Quantum Enveloping Algebras and Lie bi-algebras

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Content

My main references are

1. Majid, S., Foundations of Quantum Group Theory, Cambridge University Press, 1995,
2. Chari, V., Pressley, A., A Guide to Quantum Groups, Cambridge University Press, 1994.

The proof of all propositions I state can be found here, except if stated otherwise.

Content

1. Lie bialgebras and the classical double
2. Hopf algebras and the quantum double
3. Quantization of Lie and Hopf algebras
4. Example: the quantum Heisenberg algebra

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Part I: Lie bialgebras and the classical double

Lie bialgebras

Definition 1

(Lie bialgebra) A Lie bi algebra $(\mathfrak{g}, [,], \delta)$ is a vector space L over a field k together with a bilinear map $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ (the bracket) and a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ (the cobracket) satisfying the following axioms:

1. $[X, X] = 0 \ \forall X \in \mathfrak{g}$
2. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \ \forall X, Y, Z \in \mathfrak{g}$
3. δ is skew-symetric
4. $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a bracket on the dual Lie algebra $\mathfrak{g}^*$
5. $\delta([X, Y]) = X.\delta(Y) - Y.\delta(X)$

In this notation, $X.\delta(Y) = (ad_X \otimes 1 + 1 \otimes ad_X)(\delta(Y))$, and $ad_X(Y) = [X, Y]$, for all $X, Y \in \mathfrak{g}$, and $\delta(a) = \sum a_1 \otimes a_2$.

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Lie bialgebras

δ is a 1 cocycle, so look at the cases when δ is a coboundary: $\delta(X) = X.r$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$, and for all $X \in \mathfrak{g}$, where r obeys (if we write $r = \sum r_{12} = \sum r^{[1]} \otimes r^{[2]}$):

1. $r_{12} + r_{21}$ is a invariant under the action of $\mathfrak{g}$.
2. $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$.

Here $[r_{12}, s_{13}] = \sum [r^{[1]}, s^{[1]}] \otimes r^{[2]} \otimes s^{[2]}$.Conditon 2 is called the classical Yang-Baxter equation, and r is called the classical r-matrix. If the Lie-bialgebra structure arises from a classical r-matrix, then we call the Lie bialgebra quasitriangular.

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Lie bialgebra pairing

(page 350 of Majid)

Definition 2

(Lie bialgebra pairing) Let $(\mathfrak{g}, [\,,\,], \delta)$ be a finite dimensional Lie bialgebra. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$ viewed as vectorspace with pairing $\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to k$. Then the following relations define a Lie bialgebra structure on $\mathfrak{g}^*$:

$$\langle [a, b], c \rangle := \langle a \otimes b, \delta c \rangle \tag{1}$$

$$\langle \delta a, b \otimes c \rangle := \langle a, [c, d] \rangle \tag{2}$$

for all $a, b \in \mathfrak{g}^*$, and $c, d \in \mathfrak{g}$. Two Lie bialgebras are said to be dually paired if their Lie brackets and Lie cobrackets are related in this way.

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Classical double

The theorem is also true for infinite dimensional pairing.

Definition 3

Let $\mathfrak{g}$ be a finite dimensional Lie bialgebra with dual $\mathfrak{g}^*$. Then the classical double $D(\mathfrak{g})$ is a quasitriangular Lie bialgebra built on the vector space $\mathfrak{g}^* \oplus \mathfrak{g}$ with bracket, cobracket and r-matrix (here e_a is a basis of $\mathfrak{g}$ and f^a its dual basis):

$$
\begin{aligned}
[a \oplus b, c \oplus d]_D &= \left([c, a] + \sum c_1 \langle c_2, b\rangle - a_1 \langle a_2, d\rangle\right) \\
&\quad \oplus \left([b, d] + \sum b_1 \langle c, b_2\rangle - d_1 \langle a, d_2\rangle\right) && (3) \\
\delta_D(a \oplus b) &= \sum (a_1 \oplus 0) \otimes (a_2 \oplus 0) + \\
&\quad \sum (0 \oplus b_1) \otimes (0 \oplus b_2), && (4) \\
r_D &= \sum_a (f^a \oplus 0) \otimes (0 \oplus e_a) && (5)
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Note that $\mathfrak{g}^*$ has the negated (opposite) bracket in $D(\mathfrak{g})$.

