x,\xi}$ is defined as follows. Write $w \in \mathcal{A}(\mathcal{X}, X)$ as a sum of terms of the form uw' where $u \in \Lambda(\xi, x)$ and $w' \in \mathcal{A}(\mathcal{X} \setminus \xi, X \setminus x)$, and map u to 1 if it is 1 or $x\xi$ and to 0 is if is ξ or x:

$$1w' \mapsto w', \qquad \xi w' \mapsto 0, \qquad xw' \mapsto 0, \qquad x\xi w' \mapsto w'.$$

Proposition. $\mathcal{A}$ is a contraction algebra.

Alternative Formulations.

- $$c_{x,\xi}w = \iota_\xi \iota_x e^{x\xi} w,$$ where $\iota.$ denotes interior multiplication.
- Using Fermionic integration, $$c_{x,\xi}w = \int e^{x\xi} w \, d\xi dx.$$
- $c_{x,\xi}$ represents composition in exterior algebras! With $X^* := \{x^* : x \in X\}$, we have that $\mathrm{Hom}(\Lambda X, \Lambda Y) \cong \Lambda(X^* \sqcup Y)$ and the following square commutes:

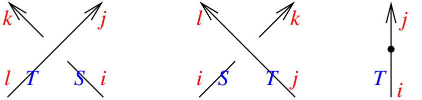
$$\begin{array}{ccc} \mathrm{Hom}(\Lambda X, \Lambda Y) \otimes \mathrm{Hom}(\Lambda Y, \Lambda Z) & \xrightarrow{/\!/} & \mathrm{Hom}(\Lambda X, \Lambda Z) \\ \updownarrow & & \updownarrow \\ \Lambda(X^* \sqcup Y \sqcup Y^* \sqcup Z) & \xrightarrow{\prod_{y \in Y} c_{y,y^*}} & \Lambda(X^*, Z) \end{array}$$

- Similarly, $\Lambda(\mathcal{X} \sqcup X) \cong (H^*)^{\otimes \mathcal{X}} \otimes H^{\otimes X}$ where H is a 2-dimensional "state space" and H^* is its dual. Under this identification, $c_{x,\xi}$ becomes the contraction of an H factor with an H^* factor.

We construct a morphism of coloured contraction algebras $\mathcal{A}\colon \mathcal{T} \to \mathcal{A}$ by declaring

$$
\begin{aligned}
X_{ijkl}[S,T] &\mapsto T^{-1/2}\exp\left(\begin{pmatrix}\xi_l & \xi_i\end{pmatrix}\begin{pmatrix}1 & 1-T\\ 0 & T\end{pmatrix}\begin{pmatrix}x_j\\ x_k\end{pmatrix}\right)\\
\bar{X}_{ijkl}[S,T] &\mapsto T^{1/2}\exp\left(\begin{pmatrix}\xi_i & \xi_j\end{pmatrix}\begin{pmatrix}T^{-1} & 0\\ 1-T^{-1} & 1\end{pmatrix}\begin{pmatrix}x_k\\ x_l\end{pmatrix}\right)\\
P_{ij}[T] &\mapsto \exp(\xi_i x_j)
\end{aligned}
$$

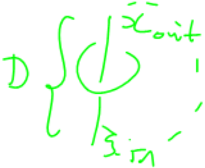
with



$X_{ijkl}[S,T]$ $\qquad$ $\bar{X}_{ijkl}[S,T]$ $\qquad$ $P_{ij}[T]$

(Note that the matrices appearing in these formulas are the Burau matrices).

Theorem.



If D is a classical link diagram with k components coloured $T_1, \ldots, T_k$ whose first component is open and the rest are closed, if MVA is the multivariable Alexander polynomial of the closure of D (with these colours), and if ρ_j is the counterclockwise rotation number of the jth component of D, then

$$\mathcal{A}(D) = T_1^{-1/2}(T_1 - 1)\left(\prod_j T_j^{\rho_j/2}\right) \cdot MVA \cdot (1 + \xi_{\text{in}} \wedge x_{\text{out}}).$$

($\mathcal{A}$ vanishes on closed links).

3. An Implementation of $\mathcal{A}$

If I didn't implement I wouldn't believe myself.

Written in Mathematica [Wo], available as the notebook `Alpha.nb` at http://drorbn.net/mo21/ap. Code lines are highlighted in grey, demo lines are plain. We start with an implementation of elements ("`Wedge`") of exterior algebras, and of the wedge product ("`WP`"):

```
WP[Wedge[u___], Wedge[v___]] := Signature[{u, v}] * Wedge @@ Sort[{u, v}];
WP[0, _] = WP[_, 0] = 0;
WP[A_, B_] :=
  Expand[Distribute[A ** B] /.
    (a_. * u_Wedge) ** (b_. * v_Wedge) :> a b WP[u, v]];
```

```
WP[Wedge[] + Wedge[a] - 2 b∧a, Wedge[] - 3 Wedge[b] + 7 c∧d]
```

Wedge[] + Wedge[a] − 3 Wedge[b] − a ∧ b + 7 c ∧ d + 7 a ∧ c ∧ d + 14 a ∧ b ∧ c ∧ d

$$e^{A} = \sum \frac{A^{\wedge n}}{n!}$$

We then define the exponentiation map in exterior algebras (“WExp”) by summing the series and stopping the sum once the current term (“t”) vanishes:

```
WExp[A_] := Module[{s = Wedge[], t = Wedge[], k = 0},
  While[t =!= 0, s += (t = Expand[WP[t, A] / (++k)])]; s]
```

```
WExp[a ∧ b + c ∧ d + e ∧ f]
```

```
Wedge[] + a ∧ b + c ∧ d + e ∧ f + a ∧ b ∧ c ∧ d + a ∧ b ∧ e ∧ f + c ∧ d ∧ e ∧ f + a ∧ b ∧ c ∧ d ∧ e ∧ f
```

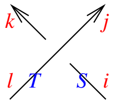
Contractions!

```
c_{x_,y_}[w_Wedge] := Module[{i, j},
    {i} = FirstPosition[w, x, {0}]; {j} = FirstPosition[w, y, {0}];
    { w                                                  (i == 0) ∧ (j == 0)
    { (-1)^(i+j+If[i>j,0,1]) Delete[w, {{i}, {j}}]       (i > 0) ∧ (j > 0)
  ];
c_{x_,y_}[ℰ_] := ℰ /. w_Wedge :> c_{x,y}[w]
WExp[a ∧ b + 2 c ∧ d]
c_{d,c}@WExp[a ∧ b + 2 c ∧ d]
```

