Handout on 180331

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Dror Bar-Natan: Talks: Matemale-1804:

Solvable Approximations of the Quantum sl_2 Portfolio


With Roland van der Veen

ωεβ:=http://drorbn.net/mm18/

Our Main Theorem (loosely stated). Everything that matters in the quantum sl_2 portfolio can be continuously expressed in terms of docile perturbaed Gaussians using solvable approximations. ○

Our Main Points.

- What's the "quantum sl_2 portfolio"?
- What in it "matters" and why? (the most important question)
- What's "solvable approximation"? What's "continuously"?
- What are "docile perturbaed Gaussians"?
- Why do they matter? (2^{nd} most important)
- How proven? (docile)
- How implemented? (sacred; the work of unsung heros)

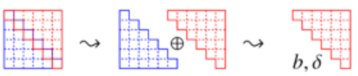
The quantum sl_2 Portfolio includes a classical universal enveloping algebra CU, its quantization QU, their tensor powers $CU^{\otimes S}$ and $QU^{\otimes S}$ with the "tensor operations" $\otimes$, their products m_k^{ij}, coproducts Δ^i_{jk} and antipodes S_i, their Cartan automophisms $C\theta\colon CU\to CU$ and $Q\theta\colon QU\to QU$, the "dequantizators" $A\mathbb{D}\colon QU\to CU$ and $S\mathbb{D}\colon QU\to CU$, and most importantly, the R-matrix R and the Drinfel'd element s. All this in any PBW basis, and change of basis maps are included.

$$R, s\in\{QU^{\otimes S}\}\circlearrowleft_{\otimes, m^{ij}_k, \Delta^i_{jk}, S_i, \theta} \xrightarrow{A\mathbb{D}, S\mathbb{D}} \{CU^{\otimes S}\}\circlearrowleft_{\otimes, m^{ij}_k, \Delta^i_{jk}, S_i, \theta}$$

Solvable Approximation. A quantized universal enveloping algebra (aka "quantum group") is an ∞-dimensional inverse limit.

untrimmed →	halfway-trimmed →	almost fully-trimmed
↕	↕	↕
too hard	"solvable approximation" (a parameter is hidden)	"finite-type"

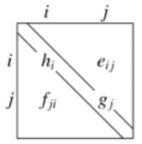
Recomposing gl_n. Half is enough! $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(◹, b, \delta)$:



$b(◹)=b\colon ◹\otimes◹\to◹$

$b(◺)\rightsquigarrow\delta\colon ◹\to◹\otimes◹$

Now define $gl_n^\epsilon:=\mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹,◹]=◹$, $[◺,◺]=\epsilon◺$, and $[◹,◺]=◺+\epsilon◹$. In detail, it is



$$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{li}e_{kj}\qquad [f_{ij},f_{kl}]=\epsilon\delta_{jk}f_{il}-\epsilon\delta_{li}f_{kj}$$
$$[e_{ij},f_{kl}]=\delta_{jk}(\epsilon\delta_{j<k}e_{il}+\delta_{il}(h_i+\epsilon g_i)/2+\delta_{i>l}f_{il})$$
$$-\delta_{li}(\epsilon\delta_{k<j}e_{kj}+\delta_{kj}(h_j+\epsilon g_j)/2+\delta_{k>j}f_{kj})$$
$$[g_i,e_{jk}]=(\delta_{ij}-\delta_{ik})e_{jk}\qquad [h_i,e_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})e_{jk}$$
$$[g_i,f_{jk}]=(\delta_{ij}-\delta_{ik})f_{jk}\qquad [h_i,f_{jk}]=\epsilon(\delta_{ij}-\delta_{ik})f_{jk}$$

Solvable Approximation (2). At $\epsilon=1$ and modulo $h=g$, the above is just gl_n. By rescaling at $\epsilon\neq0$, gl_n^ϵ is independent of ϵ. We let gl_n^k be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1}=0$. It is the "$k$-smidgen solvable approximation" of gl_n!

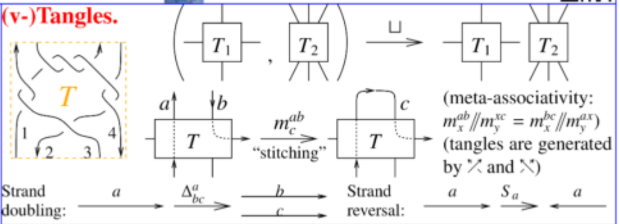
Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2:=[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}_3:=[\mathfrak{g}_2,\mathfrak{g}_2]$, …, $\mathfrak{g}_d=0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.

Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

Definition. A "docile perturbed Gaussian"

doc·ile
/ˈdäsəl/
adjective
ready to accept control or instruction; submissive.
"a cheap and docile workforce"

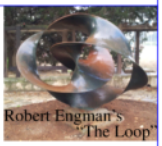
(v-)Tangles.

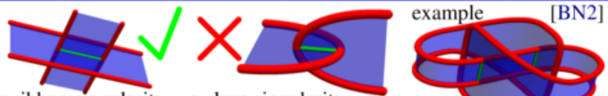


(meta-associativity: $m^{ab}_x /\!/ m^{xc}_y=m^{bc}_x /\!/ m^{ax}_y$) (tangles are generated by ⤯ and ⤰)

Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.

Claim. The knots of genus ≤2 are precisely the images of 4-component tangles via


Robert Engman's "The Loop"



A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3=\partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

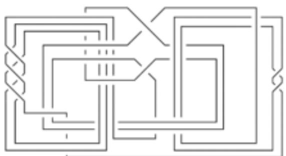
Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t)=f(t)f(1/t)$. (also for slice)


Gompf, Scharlemann, Thompson [GST]

$$U\in\mathcal{T}_n\qquad 1\in\mathcal{A}_n$$
$$\mathcal{T}_{2n}\xrightarrow{z}\mathcal{A}_{2n}\qquad \text{with } \mathcal{R}:=\kappa(\tau^{-1}(1))$$

ribbon $K\in\mathcal{T}_1$ $\quad z(K)\in\mathcal{R}\subseteq\mathcal{A}_1$



Faster is better, leaner is meaner!

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has

$$\Gamma(T)\in R_S\times M_{S\times S}(R_S)=\left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}\text{ with } R_S:=\mathbb{Z}(\{t_a\colon a\in S\}):$$

$$\left({}_a{\nearrow}_b, {}_b{\nwarrow}_a\right)\to\begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-t_a^{\pm1}\\ b & 0 & t_a^{\pm1}\end{array}\qquad T_1\sqcup T_2\to\begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array}\xrightarrow[t_a,t_b\to t_c]{m^{ab}_c}\left(\begin{array}{c|cc}(1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

(Roland: "add to A the product of column b and row a, divide by $(1-A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

$$\begin{array}{c|cc}\omega & a & S\\ \hline a & \alpha & \theta\\ S & \phi & \Xi\end{array}\xrightarrow[\substack{\mu:=T_a-1\\ \nu:=\alpha-\sigma_a\\ T_a\mapsto T_bT_c}]{q\Delta^a_{bc}}\left(\begin{array}{c|ccc}\omega & b & c & S\\ \hline b & (\sigma_a-\alpha T_a-\nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu\\ c & (T_c-1)\nu/\mu & (\alpha-\sigma_aT_a-\nu T_c)/\mu & (T_c-1)\theta/\mu\\ S & \phi & \phi & \Xi\end{array}\right)$$

$$\Big\downarrow{\scriptstyle dS^a,\ T_a\to T_a^{-1}}$$

$$\left(\begin{array}{c|cc}\alpha\omega/\sigma_a & a & S\\ \hline a & 1/\alpha & \theta/\alpha\\ S & -\phi/\alpha & (\alpha\Xi-\phi\theta)/\alpha\end{array}\right)$$

where σ…

Faddeev's Formula (In as much as we can tell, first appeared w/o proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]).

Faddeev

Quesne

Zagier

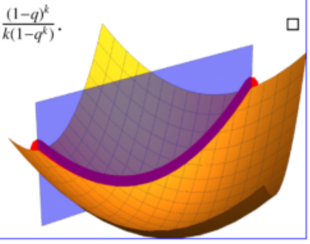
With $[n]_q := \frac{q^n-1}{q-1}$, with $[n]_q! := [1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x := \sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have

$$\log \mathbb{e}_q^x = \sum_{k\geq 1}\frac{(1-q)^k x^k}{k(1-q^k)} = x + \frac{(1-q)^2x^2}{2(1-q^2)} + \dots.$$

Proof. We have that $\mathbb{e}_q^x = \frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx} = (1+(1-q)x)\mathbb{e}_q^x$, and

$$\log \mathbb{e}_q^{qx} = \log(1+(1-q)x) + \log \mathbb{e}_q^x.$$

Writing $\log \mathbb{e}_q^x = \sum_{k\geq 1} a_k x^k$ and comparing powers of x, we get $q^k a_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. $\square$



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[Za] D. Zagier, *The Dilogarithm Function,* in Cartier, Moussa, Julia, and Vanhove (eds) *Frontiers in Number Theory, Physics, and Geometry II.* Springer, Berlin, Heidelberg, and ωεβ/Za.