

Knots in Three and Four Dimensions

Dror Bar-Natan, $\omega :=$ <http://drorbn.net/mc21>

MathCamp by Web, July 2021

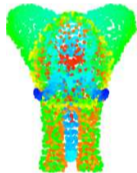
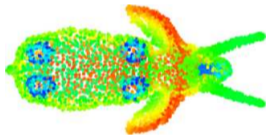
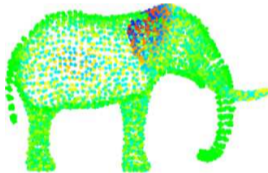
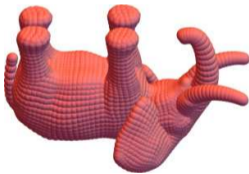
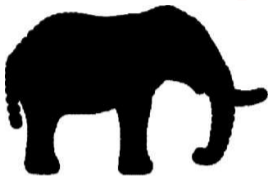
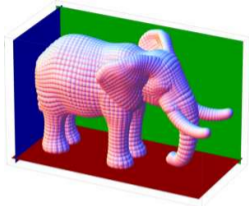
Abstract. Much as we can understand 3-dimensional objects by staring at their pictures and x-ray images and slices in 2-dimensions, so can we understand 4-dimensional objects by staring at their pictures and x-ray images and slices in 3-dimensions, capitalizing on the fact that we understand 3-dimensions pretty well. So we will spend some time staring at and understanding various 2-dimensional views of a 3-dimensional elephant, and then even more simply, various 2-dimensional views of some 3-dimensional knots. This achieved, we'll take the leap and visualize some 4-dimensional knots by their various traces in 3-dimensional space, and if we'll still have time, we'll prove that these knots are really knotted.

Thanks for inviting me to MathCamp! As most of you have never seen it, here's a picture of the lecture room:



If you can, please turn your video on! (And mic, whenever needed).

Warmup: Flatlanders View an Elephant.