Wedge[] + a ∧ b + 2 c ∧ d + 2 a ∧ b ∧ c ∧ d

−Wedge[] − a ∧ b

$\mathcal{A}$[is,os,cs,w] is also a container for the values of the $\mathcal{A}$-invariant of a tangle. In it, is are the labels of the input strands, os are the labels of the output strands, cs is an assignment of colours (namely, variables) to all the ends $\{\xi_i\}_{i\in \mathtt{is}} \sqcup \{x_j\}_{j\in \mathtt{os}}$, and w is the "payload": an element of $\Lambda(\{\xi_i\}_{i\in \mathtt{is}} \sqcup \{x_j\}_{j\in \mathtt{os}})$.



$X_{ijkl}[S,T]$

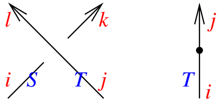
```
𝒜[X_{i_,j_,k_,l_}[S_, T_]] := 𝒜[{l, i}, {j, k}, <|ξ_i → S, x_j → T, x_k → S, ξ_l → T|>,
   Expand[T^{-1/2} WExp[Expand[{ξ_l, ξ_i}.(1  1-T
                                           0    T).{x_j, x_k}] /. ξ_a_ x_b_ :> ξ_a ∧ x_b]]];
```

```
𝒜[X_{1,2,3,4}[u, v]]
```

$$\mathcal{A}\left[\{4,1\},\{2,3\},\langle|\xi_1\to u,\ x_2\to v,\ x_3\to u,\ \xi_4\to v|\rangle,\ \frac{\mathrm{Wedge}[\,]}{\sqrt{v}}-\frac{x_2\wedge\xi_4}{\sqrt{v}}-\sqrt{v}\,x_3\wedge\xi_1-\frac{x_3\wedge\xi_4}{\sqrt{v}}+\sqrt{v}\,x_3\wedge\xi_4+\sqrt{v}\,x_2\wedge x_3\wedge\xi_1\wedge\xi_4\right]$$

```
𝒜[X̄_{i_,j_,k_,l_}] := 𝒜[X̄_{i,j,k,l}[τ_i, τ_j]];
```

The negative crossing and the “point”:



$\bar{X}_{ijkl}[S, T]$ $P_{ij}[T]$

```
𝒜[X̄i_,j_,k_,l_[S_, T_]] := 𝒜[{i, j}, {k, l}, ⟨|ξi → S, ξj → T, xk → S, xl → T|⟩,
   Expand[T^(1/2) WExp[Expand[{ξi, ξj}.( T^-1      0
                                         1 - T^-1  1 ).{xk, xl}] /. ξa_ xb_ :→ ξa ∧ xb]]];
𝒜[Xi_,j_,k_,l_] := 𝒜[Xi,j,k,l[τi, τl]];
𝒜[Pi_,j_[T_]] := 𝒜[{i}, {j}, ⟨|ξi → T, xj → T|⟩, WExp[ξi ∧ xj]];
𝒜[Pi_,j_] := 𝒜[Pi,j[τi]];
```

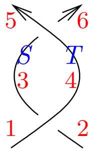
The linear structure on $\mathcal{A}$'s:

```
𝒜 /: α_ × 𝒜[is_, os_, cs_, w_] := 𝒜[is, os, cs, Expand[α w]]
𝒜 /: 𝒜[is1_, os1_, cs1_, w1_] + 𝒜[is2_, os2_, cs2_, w2_] /;
  (Sort@is1 == Sort@is2) ∧ (Sort@os1 == Sort@os2) ∧
   (Sort@Normal@cs1 == Sort@Normal@cs2) := 𝒜[is1, os1, cs1, w1 + w2]
```

Deciding if two $\mathcal{A}$'s are equal:

```
𝒜 /: 𝒜[is1_, os1_, _, w1_] ≡ 𝒜[is2_, os2_, _, w2_] :=
 TrueQ[(Sort@is1 === Sort@is2) ∧ (Sort@os1 === Sort@os2) ∧
   PowerExpand[w1 == w2]]
```

The union operation on $\mathcal{A}$'s (implemented as "multiplication"):




```
𝒜 /: 𝒜[is1_, os1_, cs1_, w1_] × 𝒜[is2_, os2_, cs2_, w2_] :=
 𝒜[is1 ⋃ is2, os1 ⋃ os2, Join[cs1, cs2], WP[w1, w2]]
```

```
Short[𝒜[X2,4,3,1[S, T]] × 𝒜[X̄3,4,6,5], 5]
```

$$\mathcal{A}\Big[\{1,2,3,4\},\ \{3,4,5,6\},\ \langle|\ \xi_2\to S,\ x_4\to T,\ x_3\to S,\ \xi_1\to T,\ \xi_3\to\tau_3,\ \xi_4\to\tau_4,\ x_6\to\tau_3,\ x_5\to\tau_4\ |\rangle,\ \frac{\sqrt{\tau_4}\ \mathrm{Wedge}[\,]}{\sqrt{T}} -$$

$$\frac{\sqrt{\tau_4}\ x_3\wedge\xi_1}{\sqrt{T}} + \sqrt{T}\ \sqrt{\tau_4}\ x_3\wedge\xi_1 - \sqrt{T}\ \sqrt{\tau_4}\ x_3\wedge\xi_2 - \frac{\sqrt{\tau_4}\ x_4\wedge\xi_1}{\sqrt{T}} - \frac{\sqrt{\tau_4}\ x_5\wedge\xi_4}{\sqrt{T}} -$$

$$\frac{x_6\wedge\xi_3}{\sqrt{T}\ \sqrt{\tau_4}} + \ll 40 \gg + \frac{\sqrt{T}\ x_3\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_3\wedge\xi_4}{\sqrt{\tau_4}} - \frac{\sqrt{T}\ x_3\wedge x_5\wedge x_6\wedge\xi_2\wedge\xi_3\wedge\xi_4}{\sqrt{\tau_4}} -$$

$$\frac{x_4\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_3\wedge\xi_4}{\sqrt{T}\ \sqrt{\tau_4}} + \frac{\sqrt{T}\ x_3\wedge x_4\wedge x_5\wedge x_6\wedge\xi_1\wedge\xi_2\wedge\xi_3\wedge\xi_4}{\sqrt{\tau_4}}\Big]$$

