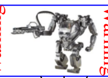
Solvable Approximations of the Quantum sl_2 Portfolio

With Roland van der Veen

ωεβ:=http://drorbn.net/mm18/

W.I.P. Warning — Brute Warning



Our Main Theorem (loosely stated). Everything that matters in the quantum sl_2 portfolio can be continuously expressed in terms of docile perturbaed Gaussians using solvable approximations. ○

Our Main Points.

- What's the "quantum sl_2 portfolio"?
- What in it "matters" and why? (the most important question)
- What's "solvable approximation"? What's "continuously"?
- What are "docile perturbaed Gaussians"?
- Why do they matter? (2^{nd} most important)
- How proven? (docile)
- How implemented? (sacred: the work of unsung heros)

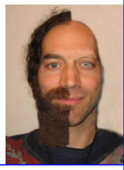
Some context and background

The quantum sl_2 Portfolio includes a classical universal enveloping algebra CU, its quantization QU, their tensor powers $CU^{\otimes S}$ and $QU^{\otimes S}$ with the "tensor operations" $\otimes$, their products m_k^{ij}, coproducts Δ^i_{jk} and antipodes S_i, their Cartan automophisms $C\theta\colon CU \to CU$ and $Q\theta\colon QU \to QU$, the "dequantizators" $A\mathbb{D}\colon QU \to CU$ and $S\mathbb{D}\colon QU \to CU$, and most importantly, the R-matrix R and the Drinfel'd element s. All this in any PBW basis, and change of basis maps are included.

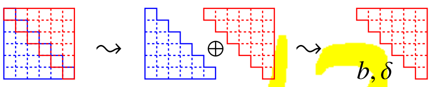
$$R, s \in \{QU^{\otimes S}\} \xrightarrow{A\mathbb{D},\, S\mathbb{D}} \{CU^{\otimes S}\}$$

Solvable Approximation. A quantized universal enveloping algebra (aka "quantum group") is an ∞-dimensional inverse limit.

untrimmed	$\longrightarrow$	halfway-trimmed	$\longrightarrow$	almost fully-trimmed
$\updownarrow$		$\updownarrow$		$\updownarrow$
too hard		"solvable approximation" (a parameter is hidden)		"finite-type"



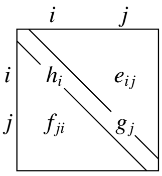
Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\nabla, b, \delta)$:



$b(\nabla) = b\colon \nabla \otimes \nabla \to \nabla$

$b(\triangle) \rightsquigarrow \delta\colon \nabla \to \nabla \otimes \nabla$

Now define $gl_n^\epsilon := \mathcal{D}(\nabla, b, \epsilon\delta)$. Schematically, this is $[\nabla, \nabla] = \nabla$, $[\triangle, \triangle] = \epsilon\triangle$, and $[\nabla, \triangle] = \triangle + \epsilon\nabla$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk}f_{il} - \epsilon\delta_{li}f_{kj}$$
$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k}e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l}f_{il}) - \delta_{li}(\epsilon\delta_{k<j}e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j}f_{kj})$$
$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik})e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})e_{jk}$$
$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik})f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik})f_{jk}$$

Solvable Approximation (2). At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n. By rescaling at $\epsilon \neq 0$, gl_n^ϵ is independent of ϵ. We let gl_n^k be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of gl_n!

Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, …, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ∇ algebra.

Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

Definition. A "docile perturbed Gaussian" in the variables $(x_i)_{i\in S}$ over the ring R is an expression of the form

docile
/ˈdäsəl/
adjective
ready to accept control or instruction; submissive.
"a cheap and docile workforce"

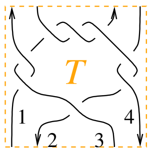
$$\mathbb{e}^{q^{ij}x_ix_j}P = \mathbb{e}^{q^{ij}x_ix_j}\left(\sum_{k\geq 0}\epsilon^k P_k\right),$$

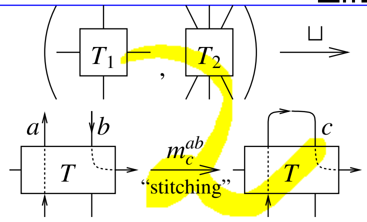
x → z in this box

where all coefficients are in R and where P is a "docile series": $\deg P_k \leq 2k$.

They matter

(v-)Tangles.

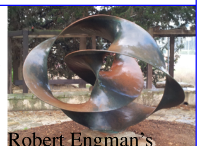




(meta-associativity: $m^{ab}_x /\!/ m^{xc}_y = m^{bc}_x /\!/ m^{ax}_y$)
(tangles are generated by $\nearrow\!\!\!\!\nwarrow$ and $\nwarrow\!\!\!\!\nearrow$)

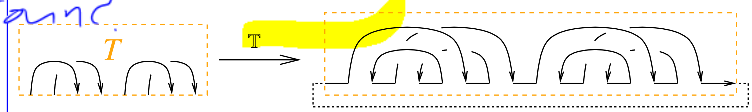
Strand doubling: $\xrightarrow{a} \;\; \xrightarrow{\Delta^a_{bc}} \;\; \xrightarrow{b} \xrightarrow{c}$ Strand reversal: $\xrightarrow{a} \;\; \xrightarrow{S_a} \;\; \xleftarrow{a}$

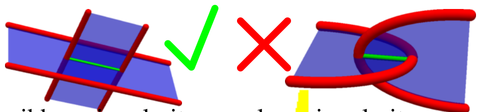
Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.



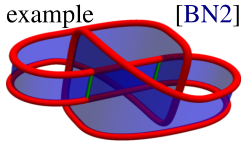
Robert Engman's "The Loop"

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via





a ribbon singularity a clasp singularity

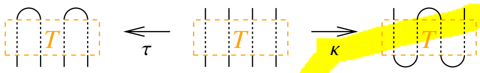


example [BN2]

A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

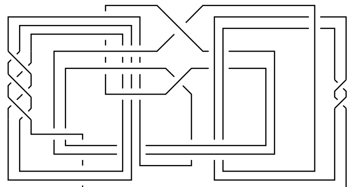


Gompf, Scharlemann, Thompson [GST]

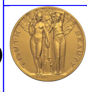
$$\begin{array}{ccccc} & U \in \mathcal{T}_n & & 1 \in \mathcal{A}_n & \\ & \tau\nearrow & & \tau\nearrow & \\ \mathcal{T}_{2n} & & \xrightarrow{z} & \mathcal{A}_{2n} & \\ & \kappa\searrow & & \kappa\searrow & \end{array}$$

with $\mathcal{R} := \kappa(\tau^{-1}(1))$

ribbon $K \in \mathcal{T}_1$ $\quad z(K) \in \mathcal{R} \subseteq \mathcal{A}_1$



Faster is better, leaner is meaner!