ω/g



ω/r

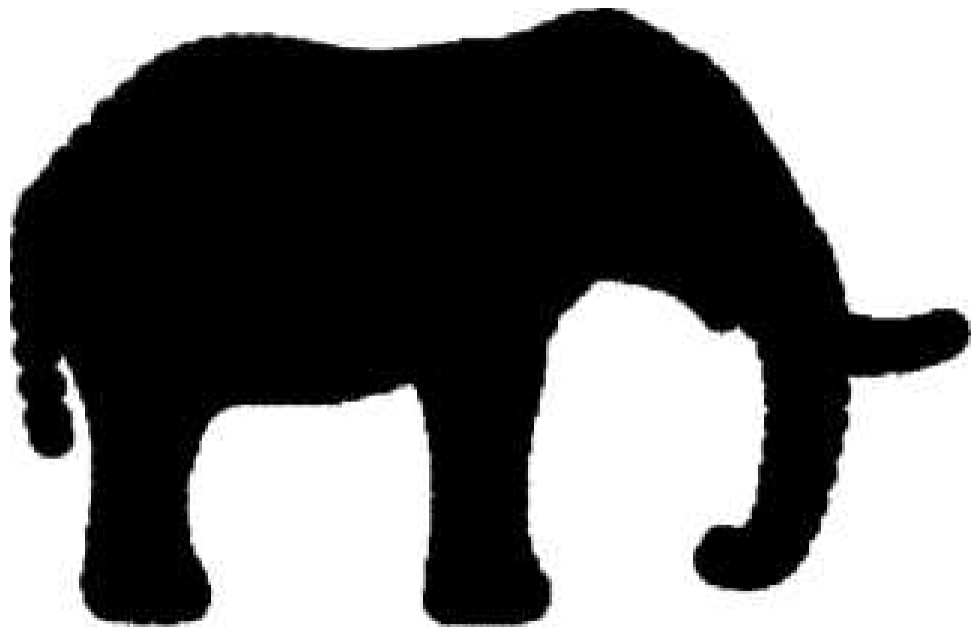


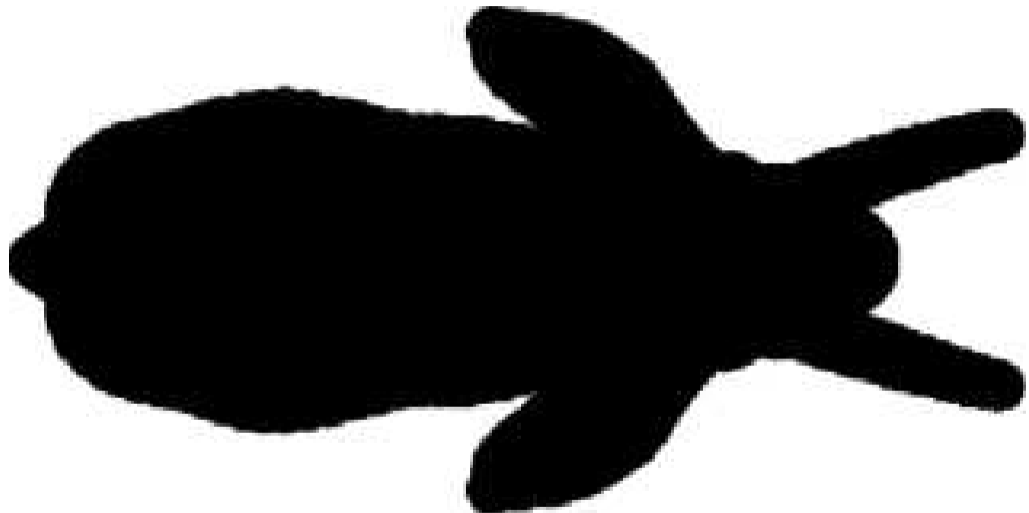
ω/b



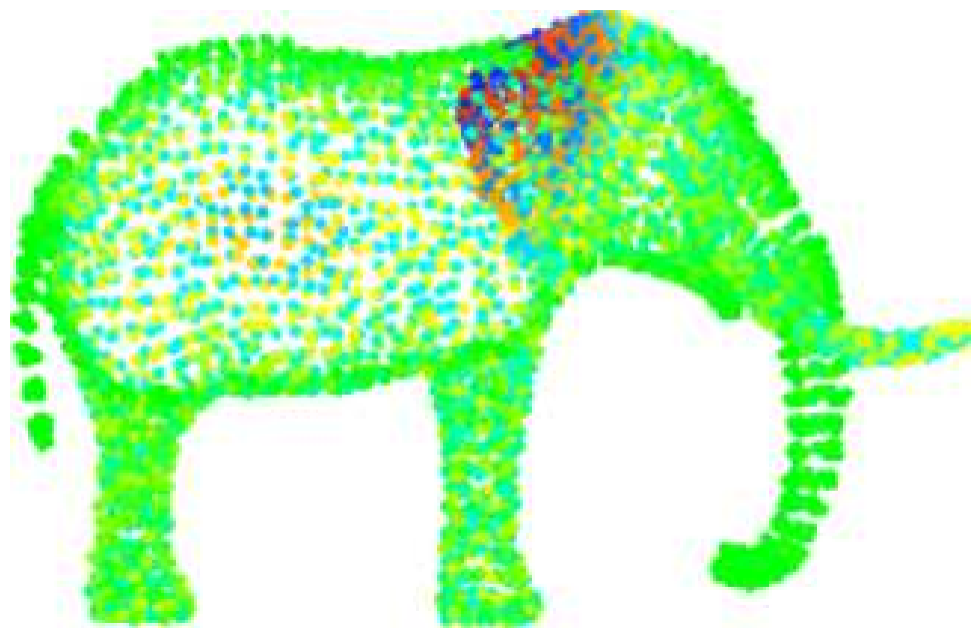
coords from $\omega/\text{Jeff2207}$

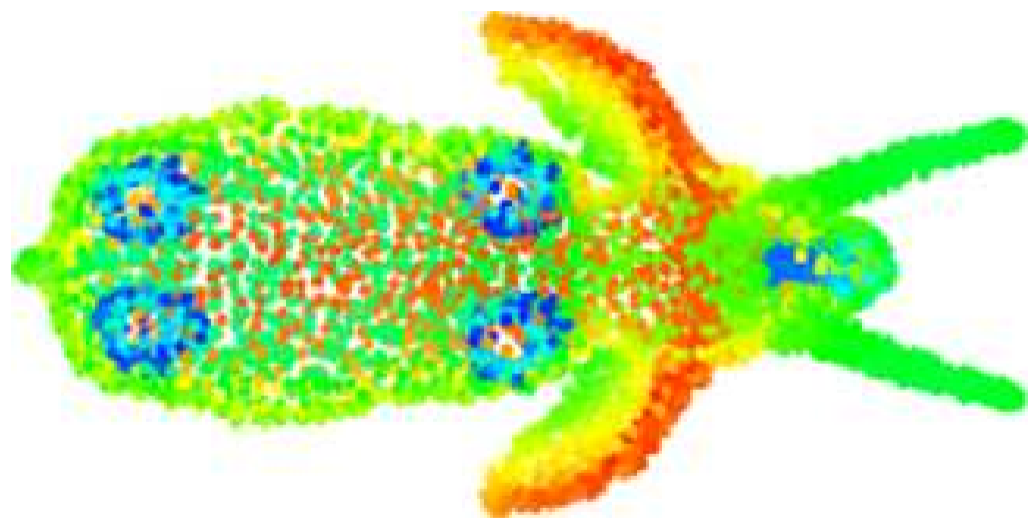


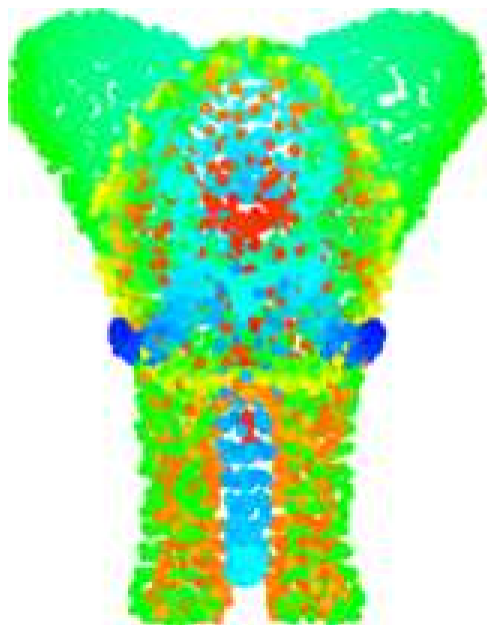














coords from [ω/Jeff2207](#)



ω/g



ω/r



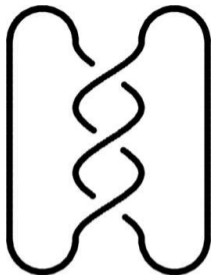
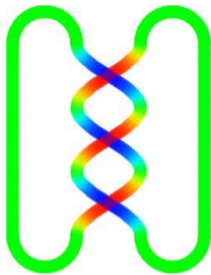
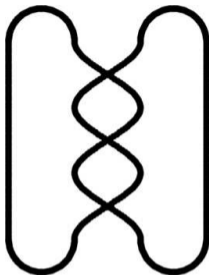
ω/b



coords from $\omega/\text{Jeff2207}$



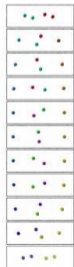
Knots.



“broken curve diagram”

