

Doodle on
this page!

Better vaxxed!

The 17 Worlds of Planar Ants

Dror Bar-Natan, <http://drorbn.net/mc21>

MathCamp by Web, July 2021

Abstract. My goal is to get you hooked, captured and unreleased until you find all 17 in real life, around you.

We all know that the plane can be filled in different periodic manners: floor tiles are often square but sometimes hexagonal, bricks are often laid in an interlaced pattern, fabrics often carry interesting patterns. A little less known is that there are precisely 17 symmetry patterns for tiling the plane; not one more, not one less. It is even less known how easy these 17 are to identify in the patterns around you, how fun it is, how common some are, and how rare some others seem to be.

Gotta Catch 'Em All!

Thanks for inviting me to MathCamp! Just to feel a little closer, here's a picture of the lecture room:

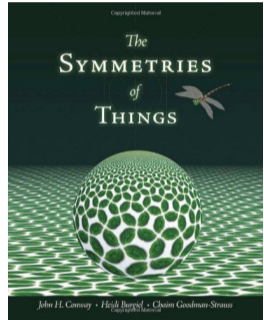


If you can, please turn your video on! (And mic, whenever needed).

Reading. An excellent book on the subject is *The Symmetries of Things* by J. H. Conway, H. Burgiel, and C. Goodman-Strauss, CRC Press, 2008.

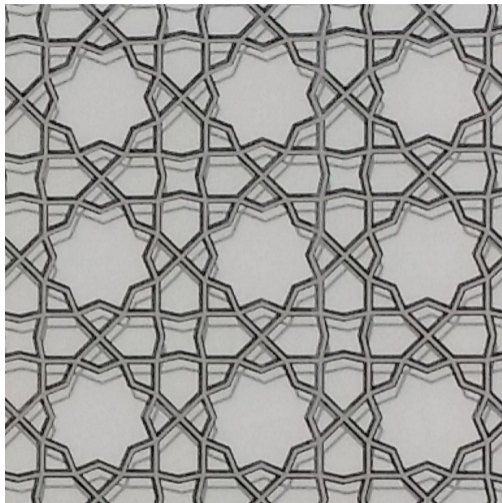
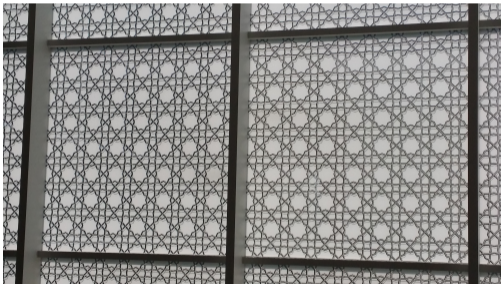
Another nice text is *Classical Tessellations and Three-Manifolds* by J. M. Montesinos, Springer-Verlag, 1987.

And another is *Tilings & Patterns* by B. Grünbaum and G. C. Shephard, Dover, 2016.



(Easy) Question 1. In what ways can you make \$2 change, using coins denominated $\$ \frac{1}{2}$, $\$ \frac{2}{3}$, $\$ \frac{3}{4}$, $\$ \frac{4}{5}$, $\$ \frac{5}{6}$, etc.?

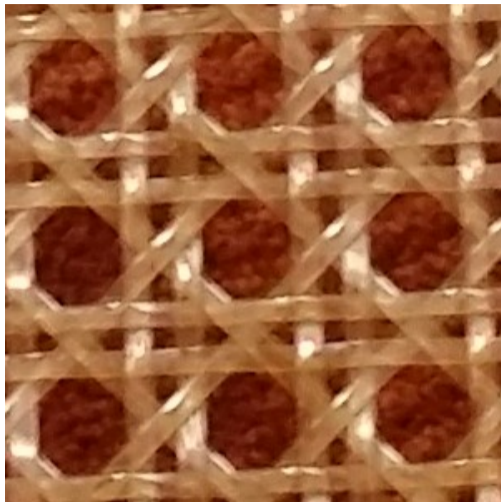
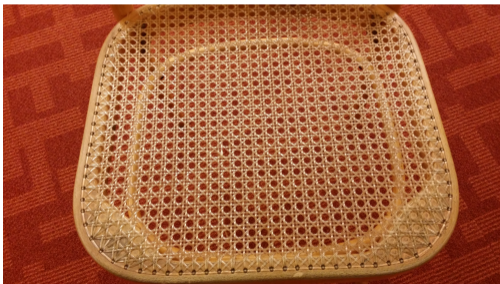
(Harder) Question 2. Why am I asking?