The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a\colon a \in S\})$:

$$\left(\nearrow\!\!\!\!\nwarrow_{a\,b},\ \nwarrow\!\!\!\!\nearrow_{b\,a}\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1 - t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi\end{array} \xrightarrow[t_a,\, t_b \to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta}\end{array}\right)$$

(Roland: "add to A the product of column b and row a, divide by $(1 - A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

$$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi\end{array} \xrightarrow[\substack{\mu := T_a - 1 \\ \nu := \alpha - \sigma_a \\ T_a \mapsto T_bT_c}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b - 1)T_c\nu/\mu & (T_b - 1)T_c\theta/\mu \\ c & (T_c - 1)\nu/\mu & (\alpha - \sigma_a T_a - \nu T_c)/\mu & (T_c - 1)\theta/\mu \\ S & \phi & \phi & \Xi\end{array}\right)$$

$$\Big\downarrow dS^a \,\Big|_{T_a \to T_a^{-1}}$$

$$\left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha\end{array}\right)$$

Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left(\nearrow\!\!\!\!\nwarrow_{a\,b},\ \nwarrow\!\!\!\!\nearrow_{b\,a}\right) = (a \to 1, b \to t_a^{\pm 1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma /\!/ m^{ab}_c = (\sigma \setminus \{a, b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a, t_b \to t_c}$.

[Vo]: A proof of the Fox-Milnor theorem using this technology (and more).

Implementation key idea: ωεβ/AlexDemo

$$(\omega, A=(\alpha_{ab})) \leftrightarrow (\omega, \lambda=\sum \alpha_{ab}t_ah_b)$$