Contractions of $\mathcal{A}$-objects:

```
c_{h_,t_}@𝒜[is_, os_, cs_, w_] := 𝒜[
    DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {x_h, ξ_t}], c_{x_h,ξ_t}[w]
  ] /. If[MatchQ[cs[ξ_t], τ_], cs[ξ_t] → cs[x_h], cs[x_h] → cs[ξ_t]];
```

$c_{4,4}\left[\mathcal{A}[X_{2,4,3,1}[S, T]] \times \mathcal{A}[\overline{X}_{3,4,6,5}]\right]$

$$\mathcal{A}\Big[\{1, 2, 3\}, \{3, 5, 6\}, \langle| \xi_2 \to S, x_3 \to S, \xi_1 \to T, \xi_3 \to \tau_3, x_6 \to \tau_3, x_5 \to T |\rangle,$$

$$\text{Wedge}[\,] - x_3 \wedge \xi_1 + T\, x_3 \wedge \xi_1 - T\, x_3 \wedge \xi_2 - x_5 \wedge \xi_1 - x_6 \wedge \xi_1 + \frac{x_6 \wedge \xi_1}{T} - \frac{x_6 \wedge \xi_3}{T} +$$

$$T\, x_3 \wedge x_5 \wedge \xi_1 \wedge \xi_2 - x_3 \wedge x_6 \wedge \xi_1 \wedge \xi_2 + T\, x_3 \wedge x_6 \wedge \xi_1 \wedge \xi_2 + x_3 \wedge x_6 \wedge \xi_1 \wedge \xi_3 -$$

$$\frac{x_3 \wedge x_6 \wedge \xi_1 \wedge \xi_3}{T} - x_3 \wedge x_6 \wedge \xi_2 \wedge \xi_3 - \frac{x_5 \wedge x_6 \wedge \xi_1 \wedge \xi_3}{T} - x_3 \wedge x_5 \wedge x_6 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3\Big]$$

Automatic and intelligent multiple contractions:

```
c@𝒜[is_, os_, cs_, w_] := Fold[c#2,#2[#1] &, 𝒜[is, os, cs, w], is ⋂ os]
𝒜[{A_𝒜}] := c[A];
𝒜[{A1_𝒜, As__𝒜}] := Module[{A2},
  A2 = First@MaximalBy[{As}, Length[A1⟦1⟧ ⋂ #⟦2⟧] + Length[A1⟦2⟧ ⋂ #⟦1⟧] &];
  𝒜[Join[{c[A1 A2]}, DeleteCases[{As}, A2]]] ]
𝒜[𝒪s_List] := 𝒜[𝒜 /@ 𝒪s]
```

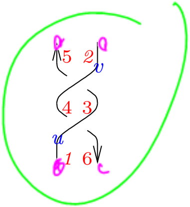
$c\left[\mathcal{A}[X_{2,4,3,1}[S, T]] \times \mathcal{A}[\bar{X}_{3,4,6,5}]\right]$

$\mathcal{A}[\{1, 2\}, \{5, 6\}, \langle| \xi_2 \to S, \xi_1 \to T, x_6 \to S, x_5 \to T |\rangle,$
$\text{Wedge}[] - x_5 \wedge \xi_1 - x_6 \wedge \xi_2 - x_5 \wedge x_6 \wedge \xi_1 \wedge \xi_2]$

$\mathcal{A}@\left\{\mathcal{A}[X_{2,4,3,1}[S, T]], \mathcal{A}[\bar{X}_{3,4,6,5}]\right\}$

$\mathcal{A}[\{1, 2\}, \{5, 6\}, \langle| \xi_2 \to S, \xi_1 \to T, x_6 \to S, x_5 \to T |\rangle,$
$\text{Wedge}[] - x_5 \wedge \xi_1 - x_6 \wedge \xi_2 - x_5 \wedge x_6 \wedge \xi_1 \wedge \xi_2]$

4. Skein relations and evaluations for $\mathcal{A}$



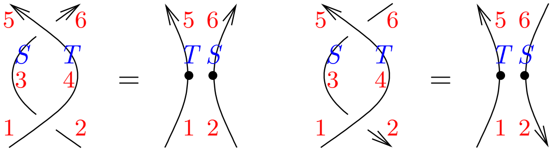
$\mathcal{A}@\{\overline{X}_{4,1,6,3}[v, u], \overline{X}_{3,2,5,4}\}$

$\mathcal{A}\Big[\{1, 2\}, \{5, 6\}, \langle| \xi_2 \to v, x_5 \to u, \xi_1 \to u, x_6 \to v |\rangle,$

$$\sqrt{u}\,\sqrt{v}\,\text{Wedge}[\,] - \frac{\sqrt{u}\,x_5\wedge\xi_1}{\sqrt{v}} + \frac{\sqrt{u}\,x_5\wedge\xi_2}{\sqrt{v}} - \sqrt{u}\,\sqrt{v}\,x_5\wedge\xi_2 + \frac{\sqrt{v}\,x_6\wedge\xi_1}{\sqrt{u}} - \sqrt{u}\,\sqrt{v}\,x_6\wedge\xi_1$$

$$\frac{\sqrt{v}\,x_6\wedge\xi_2}{\sqrt{u}} - \frac{\sqrt{u}\,x_5\wedge x_6\wedge\xi_1\wedge\xi_2}{\sqrt{v}} - \frac{\sqrt{v}\,x_5\wedge x_6\wedge\xi_1\wedge\xi_2}{\sqrt{u}} + \sqrt{u}\,\sqrt{v}\,x_5\wedge x_6\wedge\xi_1\wedge\xi_2\Big]$$

Reidemeister 2



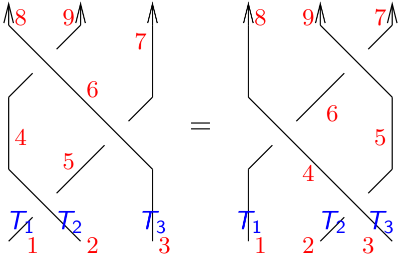
$\mathcal{A}$@{$X_{2,4,3,1}$[S, T], $\overline{X}_{3,4,6,5}$} ≡ $\mathcal{A}$@{$P_{1,5}$[T], $P_{2,6}$[S]}