A decorated window at the Aga Khan Gallery, Toronto 2014



Floor tiles at Fran's Restaurant & Bar, Toronto 2014



A chair at the Toronto Public Library, 2014



A box of tissues



A living room sofa at the Karshon's, 2012

Theorem. There are precisely 17 patterns with which to tile the plane, no more, no less. They are all made of combinations of the 10 basic features, **2**, **3**, **4**, **6**, **‡**, **§**, **‡**, **§**, **‡**, **§**, **M**, and **G**, as follows:

| ✓ | Dror's | Conway's | crystallo-graphic | ✓ | Dror's | Conway's | crystallo-graphic |
|---|-------------|----------|-------------------|---|------------|----------|-------------------|
| | 2222 | 2222 | p2 | | 3‡ | 3*3 | p31m |
| | 333 | 333 | p3 | | 2‡‡ | 2*22 | cmm |
| | 442 | 442 | p4 | | 22M | 22* | pmg |
| | 632 | 632 | p6 | | MM | ** | pm |
| | ‡‡‡‡ | *2222 | pmm | | MG | *o | cm |
| | ‡‡‡ | *333 | p3m1 | | GG | oo | pg |
| | ‡‡‡ | *442 | p4m | | 22G | 22o | pgg |
| | ‡‡‡ | *632 | p6m | | ∅ | 0 | p1 |
| | ‡‡ | 4*2 | p4g | | | | |

Video, handout, links at drorbn.net/mc21

The Basic Features.

3



rotation only

\$



rotation-reflection

Gotta catch 'em all!

M



free mirror-reflection

G



free glide-reflection



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3



rotation only



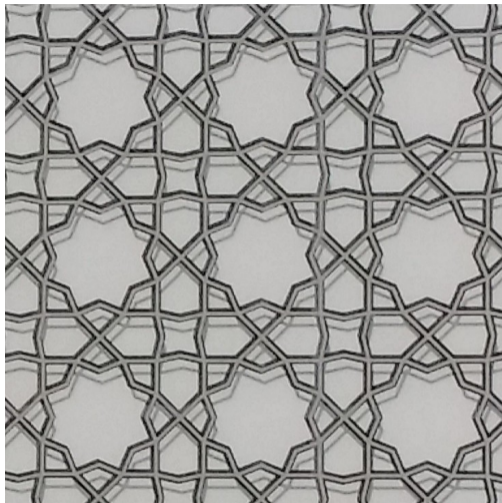
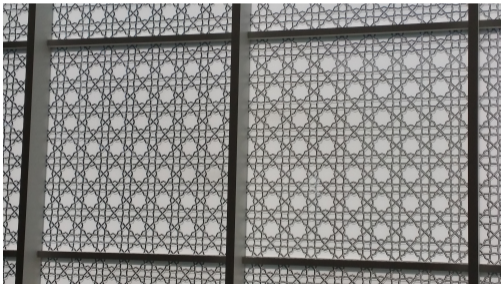
Floor tiles in a restaurant in Toronto's Baldwin Street, 2018



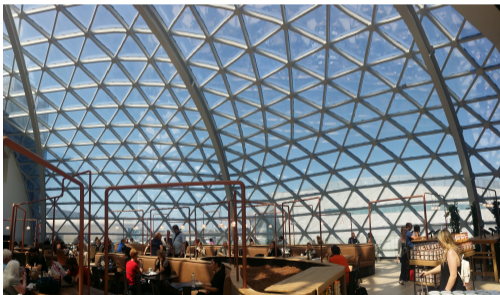
A box of tissues



rotation-reflection



A decorated window at the Aga Khan Gallery, Toronto 2014



A food court at the Roma Fiumicino airport, 2017



free mirror-reflection



A packet of tissues

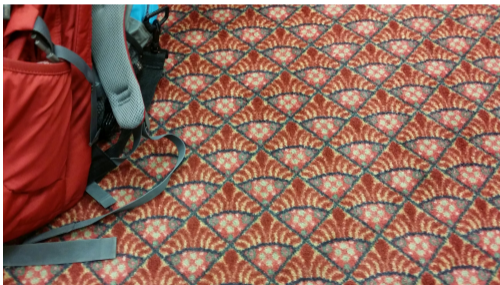


At Rick's Cafe in Toronto's Kensington Market, 2014





A living room sofa at the Karshon's, 2012



A carpet seen at Indiana University, 2016



A bike parking in Groningen, 2020

Homework.