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useful identities

Let $a, d \in \mathfrak{g}^*$ and $b, c \in \mathfrak{g}$. On $D(\mathfrak{g})$ define a pairing $\langle , \rangle_D : D(\mathfrak{g})^* \times D(\mathfrak{g}) \to k : \langle a \oplus b, d \oplus c \rangle_D = \langle a, c \rangle + \langle d, b \rangle$.
Then

$$\langle [a, b]_D, c \rangle_D = \langle a, [b, c] \rangle, \langle a, [d, c]_D \rangle_D = \langle [d, a], c \rangle.$$

Part II: Hopf algebras and the quantum double.

Definition 4

((co-)algebra) An algebra $(H, \cdot, \mu)$ over k is a vector space $(H, +, k)$ with a compatible multiplication $\cdot$ and unit map μ where

1. the multiplication $\cdot : H \otimes H \to H$ is an associative, linear map which preserves the unit,
2. the unit map $\mu : k \to H$ is a linear map with property $\cdot \circ \mu \otimes id(i \otimes a) = i \cdot a$, and $\cdot \circ id \otimes \mu(a \otimes i) = i \cdot a \; \forall a \in H, i \in k$ (or $\mu(1) = 1_H$).

A coalgebra (H, Δ, ϵ) over k is a vector space $(H, +, k)$ with a compatible comultiplication Δ and co-unit ϵ where

1. the comultiplication $\Delta : H \to H \otimes H$ is a linear, coassociative map, where coassociativity means $\Delta \otimes id \circ \Delta = id \otimes \Delta \circ \Delta$ and $\Delta(1_H) = 1_H \otimes 1_H$,
2. the counit $\epsilon : H \to k$ has property $(id \otimes \epsilon) \circ \Delta(h) = (\epsilon \otimes id) \circ \Delta(h) = h$ (so $\epsilon(1_H) = 1$).

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A coalgebra (H, Δ, ϵ) over k is a vector space $(H, +, k)$ with a compatible comultiplication Δ and co-unit ϵ where

1. the comultiplication $\Delta : H \to H \otimes H$ is a linear, coassociative map, where coassociativity means $\Delta \otimes id \circ \Delta = id \otimes \Delta \circ \Delta$ and $\Delta(1_H) = 1_H \otimes 1_H$,
2. the counit $\epsilon : H \to k$ has property $(id \otimes \epsilon) \circ \Delta(h) = (\epsilon \otimes id) \circ \Delta(h) = h$ (so $\epsilon(1_H) = 1$).

Hopf Algebras

Note that k always denotes a field of characteristic 0.

Definition 5

A Hopf algebra $(H, +, \cdot, \mu, \Delta, \epsilon, S, k)$ over k is a vector space $(H, +, k)$ which is both an algebra $(H, \cdot, \mu)$ and a coalgebra (H, Δ, ϵ), and is equipped with a linear antipode map $S : H \to H$ (which is an anti-homomorphism) obeying

1. $\Delta(gh) = \Delta(g)\Delta(h)$,
2. $\epsilon(gh) = \epsilon(g)\epsilon(h)$,
3. $\cdot(S \otimes id) \circ \Delta = \cdot(id \otimes S) \circ \Delta = \mu \circ \epsilon$

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Hopf Algebras

Definition 6

(Quasitriangular Hopf algebras) A Quasitriangular Hopf Algebra is a pair (H, R), where H is a Hopf algebra and $R \in H \otimes H$ is invertible and obeys

1. $(\Delta \otimes id)(R) = R_{12}R_{23}$ and $(id \otimes \Delta)(R) = R_{13}R_{12}$
2. $\tau \circ \Delta(h) = R\Delta(h)R^{-1}$, $\forall h \in H$, where τ is the transposition map.

Writing $R = \sum R^{(1)} \otimes R^{(2)}$, we denote $R_{ij} = \sum 1 \otimes \cdots \otimes R^{(1)} \otimes 1 \cdots \otimes R^{(2)} \otimes \cdots \otimes 1$

Definition 7

((Co-)commutative) A Hopf algebra is said to be commutative it is commutative as an algebra, and cocommutative if the co-product Δ obeys $\tau \circ \Delta = \Delta$.

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Theorem 8

Let (H, R) be a quasitriangular Hopf algebra, then R solves the equation: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, called the Quantum Yang-Baxter equation.

Ribbon Hopf algebras

Let us write $R = \sum R^{(1)} \otimes R^{(2)}$. Then define $u = \sum (SR^{(2)})R^{(1)} \in H$, and $v = Su = \sum R^{(1)} SR^{(2)}$.

Theorem 9

Let (H,R) be a quasitriangular Hopf algebra with antipode S. Then S is invertible and $S^2(h) = uhu^{-1}$ for all $h \in H$, and $S^{-2}(h) = vhv^{-1}$.