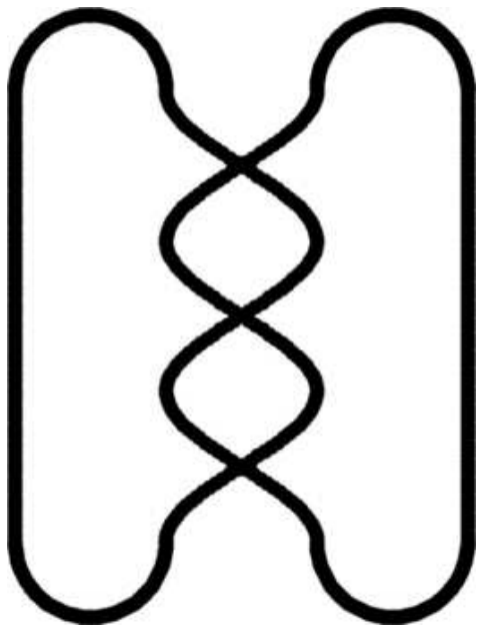


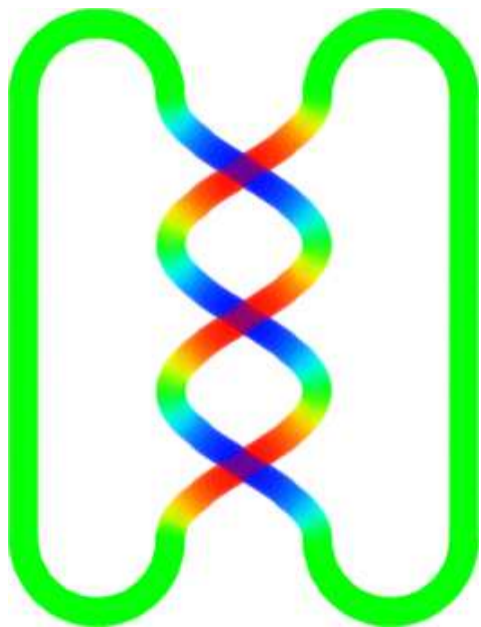
with Ester Dalvit ω /Dal

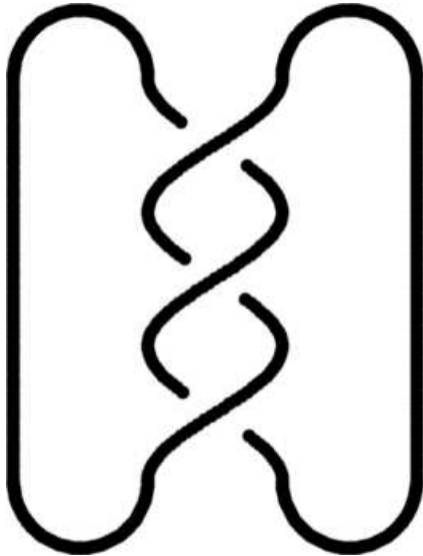


ω /M2

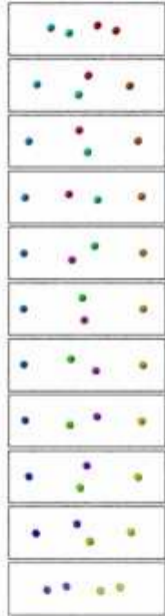
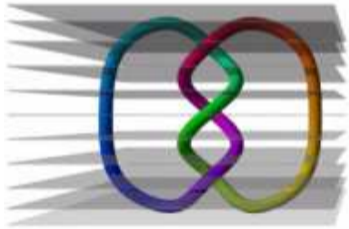
Formally, “a differentiable embedding of S^1 in \mathbb{R}^3 modulo differentiable deformations of such”.







“broken curve diagram”



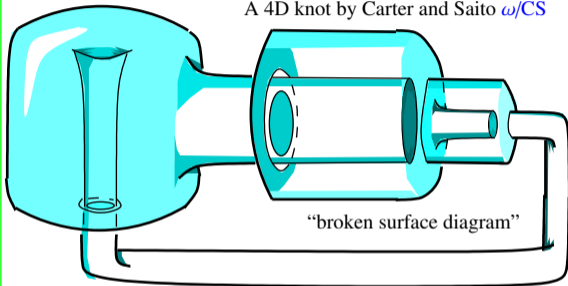
$\omega/M2$

Formally, “a differentiable embedding of S^1 in \mathbb{R}^3 modulo differentiable deformations of such”.

2-Knots / 4D Knots.

Formally, “a differentiable embedding of S^2 in \mathbb{R}^4 modulo differentiable deformations of such”.

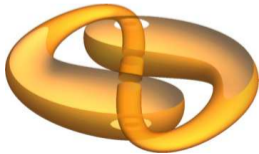
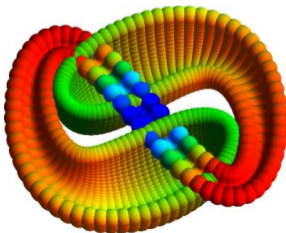
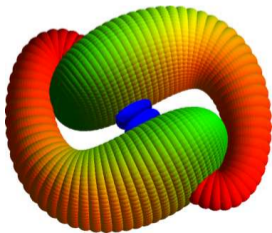
A 4D knot by Carter and Saito ω/CS



“broken surface diagram”

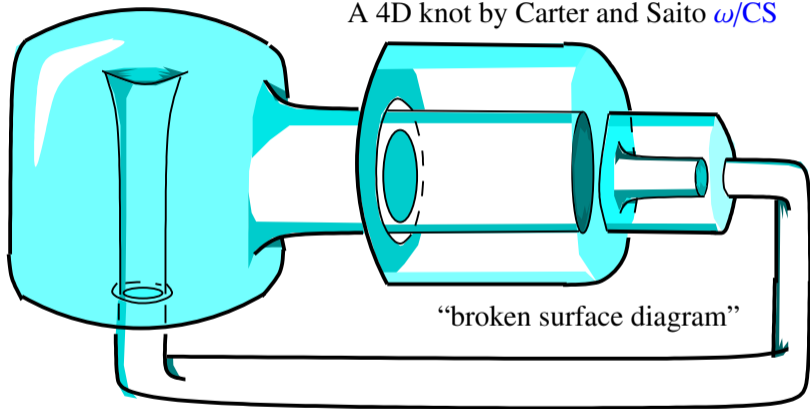


Carter, Banach, Saito



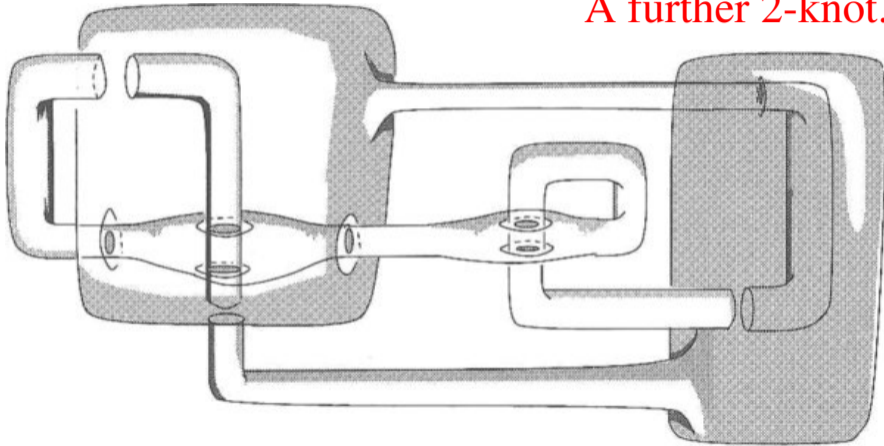
Formally, “a differentiable embedding of S^2 in \mathbb{R}^4 modulo differentiable deformations of such”.