True

$\mathcal{A}$@{$\overline{X}_{3,1,2,4}$[S, T], $X_{6,5,3,4}$} ≡ $\mathcal{A}$@{$P_{1,5}$[T], $P_{6,2}$[S]}

True

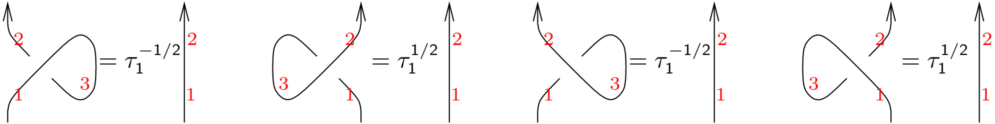
Reidemeister 3



$\mathcal{A}$@{$X_{2,5,4,1}$[T_2, T_1], $X_{3,7,6,5}$[T_3, T_1], $X_{6,9,8,4}$} ≡
$\mathcal{A}$@{$X_{3,5,4,2}$[T_3, T_2], $X_{4,6,8,1}$[T_3, T_1], $X_{5,7,9,6}$}

True

Reidemeister 1

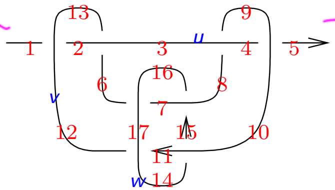


$$\left\{\mathcal{A}@\{X_{3,3,2,1}\} \equiv \tau_1^{-1/2}\,\mathcal{A}@\{P_{1,2}\},\ \mathcal{A}@\{X_{1,2,3,3}\} \equiv \tau_1^{1/2}\,\mathcal{A}@\{P_{1,2}\},\right.$$
$$\left.\mathcal{A}@\{\overline{X}_{1,3,3,2}\} \equiv \tau_1^{-1/2}\,\mathcal{A}@\{P_{1,2}\},\ \mathcal{A}@\{\overline{X}_{3,1,2,3}\} \equiv \tau_1^{1/2}\,\mathcal{A}@\{P_{1,2}\}\right\}$$

{True, True, True, True}

(So we have an invariant, up to rotation numbers).

The Relation with the Multivariable Alexander Polynomial



$\texttt{MVA} = u^{-1/2}\, v^{-1/2}\, w^{-1/2}\, (u-1)\, (v-1)\, (w-1);$

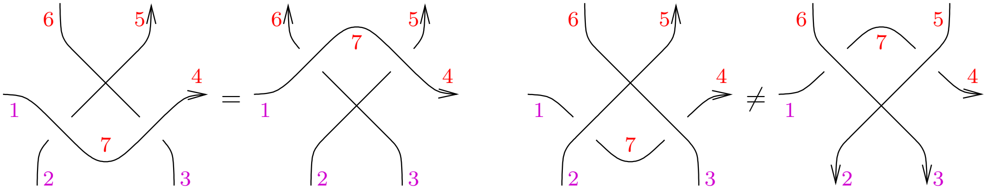
$\texttt{A} = \{\overline{X}_{1,12,2,13}[u, v],\ \overline{X}_{13,2,6,3},\ X_{8,4,9,3},\ X_{4,10,5,9},\ X_{6,17,7,16}[v, w],$
$\quad X_{15,8,16,7},\ \overline{X}_{14,10,15,11},\ \overline{X}_{11,17,12,14}\}\ \texttt{//}\ \mathcal{A}\ \texttt{//}\ \texttt{Last}\ \texttt{//}\ \texttt{Factor}$

$$\frac{(-1+u)^2\,(-1+v)\,(-1+w)\,(\text{Wedge}[\,] - x_5 \wedge \xi_1)}{u\, v}$$

$\texttt{A} == u^{-1/2}\, (u-1)\, u^0\, v^{-1/2}\, w^{1/2}\, \texttt{MVA}\, (\text{Wedge}[\,] - x_5 \wedge \xi_1)$

True

Overcrossings Commute but Undercrossings don't



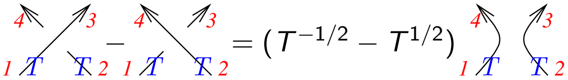
$\mathcal{A}$@$\{X_{2,7,5,1},\, X_{3,4,6,7}\} \equiv$ $\mathcal{A}$@$\{X_{3,7,6,1},\, X_{2,4,5,7}\}$

True

$\mathcal{A}$@$\{\overline{X}_{1,2,7,5},\, \overline{X}_{7,3,4,6}\} \equiv$ $\mathcal{A}$@$\{\overline{X}_{1,3,7,6},\, \overline{X}_{7,2,4,5}\}$

False

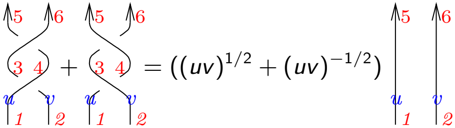
The Conway Relation (see [Co])

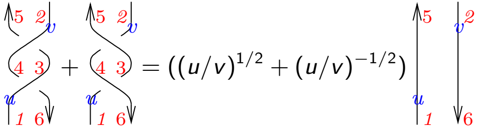


$$\mathcal{A}@\{X_{2,3,4,1}[T, T]\} - \mathcal{A}@\left\{\overline{X}_{1,2,3,4}[T, T]\right\} \equiv \left(T^{-1/2} - T^{1/2}\right) \mathcal{A}@\{P_{1,4}[T], P_{2,3}[T]\}$$

True

Conway's Second Set of Identities (see [Co])





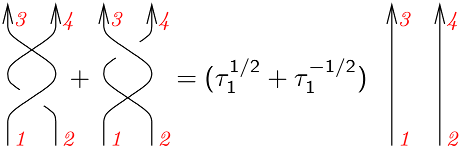

```
𝒜@{X_{2,4,3,1}[v, u], X_{4,6,5,3}} + 𝒜@{X̄_{1,2,4,3}[u, v], X̄_{3,4,6,5}} ≡
  (u^{1/2} v^{1/2} + u^{-1/2} v^{-1/2}) 𝒜@{P_{1,5}[u], P_{2,6}[v]}
```