Definition 10

(Ribbon element) A quasitriangular Hopf algebra is called a ribbon Hopf algebra if the element uv has a central square root ν, called the ribbon element, such that $\nu^2 = vu$, $S\nu = \nu$, $\epsilon\nu = 1$ and $\Delta\nu = Q^{-1}(\nu \otimes \nu)$, where $Q = R_{21}R$.

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Hopf algebra pairing

Definition 11

(Hopf Pairing) Let G, H be a Hopf algebras. H and G are said to be dually paired as Hopf algebras if they are dually paired as vector spaces, and if the multiplication, co multiplication, antipode and counit behave in the following way under the pairing $\langle , \rangle$:

$$\langle ab, c\rangle = \langle a \otimes b, \Delta c\rangle \tag{6}$$
$$\langle \Delta a, c \otimes d\rangle = \langle a, cd\rangle \tag{7}$$
$$\langle 1, c\rangle = \epsilon(c) \tag{8}$$
$$\langle a, 1\rangle = \epsilon(a) \tag{9}$$
$$\langle Sa, c\rangle = \langle a, Sc\rangle \tag{10}$$

for all $a, b \in G$ and for all $c, d \in H$. G and H are a strictly dual pair if the pairing is nondegenerate, i.e. there are no nonzero elements in G or H that pair to zero with every element in the dually paired algebra.

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Quantum Double

This works also for an infinite dimensional pairing. H^{*op} is the Hopf algebra H^* with the opposite multiplication.
We write $\Delta(a) = a_1 \otimes a_2$, omitting the summation.

Theorem 12

(Quantum Double) Let H be a finite dimensional Hopf algebra. The quantum double D(H) is a quasitriangular Hopf algebra generated by H, H^{*op} as sub Hopf algebras with the quasitriangular structure $R = \sum_a f^a \otimes e_a$, where $\{e_a\}$ is the basis of H and $\{f^a\}$ its dual basis. D(H) is realised on the vectorspace $H^* \otimes H$ with product $(a \otimes h)(b \otimes g) = \sum b_2 a \otimes h_2 g \langle Sh_1, b_1 \rangle \langle h_3, b_3 \rangle$, and tensor product unit, counit and coproduct.

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Part III: Quantization of Lie and Hopf algebras.

Universal enveloping algebra

Definition 13

Let $\mathfrak{g}$ be a Lie algebra over k.The universal enveloping algebra $U(\mathfrak{g})$ is the noncommutative algebra generated by 1 and the elements of $\mathfrak{g}$ (the tensor algebra over k) modulo the relations $[a, b] = ab - ba$ for all $a, b \in \mathfrak{g}$. The coproduct, counit and antipode are given by

$$\Delta a = a \otimes 1 + 1 \otimes a, \epsilon a = 0, Sa = -a,$$

where Δ, ϵ are extended as algebra maps, and S as an antialgebra map.

Note that this algebra is cocommutative, so we can take the R-matrix to be trivial to make $U(\mathfrak{g})$ a quasitriangular Hopfalgebra.

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Quantum Universal enveloping algebras

Definition 14

A deformation of a Hopf algebra $(H, i, \mu, \epsilon, \Delta, S)$ over a field k is a topological Hopf algebra $(H_h, i_h, \mu_h, \epsilon_h, \Delta_h, S_h)$ over the ring $k[[h]]$ of formal power series in h over k, such that

1. H_h is isomorphic to $H[[h]]$ as a $k[[h]]$ module.
2. $\mu_h = \mu$ mod h, $\Delta_h = \Delta$ mod h.

Two Hopf algebra deformations are said to be equivalent if there is an isomorphism f_h of Hopf algebras over $k[[h]]$ which is the identity (mod h).

Definition 15

(Quantized universal enveloping algebra (QUE)) A Hopf algebra deformation of the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is called a quantized universal enveloping algebra, or QUE algebra.

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co-Poisson Hopf algebras

We have extended the bracket of a Lie bialgebra $\mathfrak{g}$ to $U(\mathfrak{g})$, and we have equipped $U(\mathfrak{g})$ with a Hopf algebra structure, but we have not yet extended δ to U(g).