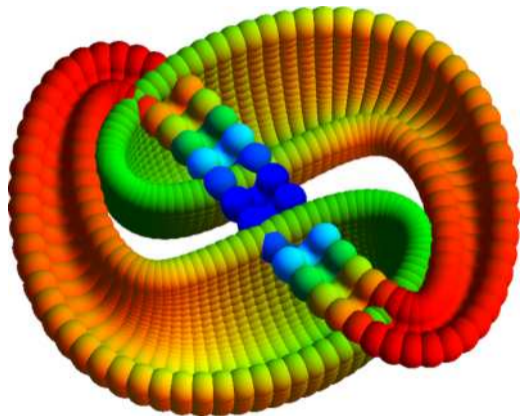
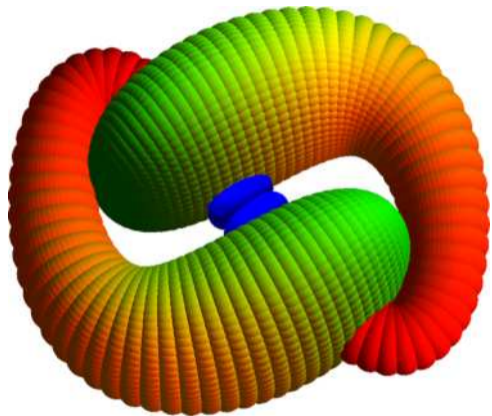
A 4D knot by Carter and Saito ω/CS

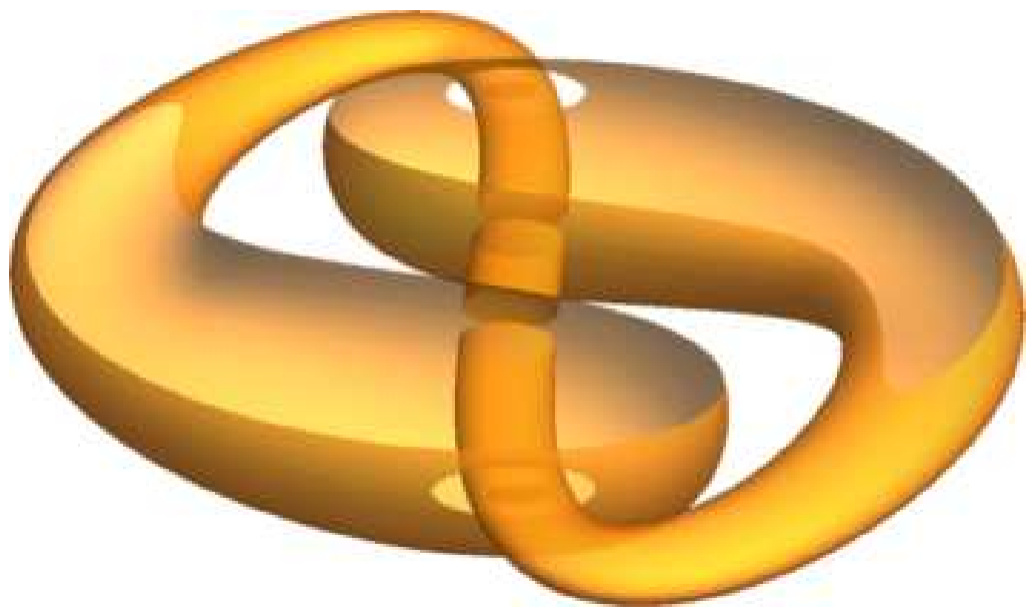


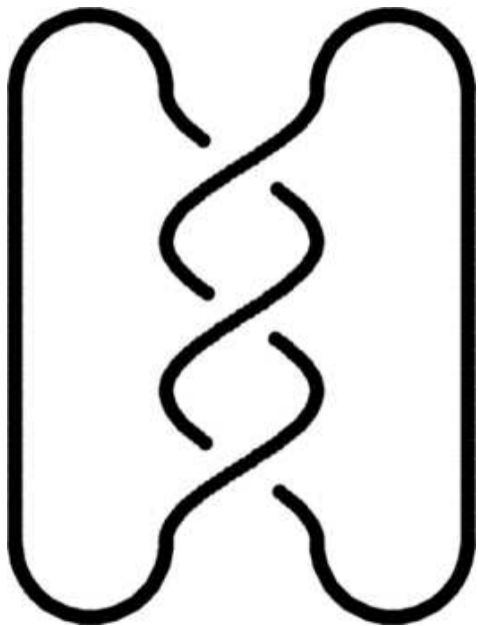
"broken surface diagram"

A further 2-knot.