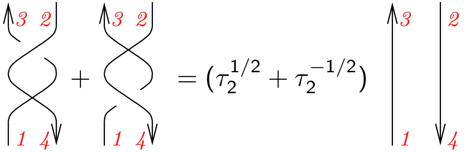
True

```
𝒜@{X̄_{4,1,6,3}[v, u], X̄_{3,2,5,4}} + 𝒜@{X_{1,6,3,4}[u, v], X_{2,5,4,3}} ≡
  (u^{1/2} v^{-1/2} + u^{-1/2} v^{1/2}) 𝒜@{P_{1,5}[u], P_{2,6}[v]}
```

True

Virtual versions (Archibald, [Ar])





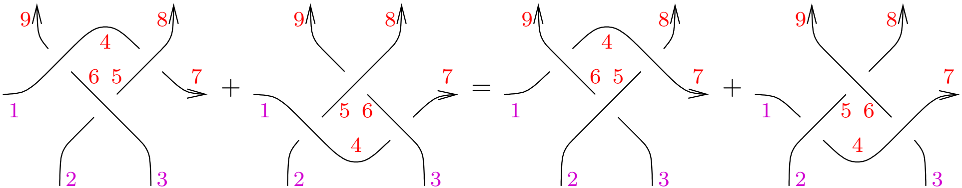
$\mathcal{A}$@$\{X_{2,3,4,1}\}$ + $\mathcal{A}$@$\{\overline{X}_{2,1,4,3}\}$ ≡ $\left(\tau_1^{1/2} + \tau_1^{-1/2}\right)$ $\mathcal{A}$@$\{P_{1,3}, P_{2,4}\}$

True

$\mathcal{A}$@$\{\overline{X}_{1,2,3,4}\}$ + $\mathcal{A}$@$\{X_{1,4,3,2}\}$ ≡ $\left(\tau_2^{1/2} + \tau_2^{-1/2}\right)$ $\mathcal{A}$@$\{P_{1,3}, P_{2,4}\}$

True

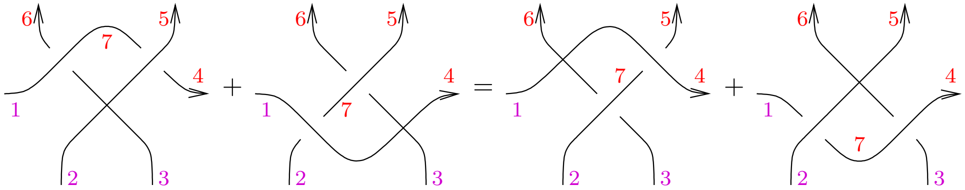
Conway's Third Identity (see [Co])



$$\mathcal{A}@\{X_{6,4,9,1}, \overline{X}_{4,5,7,8}, \overline{X}_{2,3,5,6}\} + \mathcal{A}@\{X_{2,4,5,1}, \overline{X}_{4,3,7,6}, X_{6,8,9,5}\} \equiv$$
$$\mathcal{A}@\{\overline{X}_{1,6,4,9}, X_{5,7,8,4}, X_{3,5,6,2}\} + \mathcal{A}@\{\overline{X}_{1,2,4,5}, X_{3,7,6,4}, \overline{X}_{5,6,8,9}\}$$

```
True
```

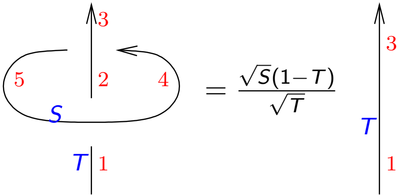
Virtual version (Archibald, [Ar])



$\mathcal{A}$@$\left\{X_{3,7,6,1}, \overline{X}_{7,2,4,5}\right\}$ + $\mathcal{A}$@$\{X_{2,4,7,1}, X_{3,5,6,7}\}$ ≡
$\mathcal{A}$@$\{X_{3,7,6,2}, X_{7,4,5,1}\}$ + $\mathcal{A}$@$\left\{\overline{X}_{1,2,7,5}, X_{3,4,6,7}\right\}$

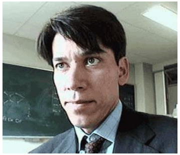
True

Jun Murakami's Fifth Axiom (see [Mu])

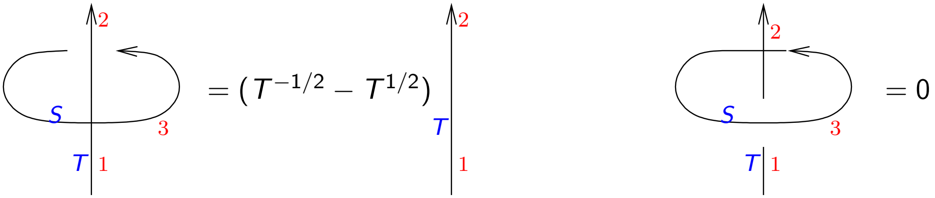


$$\mathcal{A}@\{X_{1,4,2,5}[T,\ S],\ X_{4,3,5,2}\} \equiv \frac{\sqrt{S}\ (1-T)}{\sqrt{T}}\ \mathcal{A}@\{P_{1,3}[T]\}$$

True



Virtual versions (Archibald, [Ar])



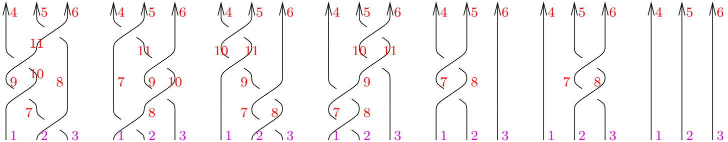

```
𝒜@{X3,2,3,1[S, T]} ≡ (T^-1/2 - T^1/2) 𝒜@{P1,2[T]}
```

True

```
𝒜@{X1,3,2,3}
```

𝒜[{1}, {2}, ⟨| $\xi_1 \to \tau_1$, $x_2 \to \tau_1$ |⟩, 0]

Jun Murakami's Third Axiom (see [Mu])



$\mathcal{A}_{2112} = \mathcal{A}@\{X_{3,8,7,2}, X_{7,10,9,1}, X_{10,11,4,9}, X_{8,6,5,11}\};$