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_ b_ → c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂_{t_a,h_a} λ  ∂_{t_a,h_b} λ  ∂_{t_a} λ;
                           ∂_{t_b,h_a} λ  ∂_{t_b,h_b} λ  ∂_{t_b} λ;
                           ∂_{h_a} λ      ∂_{h_b} λ      λ) /. (t | h)_{a|b} → 0;
  Γ[(μ = 1 - β) ω, {t_c, 1}.(γ + α δ / μ   ϵ + δ θ / μ; ϕ + α ψ / μ   Ξ + ψ θ / μ).{h_c, 1}]
    /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_ b_} := Γ[1, {t_a, t_b}.(1  1 - T_a; 0  T_a).{h_a, h_b}];
Rm_{a_ b_} := Rp_{ab} /. T_a → 1 / T_a;
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ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_{a_} ⧴ a, ∞];
   M = Outer[Factor[∂_{h_{#1} t_{#2}} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
```

Runs.

Meta-Associativity

$$\zeta = \Gamma\left[\omega,\ \{t_1, t_2, t_3, t_s\}.\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}.\{h_1, h_2, h_3, h_s\}\right];$$

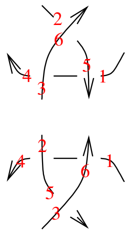
$(\zeta \mathbin{//} m_{12\to1} \mathbin{//} m_{13\to1}) == (\zeta \mathbin{//} m_{23\to2} \mathbin{//} m_{12\to1})$

True

R3 ... divide and conquer!

$\{Rm_{51}\, Rm_{62}\, Rp_{34} \mathbin{//} m_{14\to1} \mathbin{//} m_{25\to2} \mathbin{//} m_{36\to3},$
$Rp_{61}\, Rm_{24}\, Rm_{35} \mathbin{//} m_{14\to1} \mathbin{//} m_{25\to2} \mathbin{//} m_{36\to3}\}$

$$\left\{\begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}, \begin{pmatrix}1 & h_1 & h_2 & h_3\\ t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1\end{pmatrix}\right\}$$

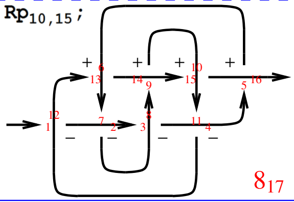


$z = Rm_{12,1}\, Rm_{27}\, Rm_{83}\, Rm_{4,11}\, Rp_{16,5}\, Rp_{6,13}\, Rp_{14,9}\, Rp_{10,15};$

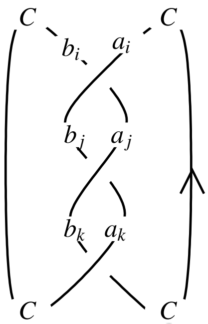
$\text{Do}[z = z \mathbin{//} m_{1k\to1},\ \{k, 2, 16\}];$

z

$$\begin{pmatrix}11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8T_1 + 4T_1^2 - T_1^3 & h_1\\ t_1 & 1\end{pmatrix}$$



8_{17}



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

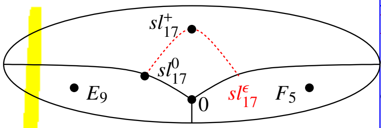
form

$$Z = \sum_{i,j,k} Ca_ib_ja_kC^2b_ia_jb_kC.$$

Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon/(\epsilon^{k+1}=0)$.

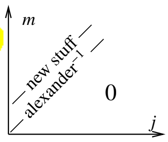


Melvin, Morton, Garoufalidis

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot\omega(K)(e^\hbar) = 1$.



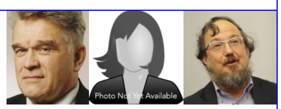
"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q-q^{-1})\omega(K)(q^d)}\left(1 + \sum_{k=1}^\infty \frac{(q-1)^k\rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$

Prior art. Some amazing computations by Rozansky and Overbay in [Ro2, Ro3] and in [Ov].

Rozansky, Overbay

Faddeev's Formula (In as much as we can tell, first appeared w/o proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n-1}{q-1}$, with $[n]_q! := [1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x := \sum_{n\geq0}\frac{x^n}{[n]_q!}$, we have



Faddeev Quesne Zagier

$$\log \mathbb{e}_q^x = \sum_{k\geq1}\frac{(1-q)^kx^k}{k(1-q^k)} = x + \frac{(1-q)^2x^2}{2(1-q^2)} + \dots.$$

Proof. We have that $\mathbb{e}_q^x = \frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx} = (1+(1-q)x)\mathbb{e}_q^x$, and

$$\log \mathbb{e}_q^{qx} = \log(1+(1-q)x) + \log \mathbb{e}_q^x.$$

Writing $\log \mathbb{e}_q^x = \sum_{k\geq1}a_kx^k$ and comparing powers of x, we get $q^ka_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

Propaganda

Balance columns & compress.

$\mathcal{GDO}$-Categories. Given $\mathfrak{g}$ with basis $B = \{x, y, \ldots\}$, consider the following diagram:

$$\begin{tikzcd}
\mathbb{Q} = \hat{\mathcal{U}}_{(q)}\left(\bigoplus_0 \mathfrak{g}\right) \arrow[r, "Z"] & \hat{\mathcal{U}}_{(q)}(\mathfrak{g}) \arrow[r, bend left, "\Delta"] & \hat{\mathcal{U}}_{(q)}\left(\bigoplus_2 \mathfrak{g}\right) \arrow[l, bend left, "m"] \\
\hat{\mathcal{S}}(\emptyset) \arrow[u] \arrow[r, "Z"] & \hat{\mathcal{S}}(B) \arrow[u, bend left, "\mathbb{O}(xy\ldots:\,\cdot)"] \arrow[u, bend right, "\mathbb{O}(yx\ldots:\,\cdot)"'] \arrow[r, bend left, "\Delta"] \arrow[loop below, "SW_{xy}"] & \hat{\mathcal{S}}(B_1, B_2) \arrow[l, bend left, "m"] \arrow[u, "\mathbb{O}(y_1x_1\ldots\otimes y_2x_2\ldots:\,\cdot)"']
\end{tikzcd}$$

Hence Z, SW_{xy}, m, Δ, (and likewise S and θ) are morphisms in the *completion* of the monoidal category $\mathcal{F}$ whose objects are finite sets B and whose morphism are $\mathrm{mor}_{\mathcal{F}}(B, B') := \mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B) \to \mathcal{S}(B')) = \mathcal{S}(B^*, B')$ (by convention, $x^* = \xi$, $y^* = \eta$, etc.). Ergo we need to *consolidate* (at least parts of) said completion.

Aside. "Consolidate" means "give a finite name to an infinite object, and figure out how to sufficiently manipulate such finite names". E.g., solving $f'' = -f$ we encounter and set $\sum \frac{(-1)^k x^{2k}}{(2k)!} \rightsquigarrow \cos x$, $\sum \frac{(-1)^k x^{2k+1}}{(2k+1)!} \rightsquigarrow \sin x$, and then $\cos^2 x + \sin^2 x = 1$ and $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

The Composition Law. If

$$\mathcal{S}(B_0) \xrightarrow[{}^t f \in \mathbb{Q}[\![\zeta_{0i}, z_{1j}]\!]]{f} \mathcal{S}(B_1) \xrightarrow[{}^t g \in \mathbb{Q}[\![\zeta_{1j}, z_{2k}]\!]]{g} \mathcal{S}(B_2)$$

then ${}^t(f /\!/ g) = {}^t(g \circ f) = \left(g|_{\zeta_{1j} \to \partial_{z_{1j}}} f \right)_{z_{1j} = 0}$.

Examples.

1. The 1-variable identity map $I \colon \mathcal{S}(z) \to \mathcal{S}(z)$ is given by ${}^t I_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by ${}^t I_n = \mathbb{e}^{z_1\zeta_1 + \cdots + z_n\zeta_n}$.
2. The "$z_i \to z_j$ variable rename map $\sigma^i_j \colon \mathcal{S}(z_i) \to \mathcal{S}(z_j)$ becomes ${}^t\sigma^i_j = \mathbb{e}^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m^{ij}_k \colon \mathcal{S}(z_i, z_j) \to \mathcal{S}(z_k)$" has ${}^t m = \mathbb{e}^{z_k(\zeta_i + \zeta_j)}$.
4. The "archetypal coproduct $\delta^i_{jk} \colon \mathcal{S}(z_i) \to \mathcal{S}(z_j, z_k)$", given by $z_i \to z_j + z_k$ or $\Delta z = z \otimes 1 + 1 \otimes z$, has ${}^t\Delta = \mathbb{e}^{(z_j + z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1 x_2} \in \mathcal{U}_q \otimes \mathcal{U}_q$. The "baby R-matrix" is ${}^t R = \mathbb{e}^{\hbar yx} \in \mathcal{S}(y, x)$.

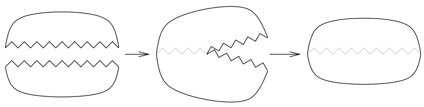
Proposition. If $F \colon \mathcal{S}(B) \to \mathcal{S}(B')$ is linear and "continuous", then ${}^t F = \exp\left(\sum_{z_i \in B} \zeta_i z_i\right) /\!/ F$.

The Heisenberg Example. The "Weyl form of the canonical commutation relations" states that if $[y, x] = t$ and t is central, then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. Thus with

$$\begin{tikzcd}
\mathcal{S}(t, y, x) \arrow[loop left, "SW_{xy}"] \arrow[r, shift left, "\mathbb{O}_{xy}"] \arrow[r, shift right, "\mathbb{O}_{yx}"'] & \mathcal{U}(t, u, x)
\end{tikzcd}$$

we have ${}^t SW_{xy} = \mathbb{e}^{\tau t + \eta y + \xi x - \eta\xi t}$.

The Zipping Issue (between unbound and bound lies half-zipped).



Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwsie convergent, set

$$\left\langle P(\zeta^j, z_i) \right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i = 0}.$$

(E.g., if $P = \sum a_{nm}\zeta^n z^m$ then $\langle P \rangle_\zeta = \sum n! a_{nn}$).

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{\eta^i z_i + y_j \zeta^j} \right\rangle_{(\zeta^j)} = \left\langle P(z_i + y_i, \zeta^j)\mathbb{e}^{\eta^i(z_i + y_i)} \right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{c + \eta^i z_i + y_j\zeta^j + q^i_j z_i \zeta^j} \right\rangle_{(\zeta^j)} = \det(\tilde{q}) \left\langle P(\tilde{q}^k_i(z_k + y_k), \zeta^j)\mathbb{e}^{c + \eta^i \tilde{q}^k_i(z_k + y_k)} \right\rangle_{(\zeta^j)}$$

where $\tilde{q}$ is the inverse matrix of $1 - q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar = 0$ and at $\partial_\hbar$).

Implementation. ωεβ/ZipBindDemo

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
{z*, x*, y*} = {ζ, ξ, η}; {ζ*, ξ*, η*} = {z, x, y};
(u_i_)* := (u*)_i;
Zip_{}[P_] := P;
Zip_{ζ_,ζs___}[P_] :=
  (Expand[P // Zip_{ζs}] /. f_. ζ^d_. :> ∂_{ζ*,d} f) /. ζ* → 0
Zip_{ζ}[(a ζ^6 + ζ + 3) (z^5 e^z + 7 z) + 99 b]
7 + 720 a + 99 b
Zip_{ξ,η}[ξ^3 η^3 e^(a x + b y + c x y)]
a^3 b^3 + 9 a^2 b^2 c + 18 a b c^2 + 6 c^3
(* 𝔼[Q,P] means e^Q P *)
𝔼 /: Zip_{ζs_List}@𝔼[Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule;
   Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[Q2, Det[qt] e^-Q2 Zip_{ζs}[e^Q1 (P /. zrule)]]];
Eh = 𝔼[h Σ_{i=1}^{3} Σ_{j=1}^{3} a_{10 i+j} x_i ξ_j, Σ_{i=1}^{3} f_i[x_1, x_2, x_3] ξ_i];
E1 = Eh /. h → 1
𝔼[a_11 x_1 ξ_1 + a_21 x_2 ξ_1 + a_31 x_3 ξ_1 + a_12 x_1 ξ_2 +
  a_22 x_2 ξ_2 + a_32 x_3 ξ_2 + a_13 x_1 ξ_3 + a_23 x_2 ξ_3 + a_33 x_3 ξ_3,
 ξ_1 f_1[x_1, x_2, x_3] + ξ_2 f_2[x_1, x_2, x_3] + ξ_3 f_3[x_1, x_2, x_3]]
Short[lhs = Zip_{ξ_1,ξ_2}@E1, 5]
𝔼[((a_13 ((-1 + a_22) a_31 - a_21 a_32) + a_12 (-a_23 a_31 + a_21 a_33) +
      (-1 + a_11) (a_23 a_32 - (-1 + a_22) a_33)) x_3 ξ_3) /
   (-1 + a_12 a_21 - a_11 (-1 + a_22) + a_22),
  (<<17>> + a_21 <<1>>) / (-1 + a_12 a_21 - a_11 (-1 + a_22) + a_22)^2]
lhs == Zip_{ξ_1}@Zip_{ξ_2}@E1 == Zip_{ξ_2}@Zip_{ξ_1}@E1
True
Short[
 lhs = Normal[Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]] //
   Zip_{ξ_1,ξ_2}, 5]
h a_13 ξ_3 f_1[0, 0, x_3] + 2 h^2 a_11 a_13 ξ_3 f_1[0, 0, x_3] +
 3 h^3 a_11^2 a_13 ξ_3 f_1[0, 0, x_3] + 2 h^3 a_12 a_13 a_21 ξ_3 f_1[0, 0, x_3] +
 h^2 a_13 a_22 ξ_3 f_1[0, 0, x_3] + <<337>> +
 1/6 h^3 a_31^3 x_3^3 ξ_3 f_3^(3,0,0)[0, 0, x_3] + 1/2 h^3 a_31^2 a_32 x_3^3 f_1^(3,1,0)[0, 0, x_3] +
 1/6 h^3 a_31^3 x_3^3 f_2^(3,1,0)[0, 0, x_3] + 1/6 h^3 a_31^3 x_3^3 f_1^(4,0,0)[0, 0, x_3]
rhs =
  Normal[Zip_{ξ_1,ξ_2}@Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]];
Simplify[lhs == rhs]
True
```

```
𝔼 /: 𝔼[Q1_, P1_] 𝔼[Q2_, P2_] := 𝔼[Q1 + Q2, P1 * P2];
Bind_{ζs_List}[L_𝔼, R_𝔼] := Module[{n, hideζs, hidezs},
   hideζs = Table[ζs⟦i⟧ → ζ_{n@i}, {i, Length@ζs}];
   hidezs = Table[ζs⟦i⟧* → z_{n@i}, {i, Length@ζs}];
   Zip_{ζs/.hideζs}[(L /. hidezs) (R /. hideζs)] ];
Bind_{ξ2}[𝔼[ξ (x1 + x2), 1], 𝔼[ξ2 (x2 + x3), 1]]
𝔼[ξ (x1 + x2 + x3), 1]
Bind_{ξ2}[𝔼[(ξ2 + ξ3) x2, 1], 𝔼[(ξ1 + ξ2) x, 1]]
𝔼[x (ξ1 + ξ2 + ξ3), 1]
```