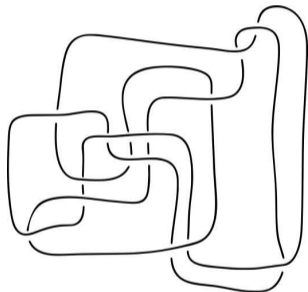








Some Unknots



Thistlethwaite's unknot



ω/U

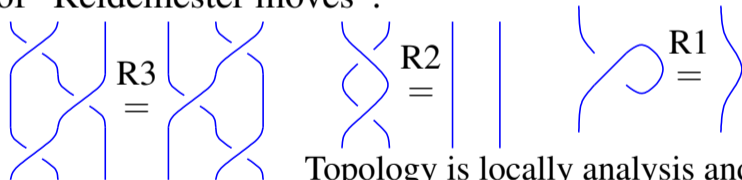


Scharein's relaxation



Haken's unknot

Reidemeister's Theorem. (a) Every knot has a “broken curve diagram”, made only of curves and “crossings” like \times . (b) Two knot diagrams represent the same 3D knot iff they differ by a sequence of “Reidemeister moves”:



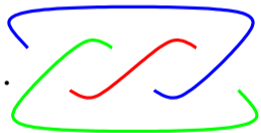
Kurt Reidemeister

Topology is locally analysis and globally algebra

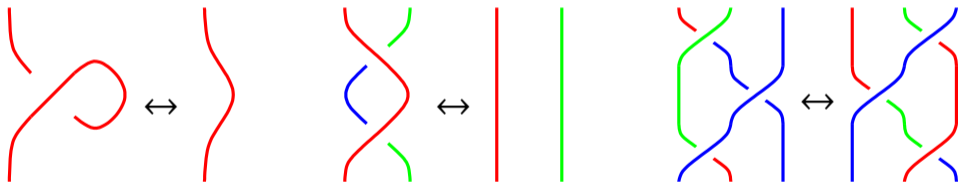
3-Colourings. Colour the arcs of a broken arc diagram in **RGB** so that every crossing is either mono-chromatic or tri-chromatic. Let $\lambda(K)$ be the number of such 3-colourings that K has.

Example. $\lambda(\bigcirc) = 3$ while $\lambda(\text{Ⓞ}) = 9$; so $\bigcirc \neq \text{Ⓞ}$.

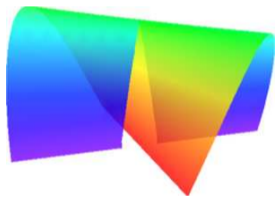
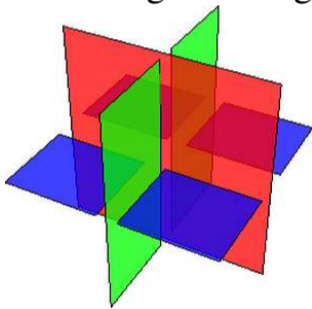
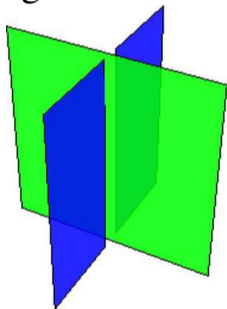
Riddle. Is $\lambda(K)$ always a power of 3?



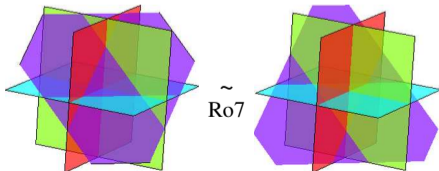
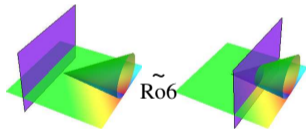
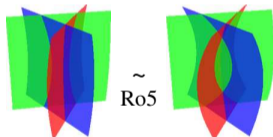
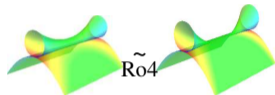
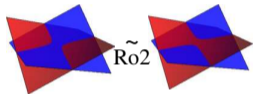
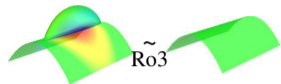
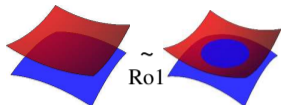
Proof sketch. It is enough to show that for each Reidemeister move, there is an end-colours-preserving bijection between the colourings of the two sides. E.g.:



Theorem. Every 2-knot can be represented by a “broken surface diagram” made of the following basic ingredients,

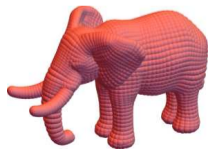
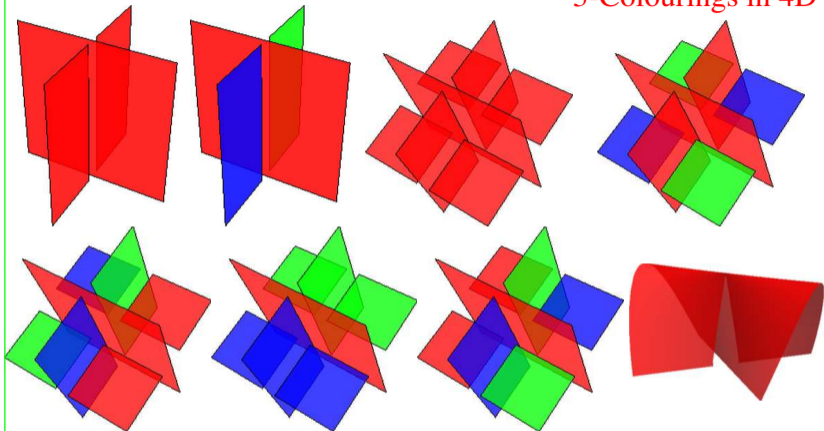


... and any two representations of the same knot differ by a sequence of the following “Roseman moves”:

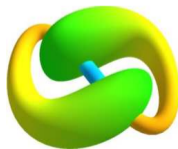


D. Roseman

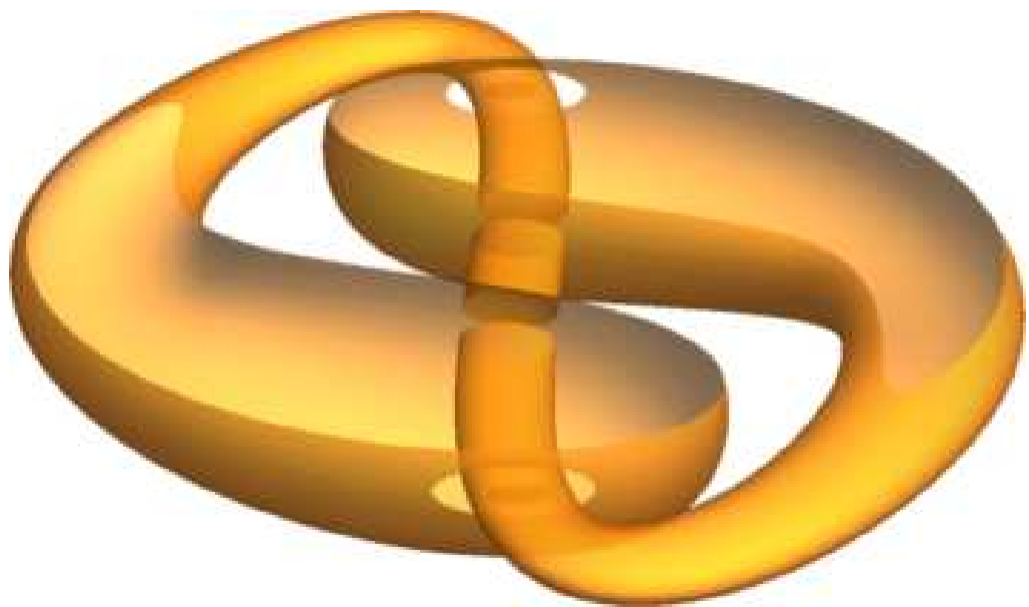
3-Colourings in 4D



→ 3

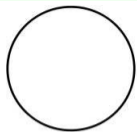


→ 9



A Knot Table

There are many
more!



