Pensieve header: Notebook for DBN Lecture 3 in Matemale, April 2018. See http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/.

Today. First Dror, then Roland.

Unrelated Question. Anybody wants a French (azerty) keyboard for a surface pro?

Before we Start...

Please think about our Partial To do List!

A Partial To Do List.

- Complete all "docility" arguments by identifying a "contained" docile substructure.
- Understand denominators and get rid of them.
- See if much can be gained by including P in the exponential: $\mathbb{e}^{L+Q}P \rightsquigarrow \mathbb{e}^{L+Q+P}$?
- Clean the program and make it efficient.
- Run it for all small knots and links, at $k = 2, 3$.
- Understand the centre and figure out how to read the output.
- Execute the Drinfel'd double procedure at $\mathbb{E}$-level (and thus get rid of `DeclareAlgebra` and all that is around it!).
- Extend to sl_3 and beyond.
- Do everything with `Zip` and `Bind` as the fundamentals, without ever referring back to (quantized) Lie algebras.
- Prove a genus bound and a Seifert formula.
- Obtain "Gauss-Gassner formulas" (ωεβ/NCSU).
- Relate with Melvin-Morton-Rozansky and with Rozansky-Overbay.
- Understand the braid group representations that arise.
- Find a topological interpretation. The Garoufalidis-Rozansky "loop expansion" [GR]?
- Figure out the action of the Cartan automorphism.
- Disprove the ribbon-slice conjecture!
- Figure out the action of the Weyl group.
- Do everything at the "arrow diagram" level of finite-type invariants of (rotational) virtual tangles.
- What else can you do with the "solvable approximations"?
- And with the "Gaussian zip and bind" technology?

Recall the Zipping Formula...

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{\eta^i z_i + y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i+y_i,\zeta^j)\mathbb{e}^{\eta^i(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i + y_j\zeta^j + q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde{q})\left\langle P(\tilde{q}^k_i(z_k+y_k),\zeta^j)\mathbb{e}^{c+\eta^i\tilde{q}^k_i(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde{q}$ is the inverse matrix of $1-q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar = 0$ and at $\partial_\hbar$).

... and that we only need to compute operations on exponentials...

Proposition. If $F\colon \mathcal{S}(B) \to \mathcal{S}(B')$ is linear and "continuous", then ${}^tF = \exp\left(\sum_{z_i\in B}\zeta_i z_i\right)/\!\!/F$.

Our Algebra QU

QU = $\langle y, a, x, t\rangle$, with t central and subject to the relations $[a, x] = \gamma x$, $[a, y] = -\gamma y$, and $x y - q y x = \frac{1-T A^2}{\hbar}$, where $q = e^{\hbar\gamma\epsilon}$, $T = e^{\hbar t}$ ($t = \epsilon a - \gamma b$, as Roland derived the algebra), and $A = e^{-\hbar\epsilon a}$. For convenience we set $\mathcal{A} = e^{\gamma\alpha}$.

QU has a co-product Δ and an antipode S given by

$$\Delta(\text{y,a,x,t}) = (y_1 + T_1 y_2 A_1,\ a_1 + a_2,\ x_1 + A_1 x_2)$$

$$S(\text{y,a,x,t}) = (-T^{-1} A^{-1} y,\ -a,\ -A^{-1} x,\ -t)$$

The Program

Initialization / Utilities

DeclareAlgebra

DeclareMorphism

Meta-Operations

Implementing CU = $\mathcal{U}(\text{sl}_2^{\gamma\epsilon})$

Implementing QU = $\mathcal{U}_q(\text{sl}_2^{\gamma\epsilon})$

The representation ρ

tSW

Exponentials as needed.

Zip and Bind

Tensorial Representations

Alternative Algorithms

New Utilities (not on handout)

```
In[•]:= α2𝒜 = {e^(c_. α_i_ + b_.) :→ 𝒜_i^(c/γ) e^b, e^(c_. α + b_.) :→ 𝒜^(c/γ) e^b, e^ℰ_ :→ e^(Expand@ℰ)};
```

Some Runs

The Structure Tensors

```
In[•]:= $k = 1; γ = 1;
```

In[∘]:= $\mathbf{tm}_{1,2\to 3}$ //. α2𝒜

But that's not the form we like! Let's open Zip and Bind!

How was it computed? Let's open the tSW box!

In[∘]:= $\mathbf{tS}_1$

How was it computed? Let's open Exponentials as Needed!

In[∘]:= $\mathbf{t\Delta}_{1\to 1,2}$

In[∘]:= $\{\mathbf{tR}_{1,2},\ \overline{\mathbf{tR}}_{1,2}\}$

In[∘]:= $\{\mathbf{tC}_1,\ \overline{\mathbf{tC}}_2\}$

Some Testing

Associativity of tm.