The 2D Lie Algebra. Clever people know[1] that if $[a, x] = \gamma x$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$. Ergo with

$$SW_{ax} \circlearrowright \mathcal{S}(a,x) \underset{\mathbb{O}_{xa}}{\overset{\mathbb{O}_{ax}}{\rightrightarrows}} \mathcal{U}(a,x)$$

we have ${}^tSW_{ax} = \mathbb{e}^{\alpha a + \mathbb{e}^{-\gamma\alpha}\xi x}$.

The Real Thing. In $QU/(\epsilon^2 = 0)$ over $\mathbb{Q}[\![\hbar]\!]$ using the *yax* order, $T = \mathbb{e}^{\hbar t}$, $\bar{T} = T^{-1}$, $\mathcal{A} = \mathbb{e}^{\gamma\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

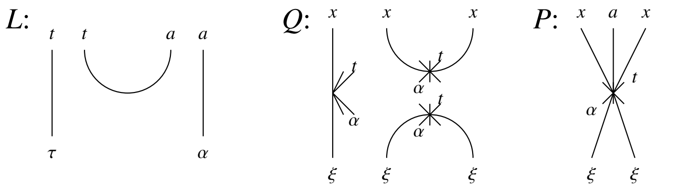
$${}^tR_{ij} = \mathbb{e}^{\hbar(y_i x_j - t_i a_j/\gamma)}\left(1 + \epsilon\hbar\left(a_i a_j/\gamma - \gamma\hbar^2 y_i^2 x_j^2/4\right)\right)$$

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$${}^tm = \mathbb{e}^{(\alpha_1+\alpha_2)a + \eta_2\xi_1(1-T)/\hbar + (\xi_1\bar{\mathcal{A}}_2+\xi_2)x + (\eta_1+\eta_2\bar{\mathcal{A}}_1)y}\,(1+\epsilon\lambda_m),$$

where $\lambda_m = 2a\eta_2\xi_1 T + \frac{1}{4}\gamma\eta_2^2\xi_1^2\left(3T^2-4T+1\right)/\hbar - \frac{1}{2}\gamma\eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2 - \frac{1}{2}\gamma\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1 + \gamma\eta_2\xi_1 xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$. Similar formulas delight us for ${}^t\Delta$ and ${}^tS$.