Definition 16

(Co-Poisson Hopf algebras) A Co-Poisson Hopf algebra over a commutative ring k is a co-commutative Hopf algebra H with a skew symmetric k-module map $\delta : H \to H \otimes H$ (the poisson co-bracket) satisfying:

1. $\sigma \circ \delta \otimes id \circ \delta = 0$, where σ means summing over cyclic permutations of the tensor product.
2. $(\Delta \otimes id)\delta = (id \otimes \delta)\Delta + \sigma_{23}(\delta \otimes id)\Delta$, where σ_{23} means switching the second and third factor.
3. For all $a, b \in H$, $\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)$.

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co-Poisson Hopf algebras

We have extended the bracket of a Lie bialgebra $\mathfrak{g}$ to $U(\mathfrak{g})$, and we have equipped $U(\mathfrak{g})$ with a Hopf algebra structure, but we have not yet extended δ to U(g).

Definition 16

(Co-Poisson Hopf algebras) A Co-Poisson Hopf algebra over a commutative ring k is a co-commutative Hopf algebra H with a skew symmetric k-module map $\delta : H \to H \otimes H$ (the poisson co-bracket) satisfying:

1. $\sigma \circ \delta \otimes id \circ \delta = 0$, where σ means summing over cyclic permutations of the tensor product.
2. $(\Delta \otimes id)\delta = (id \otimes \delta)\Delta + \sigma_{23}(\delta \otimes id)\Delta$, where σ_{23} means switching the second and third factor.
3. For all $a, b \in H$, $\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b)$.

co-Poisson Hopf algebras

Theorem 17

Let $\mathfrak{g}$ be a Lie bialgebra over a field k of characteristic zero. Then the Lie co-bracket extends uniquely to a Poisson co-bracket δ on $U(\mathfrak{g})$, making $U(\mathfrak{g})$ a co-Poisson Hopf algebra.
Conversely, if $U(\mathfrak{g})$ has a Poisson co-bracket δ, then $\delta|_{\mathfrak{g}}$ is a Lie cobracket on $\mathfrak{g}$.

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Quantum Universal enveloping algebras

Definition 18

(Quantization of Hopf algebra) Let A be a cocommutative co-Poisson-Hopf algebra over a field k of characteristic zero, and let δ be its Poisson co-bracket. A Quantization of A is a Hopf algebra deformation A_h of A such that

$$\delta(x) = \frac{\Delta_h(a) - \Delta_h^{op}(a)}{h} \pmod{h},$$

where $x \in A$ and $a \in A_h$ such that $x = a \pmod{h}$, and $\Delta^{op} = \tau \circ \Delta$ is the opposite co-bracket.
A quantization of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a quantization $U_h(\mathfrak{g})$ of its universal enveloping algebra $U(\mathfrak{g})$ equipped with the co-Poisson-Hopf structure. Conversely, $(\mathfrak{g}, \delta)$ is called the classical limit of the QUE algebra $U_h(\mathfrak{g})$.

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Quantum Universal enveloping algebras

Let us state a few things:

Theorem 19

The quantization of a Lie bi algebra is a quantized universal enveloping algebra.

Theorem 20

Let $(U_h(\mathfrak{g}), R_h)$ be a QUE algebra that is quasitriangular as a Hopf algebra, and has $R_h = 1 \otimes 1 \pmod h$. Then if we define $r \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ as $r = \frac{R_h - 1 \otimes 1}{h} \pmod h$, $r \in \mathfrak{g} \otimes \mathfrak{g}$, and the classical limit of $U_h(\mathfrak{g})$ is a quasitriangular Lie bialgebra with classical r-matrix r.

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Existence

The following theorem is due to Drinfeld (1983).

Theorem 21

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra, and let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be the classical r-matrix. Then there exists a deformation $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ whose classical limit is $\mathfrak{g}$ with the Lie bialgebra structure defined by r. Moreover, $U_h(\mathfrak{g})$ is a triangular Hopf algebra (i.e. a quasitriangular Hopf algebra with $R_{21} = R^{-1}$) and is isomorphic to $U(\mathfrak{g})[[h]]$ as an algebra over $\mathbb{R}[[h]]$.

Part IV: the quantum Heisenberg algebra.

Summary

We have

1. defined a Hopf algebra and its deformation,
2. defined a Lie bialgebra and its quantization,
3. looked at quasitriangular Hopf algebras and Lie bialgebras and the relations between the two,
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Part V: $U_q(sl_2)$

Running example: $sl_2(\mathbb{C})$

Let us consider the Lie algebra $sl_2(\mathbb{C})$ generated by $\{H, X^+, X^-\}$ and the relations

$$[H, X^{\pm}] = \pm 2X^{\pm},\ [X^+, X^-] = H.$$

Then sl_2 becomes a quasitriangular Lie bialgebra if we set

$$\delta(H) = 0,\ \delta(X^{\pm}) = X^{\pm} \wedge H = X^{\pm} \otimes H - H \otimes X^{\pm}, r = X^+ \wedge X^-.$$

Note that we can define the Lie bi-subalgebras $b^{\pm} = span\{H, X^{\pm}\}$ of $sl_2(\mathbb{C})$.