$\mathcal{A}_{1221} = \mathcal{A}@\{X_{2,8,7,1}, X_{3,10,9,8}, X_{10,6,11,9}, X_{11,5,4,7}\};$

$\mathcal{A}_{2211} = \mathcal{A}@\{X_{3,8,7,2}, X_{8,6,9,7}, X_{9,11,10,1}, X_{11,5,4,10}\};$

$\mathcal{A}_{1122} = \mathcal{A}@\{X_{2,8,7,1}, X_{8,9,4,7}, X_{3,11,10,9}, X_{11,6,5,10}\};$

$\mathcal{A}_{11} = \mathcal{A}@\{X_{2,8,7,1}, X_{8,5,4,7}, P_{3,6}\};$ $\mathcal{A}_{22} = \mathcal{A}@\{X_{3,8,7,2}, X_{8,6,5,7}, P_{1,4}\};$

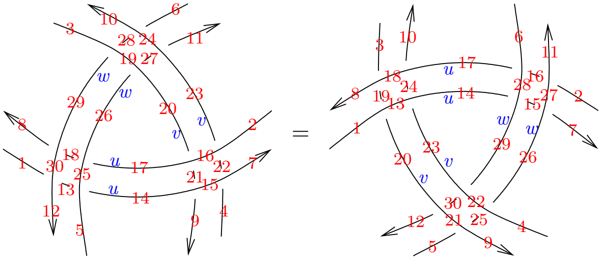
$\mathcal{A}_{\emptyset} = \mathcal{A}@\{P_{1,4}, P_{2,5}, P_{3,6}\};$

$g_+[z\_] := z^{1/2} + z^{-1/2};$ $g_-[z\_] := z^{1/2} - z^{-1/2};$

$g_+[\tau_1]\, g_-[\tau_2]\, \mathcal{A}_{2112} - g_-[\tau_2]\, g_+[\tau_3]\, \mathcal{A}_{1221} - g_-[\tau_3 / \tau_1]\, (\mathcal{A}_{2211} + \mathcal{A}_{1122}) +$
$\quad g_-[\tau_2\, \tau_3 / \tau_1]\, g_+[\tau_3]\, \mathcal{A}_{11} - g_+[\tau_1]\, g_-[\tau_1\, \tau_2 / \tau_3]\, \mathcal{A}_{22} \equiv g_-\left[\tau_3^2 / \tau_1^2\right] \mathcal{A}_{\emptyset}$

True

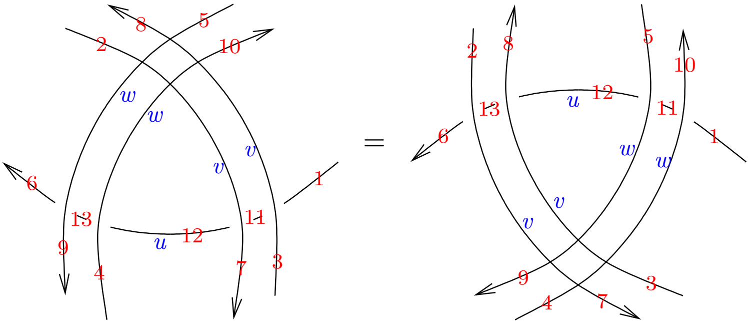
The Naik-Stanford Double Delta Move (see [NS])



$$\texttt{Timing}\Big[\mathcal{A}@\Big\{X_{6,10,28,24}[w, v],\ \bar{X}_{28,3,29,19}[w, v],\ X_{26,20,27,19}[w, v],\ \bar{X}_{27,23,11,24}[w, v],$$
$$X_{1,12,13,30}[u, w],\ \bar{X}_{13,5,14,25}[u, w],\ X_{17,26,18,25}[u, w],\ \bar{X}_{18,29,8,30}[u, w],$$
$$X_{4,7,22,15}[v, u],\ \bar{X}_{22,2,23,16}[v, u],\ X_{20,17,21,16}[v, u],\ \bar{X}_{21,14,9,15}[v, u]\Big\} \equiv$$
$$\mathcal{A}@\Big\{X_{5,9,25,21}[w, v],\ \bar{X}_{25,4,26,22}[w, v],\ X_{29,23,30,22}[w, v],\ \bar{X}_{30,20,12,21}[w, v],$$
$$X_{2,11,16,27}[u, w],\ \bar{X}_{16,6,17,28}[u, w],\ X_{14,29,15,28}[u, w],\ \bar{X}_{15,26,7,27}[u, w],$$
$$X_{3,8,19,18}[v, u],\ \bar{X}_{19,1,20,13}[v, u],\ X_{23,14,24,13}[v, u],\ \bar{X}_{24,17,10,18}[v, u]\Big\}\Big]$$

{251.156, True}

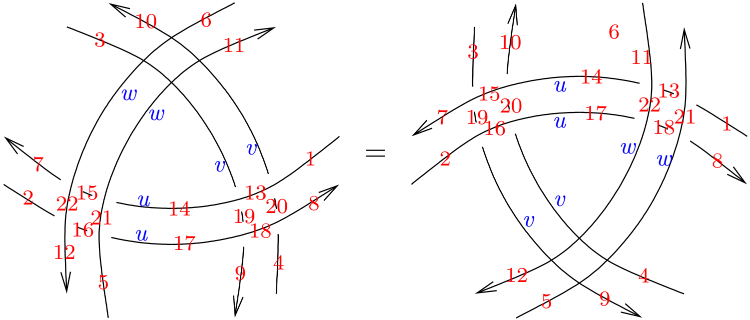
Virtual Version 1 (Archibald, [Ar])



$\mathcal{A}$@{$X_{1,8,11,3}$[u, v], $\overline{X}_{11,2,12,7}$[u, v], $X_{12,10,13,4}$[u, w], $\overline{X}_{13,5,6,9}$[u, w]} ≡
$\mathcal{A}$@{$X_{1,10,11,4}$[u, w], $\overline{X}_{11,5,12,9}$[u, w], $X_{12,8,13,3}$[u, v], $\overline{X}_{13,2,6,7}$[u, v]}