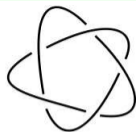
Unknot



3_1



4_1



5_1



5_2



6_1



6_2



6_3



7_1



7_2



7_3



7_4



7_5



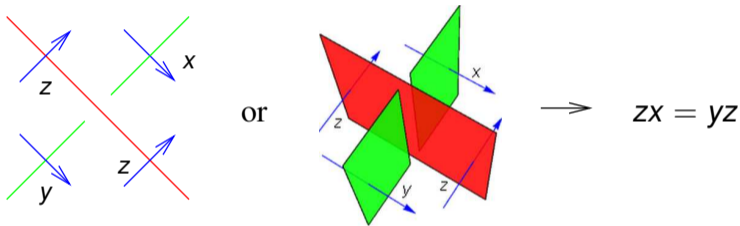
7_6



7_7

ω/KT

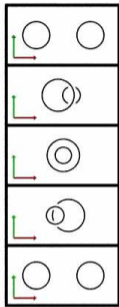
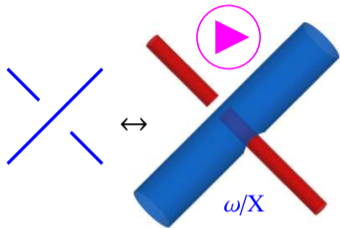
A Stronger Invariant. There is an assignment of groups to knots / 2-knots as follows. Put an arrow “under” every un-broken curve / surface in a broken curve / surface diagram and label it with the name of a group generator. Then mod out by relations as below.



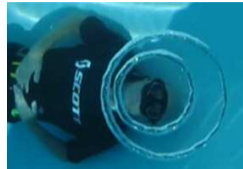
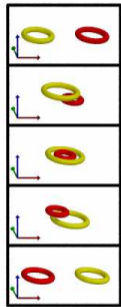
Facts. The resulting “Fundamental group” $\pi_1(K)$ of a knot / 2-knot K is a very strong but not very computable invariant of K . Though it has computable projections; e.g., for any finite G , count the homomorphisms from $\pi_1(K)$ to G .

Exercise. Show that $|\text{Hom}(\pi_1(K) \rightarrow S_3)| = \lambda(K) + 3$.

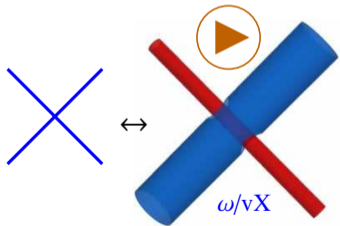
Some Movies



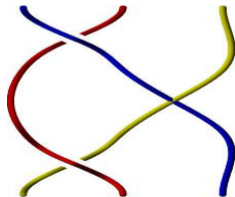
$\omega/X1$

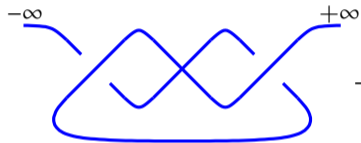


ω/Bub

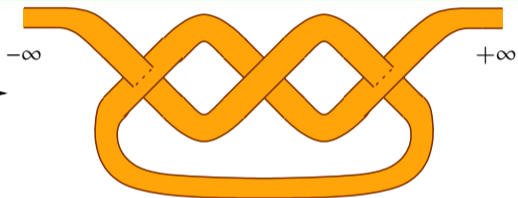


ω/F





“long w-knot diagram”

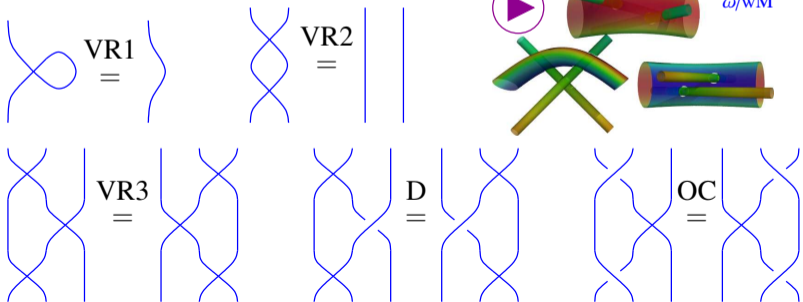


→ “simple long knotted 2D tube in 4D”

Satoh's Conjecture. (Satoh, *Virtual Knot Presentations of Ribbon Torus-Knots*, *J. Knot Theory and its Ramifications* **9** (2000) 531–542). Two long w-knot diagrams represent via the map δ the same simple long 2D knotted tube in 4D iff they differ by a sequence of R-moves as above and the “w-moves” VR1–VR3, D and OC listed below:



Shin Satoh



Some knot theory books.

- Colin C. Adams, *The Knot Book, an Elementary Introduction to the Mathematical Theory of Knots*, American Mathematical Society, 2004.
- Meike Akveld and Andrew Jobbings, *Knots Unravelled, from Strings to Mathematics*, Arbelos 2011.
- J. Scott Carter and Masahico Saito, *Knotted Surfaces and Their Diagrams*, American Mathematical Society, 1997.
- Peter Cromwell, *Knots and Links*, Cambridge University Press, 2004.
- W.B. Raymond Lickorish, *An Introduction to Knot Theory*, Springer 1997.