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Example: $U_\hbar(sl_2)$; quantization of $b^\pm$

To find $\Delta_h(H)$, we need that $\frac{\Delta_\hbar(H) - \Delta_\hbar^{op}(H)}{\hbar} \pmod{\hbar} = \delta(H) = 0$, which is satisfied by $\Delta_\hbar(H) = H \otimes 1 + 1 \otimes H$.
We can find $\Delta_\hbar(X^+)$ by taking $\Delta_\hbar(X^+) = X^+ \otimes f + g \otimes X^+$, where f and g are functions of $\hbar$.
$\Delta_\hbar$ is coassociative and an algebra homomorphism, so we get

$$\Delta_\hbar(X^+) = X^+ \otimes e^{\hbar H} + 1 \otimes X^+,\ \Delta_\hbar(X^-) = X^- \otimes 1 + e^{-\hbar H} \otimes X^+.$$

We can rewrite these expressions by defining $q = e^\hbar$:

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We now have calculated the quantizations $U_q(b^\pm)$ of the Lie bialgebras $b^\pm$. Note that the monomials $(X^\pm)^t H^s$ form a topological basis of $b^\pm$.

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Example: $U_q(sl_2)$

Let us introduce the pairing $\langle , \rangle : (U_q(b^+))^* \times U_q(b^+) \to k[[\hbar]]$. $U_q(b^+)^*$ is isomorphic to $U_q(b^-)$ (say with isomorphism $\phi : U_q(b^-) \to U_q(b^+)^*$) We use the notation $X^+ = X, \phi(X^-) = x$, and write A for $H \in U_q(b^+)$ and a for $\phi(H) \in U_q(b^+)^*$. We assume a,x is the dual basis:

$$\langle 1,1\rangle = 1, \langle 1,A\rangle = 0, \langle 1,X\rangle = 0, \langle a,1\rangle = 0, \langle x,1\rangle = 0, \quad (11)$$
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The basis

Note that we are dividing by $\hbar$, so formally we should introduce new generators $\bar{a} = \hbar a$.

After quantization, in general we have no elements corresponding to non simple roots in the Lie algebra. How to construct basis?

If the rootsystem of $\mathfrak{g}$ has no non-simple roots, we use the isomorphism $U_h(\mathfrak{g}) \to U(\mathfrak{g})[[h]]$ together with the classical PBW theorem.

In some cases $U_h(\mathfrak{g})$ contains the non simple root elements. Otherwise, use the action of the braid group (the quantum analogue of the weyl group). In our case this is not necessary!

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$U_q(sl_2)$

After executing the quantum double construction on $U_q(b^+)$ we end up with the following relations (after applying the homomorphism which sends a back to H and x back to X^-, so dividing out a part of the Cartan subalgebra of the double: $[a, A] = 0$.)

$$[X^+, X^-] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}$$

,

$$[H, X^-] = -X^-, \; [H, X^+] = X^+$$

,

We get the following R matrix from the quantum double:

$$R_h = exp(\frac{\hbar}{2} H \otimes H) \sum_{t=0}^{\infty} q^{1/2t(t+1)} \frac{(1-q^{-2})^t}{[t]_q!} (X^+)^t \otimes (X^-)^t. \tag{12}$$

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Part VI: Cohomologies.

Lie bialgebra cohomology

Definition 22

(Chevalley-Eilenburg complex) Let M be a $\mathfrak{g}$ module. Set $C^n((g), M) := Hom_k(\bigwedge^n (g), M)$, $n > 0$, and $C_0(g, M) := M$, where $\bigwedge^k \mathfrak{g}$ is the k-th exterior power of $\mathfrak{g}$. This is the Chevalley-Eilenberg cochain complex.

We define the differential on $c \in C^n(\mathfrak{g}, M)$ as

$$dc(x_1, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} x_i.c(x_1, \cdots, \hat{x}_i, \cdots, x_n) + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} c([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1}), \tag{13}$$

where $x_1, \cdots, x_{n+1} \in \mathfrak{g}$, and $x.d$ means the module action of $\mathfrak{g}$ on $d \in M$.