True

Virtual Version 2 (Archibald, [Ar])



$$\mathcal{A}@\{\bar{X}_{20,1,10,13}[v, u], X_{3,14,19,13}[v, u], X_{14,11,15,21}[u, w], \bar{X}_{15,6,7,22}[u, w], X_{2,12,16,22}[u, w], \bar{X}_{16,5,17,21}[u, w], \bar{X}_{19,17,9,18}[v, u], X_{4,8,20,18}[v, u]\} \equiv \mathcal{A}@\{X_{1,11,13,21}[u, w], \bar{X}_{13,6,14,22}[u, w], \bar{X}_{20,14,10,15}[v, u], X_{3,7,19,15}[v, u], \bar{X}_{19,2,9,16}[v, u], X_{4,17,20,16}[v, u], X_{17,12,18,22}[u, w], \bar{X}_{18,5,8,21}[u, w]\}$$

True

5. Some Problems in Heaven.

Unfortunately, $\dim \mathcal{A}(\mathcal{X}, X) = \dim \Lambda(\mathcal{X}, X) = 4^{|X|}$ is big. Fortunately, we have the following theorem, a version of one of the main results in Halacheva's thesis, [Ha1, Ha2]:

Theorem. Working in $\Lambda(\mathcal{X} \cup X)$, if $w = \omega \mathbb{e}^{\lambda}$ is a balanced Gaussian (namely, a scalar ω times the exponential of a quadratic $\lambda = \sum_{\zeta \in \mathcal{X}, z \in X} \alpha_{\zeta,z} \zeta z$), then generically so is $c_{x,\xi} \mathbb{e}^{\lambda}$.

(This is great news! The space of balanced quadratics is only $|\mathcal{X}||X|$-dimensional!)

Proof. Recall that $c_{x,\xi} \colon (1, \xi, x, x\xi)w' \mapsto (1, 0, 0, 1)w'$, write $\lambda = \mu + \eta x + \xi y + \alpha\xi x$, and ponder $\mathbb{e}^{\lambda} =$

$$\dots + \frac{1}{k!}\underbrace{(\mu + \eta x + \xi y + \alpha\xi x)(\mu + \eta x + \xi y + \alpha\xi x)\cdots(\mu + \eta x + \xi y + \alpha\xi x)}_{k \text{ factors}} + \dots .$$

Then $c_{x,\xi}\mathbb{e}^{\lambda}$ has three contributions:

- $\mathbb{e}^{\mu}$, from the term proportional to 1 (namely, independent of ξ and x) in $\mathbb{e}^{\lambda}$
- $-\alpha\mathbb{e}^{\mu}$, from the term proportional to $x\xi$, where the x and the ξ come from the same factor above.
- $\eta y\mathbb{e}^{\mu}$, from the term proportional to $x\xi$, where the x and the ξ come from different factors above.

So $c_{x,\xi}\mathbb{e}^{\lambda} = \mathbb{e}^{\mu}(1 - \alpha + \eta y) = (1 - \alpha)\mathbb{e}^{\mu}(1 + \eta y/(1 - \alpha)) = (1 - \alpha)\mathbb{e}^{\mu}\mathbb{e}^{\eta y/(1-\alpha)} = (1 - \alpha)\mathbb{e}^{\mu + \eta y/(1-\alpha)}$.

$1 + q = e^{q}$

□

Γ-calculus.

Thus we have an almost-always-defined "Γ-calculus": a contraction algebra morphism $\mathcal{T}(\mathcal{X}, X) \to R \times \mathcal{X} \otimes_{R/R} X$ whose behaviour under contractions is given by

$$c_{x,\xi}(\omega, \lambda = \mu + \eta x + \xi y + \alpha \xi x) = ((1-\alpha)\omega, \mu + \eta y/(1-\alpha)).$$

(Γ is fully defined on pure tangles – tangles without closed components – and hence on long knots).

6. An Implementation of Γ.

If I didn't implement I wouldn't believe myself.

Written in Mathematica [Wo], available as the notebook `Gamma.nb` at http://drorbn.net/mo21/ap. Code lines are highlighted in grey, demo lines are plain. We start with canonical forms for quadratics with rational function coefficients:

```
CCF[ℰ_] := Factor[ℰ];
CF[ℰ_] := Module[{vs = Union@Cases[ℰ, (ξ | x)_, ∞]},
   Total[(CCF[#[[2]]] (Times @@ vs^#[[1]])) & /@ CoefficientRules[ℰ, vs]]];
```

Multiplying and comparing Γ objects:

```
Γ /: Γ[is1_, os1_, cs1_, ω1_, λ1_] × Γ[is2_, os2_, cs2_, ω2_, λ2_] :=
 Γ[is1 ⋃ is2, os1 ⋃ os2, Join[cs1, cs2], ω1 ω2, λ1 + λ2]
Γ /: Γ[is1_, os1_, _, ω1_, λ1_] ≡ Γ[is2_, os2_, _, ω2_, λ2_] :=
 TrueQ[(Sort@is1 === Sort@is2) ∧ (Sort@os1 === Sort@os2) ∧
   Simplify[ω1 == ω2] ∧ CF@λ1 == CF@λ2]
```

No rules for linear operations!

Contractions:

```
c_{h_,t_}@Γ[is_, os_, cs_, ω_, λ_] := Module[{α, η, y, μ},
  α = ∂_{ξ_t,x_h} λ; μ = λ /. ξ_t | x_h → 0;
  η = ∂_{x_h} λ /. ξ_t → 0; y = ∂_{ξ_t} λ /. x_h → 0;
    Γ[
      DeleteCases[is, t], DeleteCases[os, h], KeyDrop[cs, {x_h, ξ_t}],
      CCF[(1 - α) ω], CF[μ + η y / (1 - α)]
    ] /. If[MatchQ[cs[ξ_t], τ_], cs[ξ_t] → cs[x_h], cs[x_h] → cs[ξ_t]]];
c@Γ[is_, os_, cs_, ω_, λ_] := Fold[c_{#2,#2}[#1] &, Γ[is, os, cs, ω, λ], is ⋂ os]
```