A generic morphism.



Implementation.

ωεβ/SL2Portfolio

```
QZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   simp /@ 𝔼[L, Q2, Det[qt] e^(-Q2) Zip_ζs[e^Q1 (P /. zrule)]] ];
QZip_{ζs_List} := QZip_{ζs,CF};
```

```
LZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. T2t /. zrule) /. Alternatives @@ zs → 0;
   simp /@
    𝔼[L2, Q2, Det[lt] e^(-L2-Q2)
     Zip_ζs[e^(L1+Q1) (P /. T2t /. zrule)]] //. t2T ];
LZip_{ζs_List} := LZip_{ζs,CF};
```

```
Bind_{}[L_, R_] := L R;
Bind_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : T | t | a | x | y)_i → v_{n@i}, {i, {is}}],
     R /. Table[(v : τ | α | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Flatten@Table[{τ_{n@i},a_{n@i}},{i,{is}}]} //
   QZip_{Flatten@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{l_List} := Bind_l; B_{is___} := Bind_{is};
Bind[ℰ_𝔼] := ℰ;
Bind[Ls__, ζs_List, R_] := Bind_ζs[Bind[Ls], R];
```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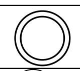
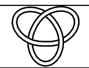
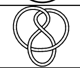
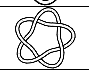
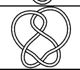
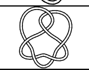
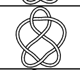
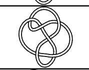
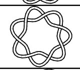
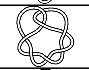
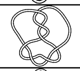
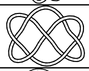
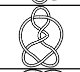
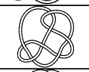
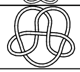
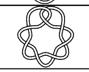
Include a full implementation

diagram	n^t_k Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral	diagram	n^t_k Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon unknotting number / amphicheiral
	0^a_1 0	1	0 / ✔ 0 / ✔		3^a_1 t	$t-1$	1 / ✘ 1 / ✘
	4^a_1 0	$3-t$	1 / ✘ 1 / ✔		5^a_1 $2t^3+3t$	t^2-t+1	2 / ✘ 2 / ✘
	5^a_2 $5t-4$	$2t-3$	1 / ✘ 1 / ✘		6^a_1 $t-4$	$5-2t$	1 / ✔ 1 / ✘
	6^a_2 t^3-4t^2+4t-4	$-t^2+3t-3$	2 / ✘ 1 / ✘		6^a_3 0	t^2-3t+5	2 / ✘ 1 / ✔
	7^a_1 $3t^5+5t^3+6t$	t^3-t^2+t-1	3 / ✘ 3 / ✘		7^a_2 $14t-16$	$3t-5$	1 / ✘ 1 / ✘
	7^a_3 $-9t^3+8t^2-16t+12$	$2t^2-3t+3$	2 / ✘ 2 / ✘		7^a_4 $32-24t$	$4t-7$	1 / ✘ 2 / ✘
	7^a_5 $9t^3-16t^2+29t-28$	$2t^2-4t+5$	2 / ✘ 2 / ✘		7^a_6 $t^3-8t^2+19t-20$	$-t^2+5t-7$	2 / ✘ 1 / ✘
	7^a_7 $8-3t$	t^2-5t+9	2 / ✘ 1 / ✘		8^a_1 $5t-16$	$7-3t$	1 / ✘ 1 / ✘

[1] Indeed $xa = (a-\gamma)x$ thus $xa^n = (a-\gamma)^n x$ thus $x\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha(a-\gamma)}x = \mathbb{e}^{-\gamma\alpha}\mathbb{e}^{\alpha a}x$ thus $x^n\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}(\mathbb{e}^{-\gamma\alpha})^n x^n$ thus $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$.