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where $x_1, \cdots, x_{n+1} \in \mathfrak{g}$, and $x.d$ means the module action of $\mathfrak{g}$ on $d \in M$.

generalised quantum double

In general we want to modify H^* in the double construction. We need a double cross product Hopf algebras.

Definition 23

Two Hopf algebras (A,H) form a matched pair if H is a right A-module coalgebra ($H \triangleleft A$) and A is a left H module coalgebra (H$\triangleright A$) obeying:

$$(hg) \triangleleft a = \sum (h \triangleleft (g_1 \triangleright a_1))(g_2 \triangleleft a_2), 1 \triangleleft a = \epsilon(a) \tag{14}$$

$$h \triangleright (ab) = \sum (h_1 \triangleright a_1)((h_2 \triangleleft a_2) \triangleright b), h \triangleright 1 = \epsilon(h) \tag{15}$$

$$\sum h_1 \triangleleft a_1 \otimes h_2 \triangleright a_2 = \sum h_2 \triangleleft a_2 \otimes h_1 \triangleright a_1. \tag{16}$$

In our case we will take the co-adjoint action Ad^* of H on H^* (or H^* on H) given by:

$$Ad^*_h(\phi) = \sum \phi_2 \langle h, (S\phi_1)\phi_3 \rangle.$$

generalised quantum double

Definition 24

A pair of matched Hopf algebras (A,H) forms a double cross product Hopf algebra built on $A \otimes H$ together with product and antipode

$$(a \otimes h)(b \otimes g) = \sum a(h_1 \triangleright b_1) \otimes (h_2 \triangleleft b_2)g, \tag{17}$$

$$S(a \otimes h) = (1 \otimes Sh)(Sa \otimes 1), \tag{18}$$

and tensor product unit, counit and coproduct $\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2$.

Lie bialgebra cohomology

Definition 25

(Lie algebra cohomology) Define the space of cocycles $Z^p(\mathfrak{g}, M) := \{c \in C^p(\mathfrak{g}, M) | dc = 0\}$ and the space of coboundaries $B^p(\mathfrak{g}, M) := \{c \in C^p(\mathfrak{g}, M) | \exists c' \in C^{p-1}(\mathfrak{g}, M) s.t.\ dc' = c\}$. Then define the Lie algebra cohomology as $H^p(\mathfrak{g}, M) := Z^p(\mathfrak{g}, M) / B^p(\mathfrak{g}, M)$.

Note that the condition $\delta([X, Y]) = X.\delta(Y) - Y.\delta(X)$ states that δ is a 1-cocycle in the Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$, with the adjoint action of $\mathfrak{g}$ on the tensor product module $\mathfrak{g} \otimes \mathfrak{g}$.

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Definition 26

(see p. 173 Chari-Pressley) Let H be a Hopf algebra. For $i, j \geq 1$, define $C^{i,j} := Hom_k(H^{\otimes i}, H^{\otimes j})$, and define $d'_{i,j} : C^{i,j} \to C^{i+1,j}$ and $d''_{i,j} : C^{i,j} \to C^{i,j+1}$ as follows (let $\gamma \in C^{i,j}$):

$$(d'\gamma)(a_1 \otimes \cdots \otimes a_{i+1}) := \Delta^{(j)}(a_1) \cdot \gamma(a_2 \otimes \cdots \otimes a_{i+1}) + \sum_{r=1}^{i} (-1)^r \gamma(a_1 \otimes \cdots \otimes a_{r-1} a_{r+1} \otimes a_{r+2} \otimes \cdots \otimes a_{i+1}) + (-1)^{i+1} \gamma(a_1 \otimes \cdots \otimes a_i).\Delta^{(j)}(a_{i+1}),$$

Hopf algebra cohomology

Definition 27

$$(d''\gamma)(a_1 \otimes \cdots \otimes a_i) :=$$
$$(\mu^{(i)} \otimes \gamma)(\Delta_{1,i+1}(a_1)\Delta_{2,i+2}(a_2)\cdots\Delta_{i,2i}(a_i))$$
$$+\sum_{r=1}^{j}(-1)^r(id^{\otimes r-1} \otimes \Delta \otimes id^{\otimes j-r})(\gamma(a_1 \otimes \cdots \otimes a_i))$$
$$+(-1)^{j+1}(\gamma \otimes \mu^{(i)})(\Delta_{1,i+1}(a_1)\Delta_{2,i+2}(a_2)\cdots\Delta_{i,2i}(a_i)).$$