The crossings and the point:

```
Γ[X_{i_,j_,k_,l_}[S_, T_]] := Γ[{l, i}, {j, k}, <|ξ_i → S, x_j → T, x_k → S, ξ_l → T|>,
    T^{-1/2}, CF[{ξ_l, ξ_i}.(1  1-T; 0  T).{x_j, x_k}]];
Γ[X̄_{i_,j_,k_,l_}[S_, T_]] := Γ[{i, j}, {k, l}, <|ξ_i → S, ξ_j → T, x_k → S, x_l → T|>,
    T^{1/2}, CF[{ξ_i, ξ_j}.(T^{-1}  0; 1-T^{-1}  1).{x_k, x_l}]];
Γ[X_{i_,j_,k_,l_}] := Γ[X_{i,j,k,l}[τ_i, τ_l]];
Γ[X̄_{i_,j_,k_,l_}] := Γ[X̄_{i,j,k,l}[τ_i, τ_j]];
```

```
Γ[P_{i_,j_}[T_]] := Γ[{i}, {j}, <|ξ_i → T, x_j → T|>, 1, ξ_i x_j];
Γ[P_{i_,j_}] := Γ[P_{i,j}[τ_i]];
```

Automatic intelligent contractions:

```
Γ[{γ_Γ}] := c[γ];
Γ[{γ1_Γ, γs__Γ}] := Module[{γ2},
  γ2 = First@MaximalBy[{γs}, Length[γ1〚1〛 ⋂ #〚2〛] + Length[γ1〚2〛 ⋂ #〚1〛] &];
  Γ[Join[{c[γ1 γ2]}, DeleteCases[{γs}, γ2]]] ]
Γ[𝒪s_List] := Γ[Γ /@ 𝒪s]
```

Conversions $\mathcal{A} \leftrightarrow \Gamma$:

```
Γ@𝒜[is_, os_, cs_, w_] := Module[{i, j, ω = Coefficient[w, Wedge[]]},
   Γ[is, os, cs, ω, Sum[Cancel[-Coefficient[w, xj ∧ ξi] ξi xj / ω],
     {i, is}, {j, os}]]
  ];
𝒜@Γ[is_, os_, cs_, ω_, λ_] :=
  𝒜[is, os, cs, Expand[ω WExp[Expand[λ] /. ξa_ xb_ :> ξa ∧ xb]]];
```

The conversions are inverses of each other:

```
Γ[{1, 2, 3}, {1, 2, 3}, {x1 → τ1, x2 → τ2, x3 → τ3, ξ1 → τ1, ξ2 → τ2, ξ3 → τ3},
  ω, a11 x1 ξ1 + a12 x2 ξ1 + a13 x3 ξ1 + a21 x1 ξ2 + a22 x2 ξ2 + a23 x3 ξ2 + a31 x1 ξ3 +
   a32 x2 ξ3 + a33 x3 ξ3];
Γ@𝒜@γ == γ
```

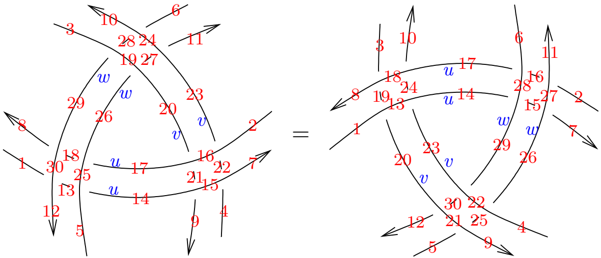
True

The conversions commute with contractions:

```
Γ@c3,3@𝒜@γ ≡ c3,3@γ
```

True

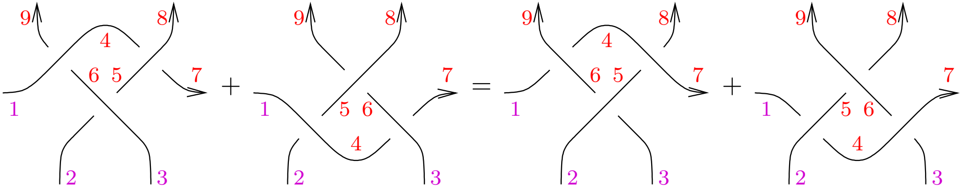
The Naik-Stanford Double Delta Move (again)



$$\text{Timing}[\Gamma@\{X_{6,10,28,24}[w,v],\ \bar{X}_{28,3,29,19}[w,v],\ X_{26,20,27,19}[w,v],\ \bar{X}_{27,23,11,24}[w,v],$$
$$X_{1,12,13,30}[u,w],\ \bar{X}_{13,5,14,25}[u,w],\ X_{17,26,18,25}[u,w],\ \bar{X}_{18,29,8,30}[u,w],$$
$$X_{4,7,22,15}[v,u],\ \bar{X}_{22,2,23,16}[v,u],\ X_{20,17,21,16}[v,u],\ \bar{X}_{21,14,9,15}[v,u]\} \equiv$$
$$\Gamma@\{X_{5,9,25,21}[w,v],\ \bar{X}_{25,4,26,22}[w,v],\ X_{29,23,30,22}[w,v],\ \bar{X}_{30,20,12,21}[w,v],$$
$$X_{2,11,16,27}[u,w],\ \bar{X}_{16,6,17,28}[u,w],\ X_{14,29,15,28}[u,w],\ \bar{X}_{15,26,7,27}[u,w],$$
$$X_{3,8,19,18}[v,u],\ \bar{X}_{19,1,20,13}[v,u],\ X_{23,14,24,13}[v,u],\ \bar{X}_{24,17,10,18}[v,u]\}]$$

{1.28125, True}

Conway's Third Identity

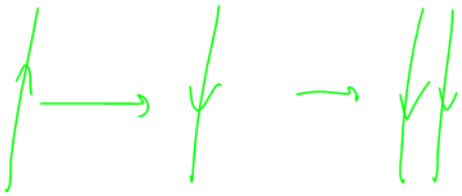


Sorry, Γ has nothing to say about that...'

What I still don't understand.

makes sense even if $\alpha = 0$!

- What becomes of $c_{x,\xi}e^{\lambda}$ if we have to divide by 0 in order to write it again as an exponentiated quadratic? Does it still live within a very small subset of $\Lambda(\mathcal{X} \sqcup X)$?
- How do cablings and strand reversals fit within $\mathcal{A}$?
- Are there "classicality conditions" satisfied by the invariants of classical tangles (as opposed to virtual ones)?



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Wolfram Language & System Documentation Center, https://reference.wolfram.com/language/.

Thank You!