Go out and find them all! At home, around the corner, a mile away. Take pictures and upload them to <http://drorbn.net/mc21/upload> using the file format **name-type-description.jpg**, where

- ▶ **name** is your name or alias.
- ▶ **type** is the type of the tiling pattern, using the Conway conventions but with the “*” replaced with the English letter “s”. In other words, **type** is one of 2222, 333, 442, 632, s2222, s333, s442, s632, 4s2, 3s3, 2s22, 22s, ss, so, oo, 22o, or 0.
- ▶ **description** is a short description.

As an example, see the file `Dror-4S2-StClairWSubway.jpg` there.

We will start our class tomorrow with a quick discussion of the patterns you will find — but I can only promise to look at whatever will be uploaded at least two hours ahead of class.

Privacy note. Whatever you upload I may post on my web site. So make sure the pictures you upload don't include anything personal.

Prize.

US\$50 if you find a 333 in time for our second meeting!

- ▶ You must find it “natural” — it can't be your own drawing, or within a book on symmetries, or in a museum that has an exhibit on symmetries (I think [MoMath](#) has one).
- ▶ In the unlikely event that more than one person will find a 333, I'll split the prize between all winners.
- ▶ I've paid the prize twice before, but it's really tough. I've only seen a 333 “in nature” three times in more than 20 years of looking for it!



_____ of the TV series _____
visiting _____

See you tomorrow!

Video, handout, links at drorbn.net/mc21

The Basic Features.

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rotation only

\$



rotation-reflection

Gotta catch 'em all!

M



free mirror-reflection

G



free glide-reflection



MATHCAMP



Theorem. There are precisely 17 patterns with which to tile the plane, no more, no less. They are all made of combinations of the 10 basic features, **2**, **3**, **4**, **6**, **‡**, **§**, **¶**, **♠**, **M**, and **G**, as follows:

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| | ‡3‡ | *632 | p6m | | ∅ | 0 | p1 |
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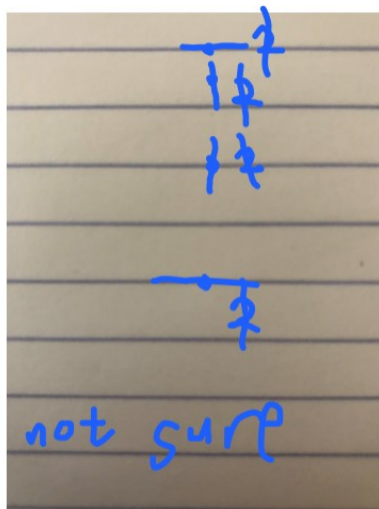
_____ of the TV series _____
visiting _____

Let's look at what you found...