diagram	n^t_k	Alexander's ω^+	Today's / Rozansky's ρ^+_1	genus / ribbon	unknotting number / amphicheiral
	8^a_2	$-t^3+3t^2-3t+3$	$2t^5-8t^4+10t^3-12t^2+13t-12$	3 / ✘	2 / ✘
	8^a_3	$9-4t$	0	1 / ✘	2 / ✔
	8^a_4	$-2t^2+5t-5$	$3t^3-8t^2+6t-4$	2 / ✘	2 / ✘
	8^a_5	$-t^3+3t^2-4t+5$	$-2t^5+8t^4-13t^3+20t^2-22t+24$	3 / ✘	2 / ✘
	8^a_6	$-2t^2+6t-7$	$5t^3-20t^2+28t-32$	2 / ✘	2 / ✘
	8^a_7	t^3-3t^2+5t-5	$-t^5+4t^4-10t^3+12t^2-13t+12$	3 / ✘	1 / ✘
	8^a_8	$2t^2-6t+9$	$-t^3+4t^2-12t+16$	2 / ✔	2 / ✘
	8^a_9	$-t^3+3t^2-5t+7$	0	3 / ✔	1 / ✔
	8^a_{10}	t^3-3t^2+6t-7	$-t^5+4t^4-11t^3+16t^2-21t+20$	3 / ✘	2 / ✘
	8^a_{11}	$-2t^2+7t-9$	$5t^3-24t^2+39t-44$	2 / ✘	1 / ✘
	8^a_{12}	$t^2-7t+13$	0	2 / ✘	2 / ✔
	8^a_{13}	$2t^2-7t+11$	$-t^3+4t^2-14t+20$	2 / ✘	1 / ✘
	8^a_{14}	$-2t^2+8t-11$	$5t^3-28t^2+57t-68$	2 / ✘	1 / ✘
	8^a_{15}	$3t^2-8t+11$	$21t^3-64t^2+120t-140$	2 / ✘	2 / ✘
	8^a_{16}	t^3-4t^2+8t-9	$t^5-6t^4+17t^3-28t^2+35t-36$	3 / ✘	2 / ✘
	8^a_{17}	$-t^3+4t^2-8t+11$	0	3 / ✘	1 / ✔
	8^a_{18}	$-t^3+5t^2-10t+13$	0	3 / ✘	2 / ✔
	8^n_{19}	t^3-t^2+1	$-3t^5-4t^2-3t$	3 / ✘	3 / ✘
	8^n_{20}	t^2-2t+3	$4t-4$	2 / ✔	1 / ✘
	8^n_{21}	$-t^2+4t-5$	$t^3-8t^2+16t-20$	2 / ✘	1 / ✘
	9^a_1	$t^4-t^3+t^2-t+1$	$4t^7+7t^5+9t^3+10t$	4 / ✘	4 / ✘
	9^a_2	$4t-7$	$30t-40$	1 / ✘	1 / ✘
	9^a_3	$2t^3-3t^2+3t-3$	$-13t^5+12t^4-25t^3+20t^2-32t+24$	3 / ✘	3 / ✘
	9^a_4	$3t^2-5t+5$	$23t^3-28t^2+46t-44$	2 / ✘	2 / ✘
	9^a_5	$6t-11$	$100-65t$	1 / ✘	2 / ✘
	9^a_6	$2t^3-4t^2+5t-5$	$13t^5-24t^4+45t^3-52t^2+68t-64$	3 / ✘	3 / ✘
	9^a_7	$3t^2-7t+9$	$23t^3-56t^2+99t-108$	2 / ✘	2 / ✘
	9^a_8	$-2t^2+8t-11$	$3t^3-16t^2+29t-28$	2 / ✘	2 / ✘
	9^a_9	$2t^3-4t^2+6t-7$	$13t^5-24t^4+55t^3-72t^2+98t-96$	3 / ✘	3 / ✘
	9^a_{10}	$4t^2-8t+9$	$-40t^3+72t^2-114t+120$	2 / ✘	2, 3 / ✘
	9^a_{11}	$-t^3+5t^2-7t+7$	$-2t^5+16t^4-41t^3+52t^2-66t+64$	3 / ✘	2 / ✘
	9^a_{12}	$-2t^2+9t-13$	$5t^3-36t^2+84t-100$	2 / ✘	1 / ✘
	9^a_{13}	$4t^2-9t+11$	$-40t^3+92t^2-154t+168$	2 / ✘	2, 3 / ✘
	9^a_{14}	$2t^2-9t+15$	$-t^3+8t^2-35t+60$	2 / ✘	1 / ✘
	9^a_{15}	$-2t^2+10t-15$	$-5t^3+40t^2-108t+136$	2 / ✘	2 / ✘
	9^a_{16}	$2t^3-5t^2+8t-9$	$-13t^5+36t^4-80t^3+120t^2-161t+168$	3 / ✘	3 / ✘
	9^a_{17}	t^3-5t^2+9t-9	$t^5-8t^4+23t^3-32t^2+28t-24$	3 / ✘	2 / ✘
	9^a_{18}	$4t^2-10t+13$	$40t^3-108t^2+193t-220$	2 / ✘	2 / ✘
	9^a_{19}	$2t^2-10t+17$	$t^3-8t^2+20t-24$	2 / ✘	1 / ✘
	9^a_{20}	$-t^3+5t^2-9t+11$	$2t^5-16t^4+47t^3-84t^2+117t-124$	3 / ✘	2 / ✘
	9^a_{21}	$-2t^2+11t-17$	$-5t^3+44t^2-127t+164$	2 / ✘	1 / ✘
	9^a_{22}	$t^3-5t^2+10t-11$	$-t^5+8t^4-24t^3+38t^2-40t+36$	3 / ✘	1 / ✘
	9^a_{23}	$4t^2-11t+15$	$40t^3-128t^2+243t-288$	2 / ✘	2 / ✘
	9^a_{24}	$-t^3+5t^2-10t+13$	$-4t^2+16t-20$	3 / ✘	1 / ✘
	9^a_{25}	$-3t^2+12t-17$	$12t^3-70t^2+153t-188$	2 / ✘	2 / ✘
	9^a_{26}	$t^3-5t^2+11t-13$	$-t^5+8t^4-31t^3+64t^2-85t+92$	3 / ✘	1 / ✘
	9^a_{27}	$-t^3+5t^2-11t+15$	$t^3-8t^2+24t-32$	3 / ✔	1 / ✘
	9^a_{28}	$t^3-5t^2+12t-15$	$t^5-8t^4+30t^3-68t^2+105t-120$	3 / ✘	1 / ✘
	9^a_{29}	$t^3-5t^2+12t-15$	$t^5-8t^4+26t^3-48t^2+59t-56$	3 / ✘	2 / ✘
	9^a_{30}	$-t^3+5t^2-12t+17$	$2t^3-10t^2+25t-32$	3 / ✘	1 / ✘
	9^a_{31}	$t^3-5t^2+13t-17$	$t^5-8t^4+33t^3-80t^2+132t-152$	3 / ✘	2 / ✘
	9^a_{32}	$t^3-6t^2+14t-17$	$-t^5+10t^4-42t^3+94t^2-133t+148$	3 / ✘	2 / ✘
	9^a_{33}	$-t^3+6t^2-14t+19$	$t^3-10t^2+30t-40$	3 / ✘	1 / ✘
	9^a_{34}	$-t^3+6t^2-16t+23$	$3t^3-18t^2+43t-56$	3 / ✘	1 / ✘
	9^a_{35}	$7t-13$	$90t-144$	1 / ✘	2, 3 / ✘
	9^a_{36}	$-t^3+5t^2-8t+9$	$-2t^5+16t^4-44t^3+66t^2-87t+88$	3 / ✘	2 / ✘
	9^a_{37}	$2t^2-11t+19$	$t^3-8t^2+22t-28$	2 / ✘	2 / ✘
	9^a_{38}	$5t^2-14t+19$	$62t^3-204t^2+382t-452$	2 / ✘	2, 3 / ✘
	9^a_{39}	$-3t^2+14t-21$	$-12t^3+84t^2-210t+268$	2 / ✘	1 / ✘
	9^a_{40}	$t^3-7t^2+18t-23$	$t^5-12t^4+57t^3-144t^2+229t-264$	3 / ✘	2 / ✘
	9^a_{41}	$3t^2-12t+19$	$3t^3-20t^2+70t-108$	2 / ✔	2 / ✘
	9^n_{42}	$-t^2+2t-1$	$-t^3+2t^2+t-4$	2 / ✘	1 / ✘
	9^n_{43}	$-t^3+3t^2-2t+1$	$-2t^5+8t^4-7t^3+2t^2-5t+4$	3 / ✘	2 / ✘
	9^n_{44}	t^2-4t+7	$-2t^2+9t-12$	2 / ✘	1 / ✘