Hopf algebra cohomology

in this definition, $\mu^{(i)}$ and $\Delta^{(j)}$ are defined as follows

$$\mu^{(i)}(a_1 \otimes \cdots \otimes a_i) = a_1 \cdot \cdots \cdot a_i \tag{19}$$

$$\Delta^{(j)}(a) = (id \otimes \cdots \otimes id \otimes \Delta) \cdots (id \otimes \Delta)(\Delta(a)). \tag{20}$$

The $\Delta_{i,j}$ means sending the coproduct to the ith and the jth coordinate. The next proposition follows by direct computation

Theorem 28

Let d' and d" be as in the definitions, then,
$d' \circ d' = d'' \circ d'' = d' \circ d'' + d'' \circ d' = 0.$

Definition 29

Let H be a Hopf algebra, and let d' and d" be as defined previously, and set $d = d'_{ij} + (-1)^i d''_{ij}$ and $C^n = \bigoplus_{i+j=n+1} C^{ij}$. Then $d : C^n \to C^{n+1}$ and (C,d) is a cochain complex with cohomology groups $H^*(H, H)$.

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Hopf algebra deformations

Let us write $a_h = a + a_1 h + a_2 h^2 + \cdots$ for an element of A_h.

Because μ_h and Δ_h are $k[[h]]$-module maps, they are determined by their values on elements of A_h for which $a_1 = a_2 = \cdots = 0$. Write

$$\mu_h(a \otimes a') = \mu(a \otimes a') + \mu_1(a \otimes a')h + \mu_2(a \otimes a')h^2 + \cdots \tag{21}$$

$$\Delta_h(a) = \Delta(a) + \Delta_1(a)h + \Delta_2(a)h^2 + \cdots \tag{22}$$

The (co-)associativity and algebra homomorphism conditions of the Hopf algebra deformation are

$$\mu_h(\mu_h(a_1 \otimes a_2) \otimes a_3) = \mu_h(a_1 \otimes \mu_h(a_2 \otimes a_3)) \tag{23}$$

$$(\Delta_h \otimes id)\Delta_h(a) = (id \otimes \Delta_h)\Delta_h(a)$$

$$\Delta_h(\mu_h(a_1 \otimes a_2)) = (\mu_h \otimes \mu_h)\Delta_h^{13}(a_1)\Delta_h^{24}(a_2).$$

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Hopf algebra deformations

Definition 30

A pair of k-module map (μ_1, Δ_1) is called a deformation (mod h^2) of a Hopf algebra H if it satisfies

$$\mu_1(a_1 a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2)a_3 = a_1\mu_1(a_2 \otimes a_3) + \mu_1(a_1 \otimes a_2 a_3)$$

$$(\Delta \otimes id)\Delta_1(a) + (\Delta_1 \otimes id)\Delta(a) = (id \otimes \Delta)\Delta_1(a) + (id \otimes \Delta_1)\Delta(a)$$

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Or more generally a deformation (mod h^{n+1}) is a 2n-tuple $(\mu_1, \cdots, \mu_n, \Delta_1, \cdots, \Delta_n)$ which satisfies the (co-)associativity and algebra homomorphism conditions (mod h^{n+1})

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Hopf algebra deformation

Theorem 31

The following relations between Hopf algebra cohomology and Hopf algebra relations hold:

1. *there is a natural bijection between $H^2(H,H)$ and the set of equivalence classes of deformation (mod h^2) of H,*
2. *If $H^2(H,H) = 0$, every deformation of H is trivial and*
3. *If $H^3(H,H) = 0$, every deformation (mod h^2) of H extends to a genuine deformation of H.*

In our case, $H^3(H,H)$, will not be trivial unfortunately.

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Poisson-Lie groups

Part IIV: Relation to Poisson groups.

Relation to Poisson groups

Definition 32

(Lie Group) A Lie group G is a smooth manifold G without boundary that is a group with a smooth multiplication map $\mu : G \times G \to G$ and a smooth inversion map $i : G \to G$.

Definition 33

(Poisson Structure) Let M be a smooth manifold of finite dimension m, and denote with C(M) the algebra of smooth real valued functions on M. A Poisson structure on M is an $\mathbb{R}$ bilinear map $\{,\} : C(M) \times C(M) \to C(M)$ (the Poisson bracket) satisfying $\forall f_1, f_2, f_3 \in C(M)$:

1. $\{f_1, f_2\} = -\{f_2, f_1\}$
2. $\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$
3. $\{f_1 f_2, f_3\} = \{f_1, f_3\} f_2 + f_1 \{f_2, f_3\}$

Relation to Poisson groups

Definition 34

(Poisson Maps) A smooth map $F : M \to N$ between Poisson manifolds is a Poisson map if it preserves the Poisson brackets of M and N: $\{f_1, f_2\}_M \circ F = \{f_1 \circ F, f_2 \circ F\}_N$.
(Product Poisson structure) The Product Poisson structure is given by $\{f_1(x, y), f_2\}_{M\times N}(x, y) = \{f_1(., y), f_2(., y)\}_M(x) + \{f_1(x, .), f_2(x, .)\}_N(y)$, where $f_1, f_2 \in C(M \times N)$.