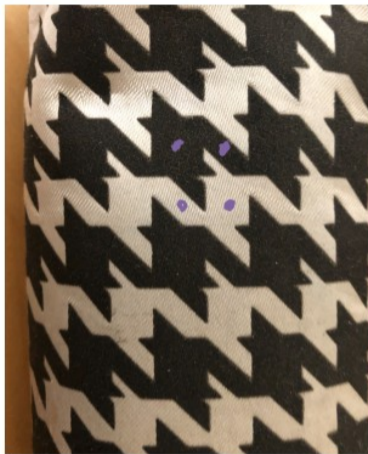
<http://drorbn.net/mc21/upload>



Dror-4S2-StClairWSubway.jpg



Janelle-2222-BinderPaper.jpg



Janelle-2222-PencilCase.jpg



Janelle-22o-MathtownFloorAtTau.jpg

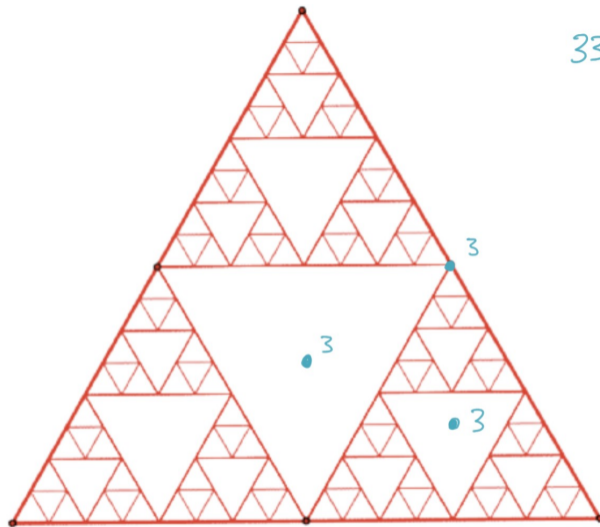


Janelle-632-TissueBox.jpg



Janelle-oo-floor.jpg

333



Maggie-333-BrokenSierpinsky.jpg

447



Maggie-442-BowlPattern.jpg



*J-Lo's picture

Ⓟ

Maggie-632-JLo'sPot.jpg

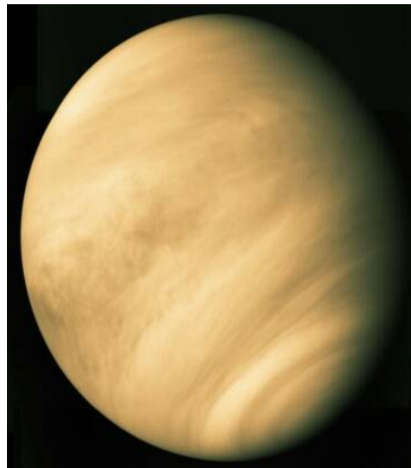


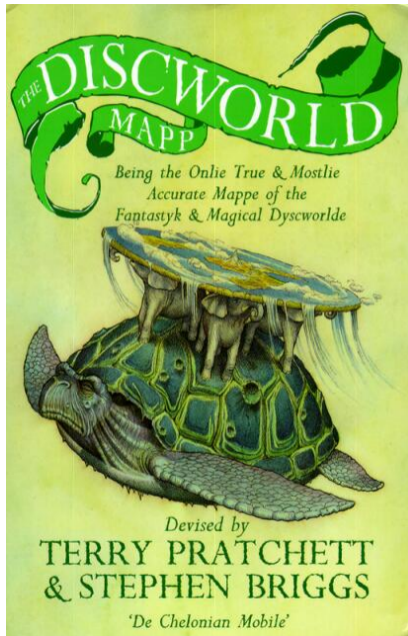
GG

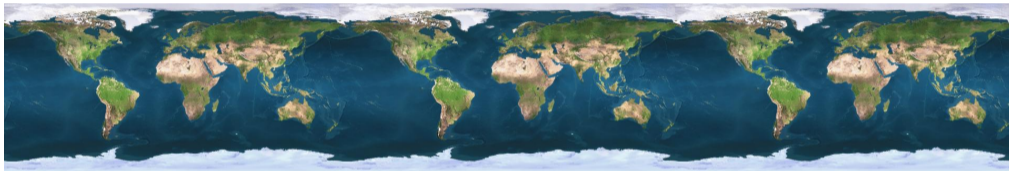
*J-Lo's
image

Maggie-GG-JLo'sBag.jpg

What if we lived on Venus?