diagram	n^t_k	Alexander's ω^+	Today's / Rozansky's ρ_1^+	genus / ribbon	unknotting number / amphicheiral
	9^n_{45}	$-t^2+6t-9$	$t^3-14t^2+47t-60$	2 / ✘	1 / ✘
	9^n_{46}	$5-2t$	$3t-12$	1 / ✔	2 / ✘
	9^n_{47}	t^3-4t^2+6t-5	$-t^5+6t^4-15t^3+16t^2-10t+12$	3 / ✘	2 / ✘
	9^n_{48}	$-t^2+7t-11$	$-t^3+12t^2-42t+52$	2 / ✘	2 / ✘
	9^n_{49}	$3t^2-6t+7$	$-21t^3+38t^2-61t+60$	2 / ✘	3 / ✘
	10^a_{1}	$9-4t$	$14t-40$	1 / ✘	1 / ✘
	10^a_{2}	$-t^4+3t^3-3t^2+3t-3$	$3t^7-12t^6+16t^5-20t^4+24t^3-24t^2+27t-24$	4 / ✘	3 / ✘
	10^a_{3}	$13-6t$	$11t-28$	1 / ✔	2 / ✘
	10^a_{4}	$-3t^2+7t-7$	$4t^3-8t^2+t+8$	2 / ✘	2 / ✘
	10^a_{5}	$t^4-3t^3+5t^2-5t+5$	$-2t^7+8t^6-20t^5+28t^4-36t^3+36t^2-39t+36$	4 / ✘	2 / ✘
	10^a_{6}	$-2t^3+6t^2-7t+7$	$9t^5-36t^4+56t^3-72t^2+81t-84$	3 / ✘	3 / ✘
	10^a_{7}	$-3t^2+11t-15$	$14t^3-72t^2+135t-160$	2 / ✘	1 / ✘
	10^a_{8}	$-2t^3+5t^2-5t+5$	$7t^5-20t^4+23t^3-28t^2+26t-24$	3 / ✘	2 / ✘
	10^a_{9}	$-t^4+3t^3-5t^2+7t-7$	$-t^7+4t^6-10t^5+20t^4-25t^3+28t^2-28t+28$	4 / ✘	1 / ✘
	10^a_{10}	$3t^2-11t+17$	$-5t^3+24t^2-71t+100$	2 / ✘	1 / ✘
	10^a_{11}	$-4t^2+11t-13$	$16t^3-52t^2+68t-72$	2 / ✘	2, 3 / ✘
	10^a_{12}	$2t^3-6t^2+10t-11$	$-5t^5+20t^4-50t^3+72t^2-89t+92$	3 / ✘	2 / ✘
	10^a_{13}	$2t^2-13t+23$	$t^3-12t^2+51t-84$	2 / ✘	2 / ✘
	10^a_{14}	$-2t^3+8t^2-12t+13$	$9t^5-52t^4+119t^3-180t^2+225t-236$	3 / ✘	2 / ✘
	10^a_{15}	$2t^3-6t^2+9t-9$	$-3t^5+12t^4-24t^3+24t^2-17t+12$	3 / ✘	2 / ✘
	10^a_{16}	$-4t^2+12t-15$	$-16t^3+56t^2-76t+80$	2 / ✘	2 / ✘
	10^a_{17}	$t^4-3t^3+5t^2-7t+9$	0	4 / ✘	1 / ✔
	10^a_{18}	$-4t^2+14t-19$	$16t^3-68t^2+121t-140$	2 / ✘	1 / ✘
	10^a_{19}	$2t^3-7t^2+11t-11$	$3t^5-16t^4+35t^3-40t^2+30t-24$	3 / ✘	2 / ✘
	10^a_{20}	$-3t^2+9t-11$	$14t^3-56t^2+88t-104$	2 / ✘	2 / ✘
	10^a_{21}	$-2t^3+7t^2-9t+9$	$9t^5-44t^4+80t^3-104t^2+121t-124$	3 / ✘	2 / ✘
	10^a_{22}	$-2t^3+6t^2-10t+13$	$-t^5+4t^4-10t^3+24t^2-37t+44$	3 / ✔	2 / ✘
	10^a_{23}	$2t^3-7t^2+13t-15$	$-5t^5+24t^4-67t^3+108t^2-137t+144$	3 / ✘	1 / ✘
	10^a_{24}	$-4t^2+14t-19$	$24t^3-116t^2+221t-268$	2 / ✘	2 / ✘
	10^a_{25}	$-2t^3+8t^2-14t+17$	$9t^5-52t^4+131t^3-232t^2+314t-344$	3 / ✘	2 / ✘
	10^a_{26}	$-2t^3+7t^2-13t+17$	$-t^5+4t^4-10t^3+28t^2-49t+60$	3 / ✘	1 / ✘
	10^a_{27}	$2t^3-8t^2+16t-19$	$5t^5-28t^4+87t^3-164t^2+229t-252$	3 / ✘	1 / ✘
	10^a_{28}	$4t^2-13t+19$	$-8t^3+36t^2-100t+136$	2 / ✘	2 / ✘
	10^a_{29}	$t^3-7t^2+15t-17$	$t^5-12t^4+52t^3-104t^2+124t-128$	3 / ✘	2 / ✘
	10^a_{30}	$-4t^2+17t-25$	$24t^3-148t^2+345t-440$	2 / ✘	1 / ✘
	10^a_{31}	$4t^2-14t+21$	$-4t^2+9t-12$	2 / ✘	1 / ✘
	10^a_{32}	$-2t^3+8t^2-15t+19$	$t^5-4t^4+13t^3-40t^2+78t-96$	3 / ✘	1 / ✘
	10^a_{33}	$4t^2-16t+25$	0	2 / ✘	1 / ✔
	10^a_{34}	$3t^2-9t+13$	$-5t^3+20t^2-52t+68$	2 / ✘	2 / ✘
	10^a_{35}	$2t^2-12t+21$	$-t^3+12t^2-47t+76$	2 / ✔	2 / ✘
	10^a_{36}	$-3t^2+13t-19$	$14t^3-88t^2+208t-264$	2 / ✘	2 / ✘
	10^a_{37}	$4t^2-13t+19$	0	2 / ✘	2 / ✔
	10^a_{38}	$-4t^2+15t-21$	$24t^3-128t^2+270t-336$	2 / ✘	2 / ✘
	10^a_{39}	$-2t^3+8t^2-13t+15$	$9t^5-52t^4+125t^3-204t^2+263t-280$	3 / ✘	2 / ✘
	10^a_{40}	$2t^3-8t^2+17t-21$	$-5t^5+28t^4-89t^3+176t^2-258t+288$	3 / ✘	2 / ✘
	10^a_{41}	$t^3-7t^2+17t-21$	$t^5-12t^4+54t^3-120t^2+157t-164$	3 / ✘	2 / ✘
	10^a_{42}	$-t^3+7t^2-19t+27$	$2t^3-8t^2+11t-12$	3 / ✔	1 / ✘
	10^a_{43}	$-t^3+7t^2-17t+23$	0	3 / ✘	2 / ✔
	10^a_{44}	$t^3-7t^2+19t-25$	$t^5-12t^4+56t^3-140t^2+220t-248$	3 / ✘	1 / ✘
	10^a_{45}	$-t^3+7t^2-21t+31$	0	3 / ✘	2 / ✔
	10^a_{46}	$-t^4+3t^3-4t^2+5t-5$	$-3t^7+12t^6-21t^5+34t^4-43t^3+52t^2-55t+56$	4 / ✘	3 / ✘
	10^a_{47}	$t^4-3t^3+6t^2-7t+7$	$-2t^7+8t^6-23t^5+38t^4-56t^3+60t^2-68t+64$	4 / ✘	2, 3 / ✘
	10^a_{48}	$t^4-3t^3+6t^2-9t+11$	$t^5-2t^4+2t^3-3t+4$	4 / ✔	2 / ✘
	10^a_{49}	$3t^3-8t^2+12t-13$	$30t^5-94t^4+196t^3-292t^2+372t-392$	3 / ✘	3 / ✘
	10^a_{50}	$-2t^3+7t^2-11t+13$	$-9t^5+44t^4-94t^3+150t^2-186t+200$	3 / ✘	2 / ✘
	10^a_{51}	$2t^3-7t^2+15t-19$	$-5t^5+24t^4-73t^3+134t^2-194t+212$	3 / ✘	2, 3 / ✘
	10^a_{52}	$2t^3-7t^2+13t-15$	$-3t^5+16t^4-37t^3+50t^2-49t+44$	3 / ✘	2 / ✘
	10^a_{53}	$6t^2-18t+25$	$93t^3-346t^2+680t-828$	2 / ✘	2, 3 / ✘
	10^a_{54}	$2t^3-6t^2+10t-11$	$-3t^5+12t^4-24t^3+26t^2-21t+16$	3 / ✘	2, 3 / ✘
	10^a_{55}	$5t^2-15t+21$	$66t^3-246t^2+488t-596$	2 / ✘	2 / ✘
	10^a_{56}	$-2t^3+8t^2-14t+17$	$-9t^5+52t^4-133t^3+234t^2-312t+340$	3 / ✘	2 / ✘
	10^a_{57}	$2t^3-8t^2+18t-23$	$-5t^5+28t^4-93t^3+194t^2-300t+340$	3 / ✘	2 / ✘
	10^a_{58}	$3t^2-16t+27$	$3t^3-28t^2+94t-140$	2 / ✘	2 / ✘
	10^a_{59}	$t^3-7t^2+18t-23$	$-t^5+12t^4-55t^3+128t^2-181t+196$	3 / ✘	1 / ✘