In[∘]:= $\mathbf{tm}_{1,2\to 2} \sim \mathbf{B}_2 \sim \mathbf{tm}_{2,3\to 1}$ //. α2𝒜

In[∘]:= $\mathbf{tm}_{1,2\to 2} \sim \mathbf{B}_2 \sim \mathbf{tm}_{2,3\to 1} \equiv \mathbf{tm}_{2,3\to 2} \sim \mathbf{B}_2 \sim \mathbf{tm}_{1,2\to 1}$

tS is an anti-homomorphism for tm.

In[∘]:= $(\mathbf{tS}_1\,\mathbf{tS}_2) \sim \mathbf{B}_{1,2} \sim \mathbf{tm}_{1,2\to 1} \equiv \mathbf{tm}_{2,1\to 1} \sim \mathbf{B}_1 \sim \mathbf{tS}_1$

Testing convolution inverse:

In[∘]:= $\mathbf{t\Delta}_{1\to 1,2} \sim \mathbf{B}_1 \sim \mathbf{tS}_1 \sim \mathbf{B}_{1,2} \sim \mathbf{tm}_{1,2\to 1}$

Testing quasi-triangular axioms

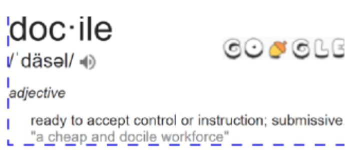
In[∘]:= $(\mathbf{t\Delta}_{1\text{->}1,2}\,\mathbf{tR}_{3,4}) \sim \mathbf{B}_{1,2,3,4} \sim (\mathbf{tm}_{1,3\to 1}\,\mathbf{tm}_{2,4\to 2}) \equiv (\mathbf{t\Delta}_{1\to 2,1}\,\mathbf{tR}_{3,4}) \sim \mathbf{B}_{1,2,3,4} \sim (\mathbf{tm}_{3,1\to 1}\,\mathbf{tm}_{4,2\to 2})$

Testing R3

In[∘]:= $(\mathbf{tR}_{2,3}\,\mathbf{tR}_{1,4}\,\mathbf{tR}_{5,6}) \sim \mathbf{B}_{1,2,3,4,5,6} \sim (\mathbf{tm}_{1,5\to 1}\,\mathbf{tm}_{2,6\to 2}\,\mathbf{tm}_{3,4\to 3}) \equiv$
$(\mathbf{tR}_{1,2}\,\mathbf{tR}_{5,3}\,\mathbf{tR}_{6,4}) \sim \mathbf{B}_{1,2,3,4,5,6} \sim (\mathbf{tm}_{1,5\to 1}\,\mathbf{tm}_{2,6\to 2}\,\mathbf{tm}_{3,4\to 3})$

Docility and why it matters:

Definition. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form



$$\mathbb{e}^{q^{ij}z_iz_j}P = \mathbb{e}^{q^{ij}z_iz_j}\left(\sum_{k\geq 0}\epsilon^k P_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 4k$.

Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$.

In our case our invariants and operations are of the form $\mathbb{e}^{L+Q}P$, where

- L is a quadratic of the form $\sum l_{z\zeta} z\zeta$, where z runs over $\{t_i, \alpha_i\}_{i\in S}$ and ζ runs over $\{\tau_i, a_i\}_{i\in S}$, with integer coefficients $l_{z\zeta}$.
- Q is a quadratic of the form $\sum q_{z\zeta} z\zeta$, where z runs over $\{x_i, \eta_i\}_{i\in S}$ and ζ runs over $\{\xi_i, y_i\}_{i\in S}$, with coefficients $q_{z\zeta}$ in the ring R_s of rational functions in $(T_i)_{i\in S}$ and $(\mathcal{A}_i)_{i\in S}$.
- $P = \sum \epsilon^k P_k$ is a docile power series in $\{y_i, a_i, x_i, \eta_i, \xi_i\}_{i\in S}$, where $\deg(y_i, a_i, x_i, \eta_i, \xi_i) = (1, 2, 1, 1, 1)$.

The Trefoil

Seeing that our invariant is poly-time and the trefoil knot is tiny, the following will compute in no time at all:

```
In[ ]:= Timing@Block[{$k = 1},
  Z = tR1,5 tR6,2 tR3,7 tC̄4 tKink̄8 tKink̄9 tKink̄10;
  Do[Z = Z ~ B1,k ~ tm1,k→1, {k, 2, 10}]; Z]
```

```
In[ ]:= Timing@Block[{$k = 1},
  Z = tR1,5 tR6,2 tR3,7 tC̄4 tKink̄8 tKink̄9 tKink̄10 /. T_ → T1;
  Do[Z = Z ~ B1,k ~ tm1,k→1 /. T_ → T1, {k, 2, 10}]; Z]
```