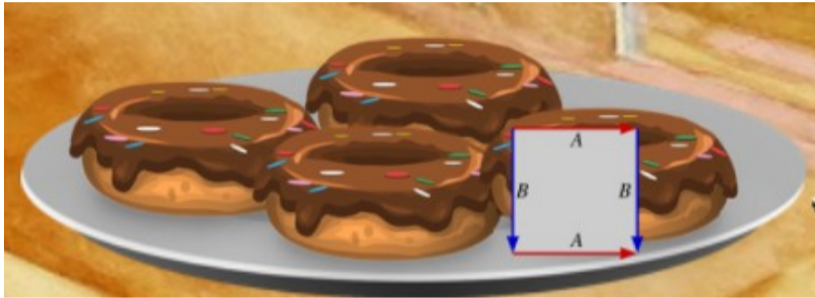






The Jerusalem Renaissance Hotel





Doughnuts stolen from the MathCamp Kitchen



A truck's tire inner tube and Itai

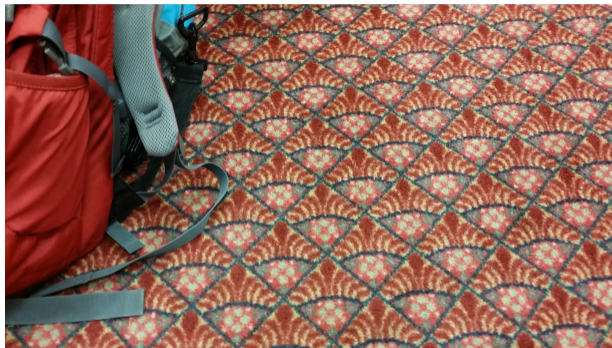


Floor of La Tortilleria, Toronto 2018



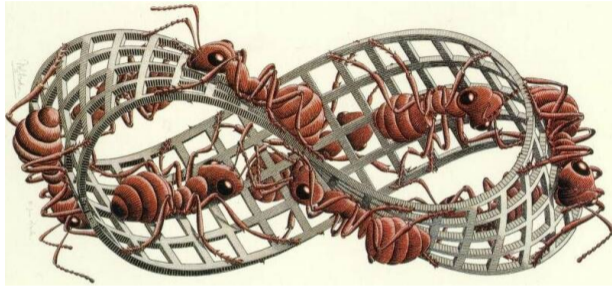


Earth on a cylinder



A carpet seen at Indiana University, 2016





A Möbius band by M. C. Escher



A living room sofa at the Karshon's, 2012





A Klein bottle from <https://www.kleinbottle.com/>



A bed cover and Annie, 2000





a $90^\circ 90^\circ 90^\circ 90^\circ$ pillow



This one earned Angela Wu C\$50 a few years ago



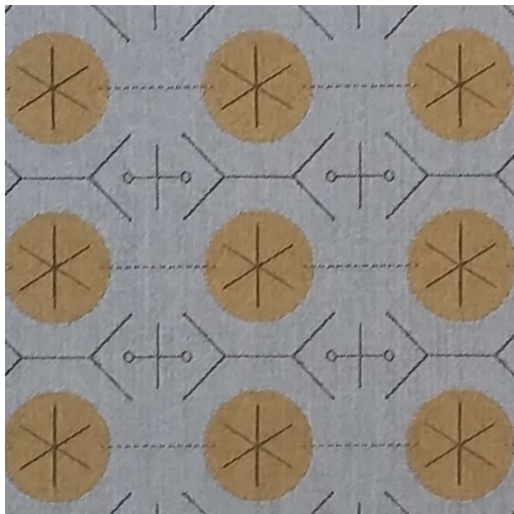


a $60^\circ 60^\circ 60^\circ$ Hamantashan

Similarly, 442 \rightarrow a $90^\circ 45^\circ 45^\circ$ pillow or cookie, and 632 \rightarrow a $90^\circ 60^\circ 30^\circ$ pillow or cookie.



Wallpaper at Bridgehead Coffee in Ottawa





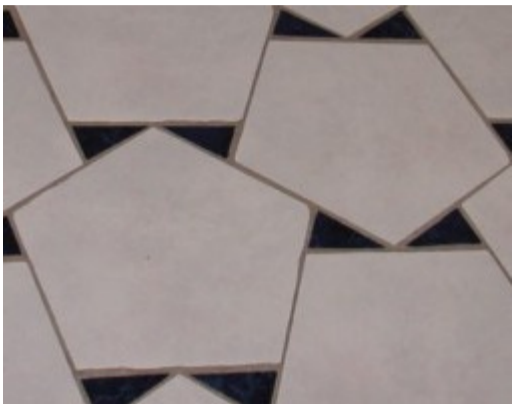
a $90^\circ 90^\circ 90^\circ 90^\circ$ rectangle

Similarly, 442 \rightarrow a $90^\circ 45^\circ 45^\circ$ triangle, 333 \rightarrow a $60^\circ 60^\circ 60^\circ$ triangle, and 632 \rightarrow a $90^\circ 60^\circ 30^\circ$ triangle.





The powder room at the Kuperberg-Zieve's



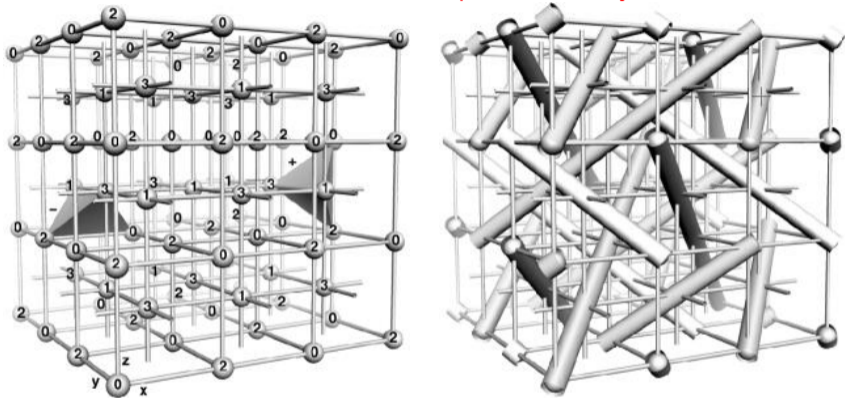


A plastic bag

Homework.

What about **4~~2~~**, **3~~3~~**, **2~~2~~~~2~~**, and **22G**?

The 230 Worlds of Spatial Monkeys



The 219 Worlds of Monkeys that Can't Tell Left From Right

(Numbers and pictures from [arXiv:math/9911185](https://arxiv.org/abs/math/9911185) by Conway, Friedrichs, Huson, and Thurston; see also <http://webmineral.com/crystall.shtml>)

Thank You!