Definition 35

A Poisson-Lie group G is a Lie group which also has a Poisson structure that is compatible with the Lie structure, i.e. the multiplication map $\mu : G \times G \to G$ is a poisson map. A homomorphism of Poisson Lie groups is a homomorphism of Lie groups that is also a Poisson map.

Relation to Poisson groups

Theorem 36

Define on a Poisson Lie group G $Ad(x)(y) = xyx^{-1}$ for all $x, y \in G$. Then the tangent space at the unit element e of G is a Liealgebra $\mathfrak{g}$ with Lie bracket $[X, Y] = T_e Ad(X)(Y)$. Define the cobracket δ by the relation $\langle X, d\{f_1, f_2\}_e \rangle = \langle \delta(X), (df_1)_1 \otimes (df_2)_e \rangle$. Then $(T_e G, [,], \delta)$ is a Lie bialgebra.

Proof: Check the definitions. (See "A Guide to Quantum Groups" by Chari, V. and Pressley, A., page 25.)

Note: if the Lie algebra arising in this case is quasitriangular, i.e. if δ is a coboundary, then one can use the classical r-matrix to define the Poisson bracket, and one can define a classical R-matrix $R \in G \times G$ which is a solution of the Quantum Yang Baxter equation: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

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Relation to Poisson groups

Theorem 36

Define on a Poisson Lie group G $Ad(x)(y) = xyx^{-1}$ for all $x, y \in G$. Then the tangent space at the unit element e of G is a Liealgebra $\mathfrak{g}$ with Lie bracket $[X, Y] = T_e Ad(X)(Y)$.Define the cobracket δ by the relation $\langle X, d\{f_1, f_2\}_e \rangle = \langle \delta(X), (df_1)_1 \otimes (df_2)_e \rangle$. Then $(T_e G, [,], \delta)$ is a Lie bialgebra.

Proof: Check the definitions. (See "A Guide to Quantum Groups" by Chari, V. and Pressley, A., page 25.)
Note: if the Lie algebra arising in this case is quasitriangular, i.e. if δ is a coboundary, then one can use the classical r-matrix to define the Poisson bracket, and one can define a classical R-matrix $R \in G \times G$ which is a solution of the Quantum Yang Baxter equation: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

Relation to Poisson groups

Definition 37

(Poisson algebra) A Poisson algebra over k is a commutative algebra A over k with a skew-symmetric k-module map $\{,\} : A \otimes A \to A$ (Poisson bracket) such that $\forall a, b, c \in A$:

1. $\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0$,
2. $\{ab, c\} = \{a, c\}b + a\{b, c\}$.

A Poisson Hopf algebra is a Poisson algebra which is also a Hopf algebra, such that the Poisson structure and the Hopf structure are compatible in the following way:

$$\forall a, b \in A,\ \{\Delta(a), \Delta(b)\}_{A\otimes A} = \Delta(\{a, b\}_A),$$

where
$\{a_1 \otimes b_1, a_2 \otimes b_2\}_{A\otimes A} = \{a_1, a_2\}_A \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}_A$.

The Poisson structure of a Poisson-Lie group is a Poisson algebra.

Lie algebra cohomology from Lie group de rahm complex

Theorem 38

The Chevalley-Eilenberg differential on $\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra belonging to the Lie group G, is equal to the De Rham differential $\Omega^*(G)$ restricted to the space of left invariant differential forms. (See wikipedia: Lie algebra cohomology)

Note that the object dual to $U(\mathfrak{g})$, the regular functions F(G) on a Poisson-Lie group G,which is a Poisson algebra, is only the completion of a Poisson-Hopf algebra, due to $F(G \times G) \neq F(G) \otimes F(G)$. This can be solved by looking at the subalgebra of finite dimensional representations Rep(G) of F(G), which is dense in F(G).

Relation between Univeral enveloping algebra and Poisson algebra on a Poisson Lie group

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Geometrical Quantization of Poisson algebras

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