diagram	n^t_k Alexander's ω^+ Today's / Rozansky's ρ^+_1	genus / ribbon unknotting number / amphicheiral	diagram	n^t_k Alexander's ω^+ Today's / Rozansky's ρ^+_1	genus / ribbon unknotting number / amphicheiral
	10^a_{60} $-t^3+7t^2-20t+29$ $5t^3-40t^2+122t-176$	3 / ✘ 1 / ✘		10^a_{61} $-2t^3+5t^2-6t+7$ $-7t^5+20t^4-27t^3+36t^2-35t+36$	3 / ✘ 2, 3 / ✘
	10^a_{62} $t^4-3t^3+6t^2-8t+9$ $-2t^7+8t^6-23t^5+40t^4-63t^3+76t^2-89t+88$	4 / ✘ 2 / ✘		10^a_{63} $5t^2-14t+19$ $66t^3-220t^2+416t-496$	2 / ✘ 2 / ✘
	10^a_{64} $-t^4+3t^3-6t^2+10t-11$ $-t^7+4t^6-11t^5+24t^4-37t^3+52t^2-60t+64$	4 / ✘ 2 / ✘		10^a_{65} $2t^3-7t^2+14t-17$ $-5t^5+24t^4-71t^3+124t^2-169t+180$	3 / ✘ 2 / ✘
	10^a_{66} $3t^3-9t^2+16t-19$ $30t^5-112t^4+279t^3-480t^2+662t-724$	3 / ✘ 3 / ✘		10^a_{67} $-4t^2+16t-23$ $24t^3-140t^2+312t-392$	2 / ✘ 2 / ✘
	10^a_{68} $4t^2-14t+21$ $8t^3-40t^2+117t-164$	2 / ✘ 2 / ✘		10^a_{69} $t^3-7t^2+21t-29$ $-t^5+12t^4-68t^3+212t^2-397t+476$	3 / ✘ 2 / ✘
	10^a_{70} $t^3-7t^2+16t-19$ $-t^5+12t^4-53t^3+114t^2-146t+152$	3 / ✘ 2 / ✘		10^a_{71} $-t^3+7t^2-18t+25$ t^3-2t^2-t+4	3 / ✘ 1 / ✘
	10^a_{72} $-2t^3+9t^2-16t+19$ $-9t^5+60t^4-167t^3+298t^2-410t+448$	3 / ✘ 2 / ✘		10^a_{73} $t^3-7t^2+20t-27$ $t^5-12t^4+65t^3-194t^2+350t-416$	3 / ✘ 1 / ✘
	10^a_{74} $-4t^2+16t-23$ $24t^3-136t^2+290t-360$	2 / ✘ 2 / ✘		10^a_{75} $-t^3+7t^2-19t+27$ $-4t^3+36t^2-117t+172$	3 / ✔ 2 / ✘
	10^a_{76} $-2t^3+7t^2-12t+15$ $-9t^5+44t^4-104t^3+184t^2-245t+272$	3 / ✘ 2, 3 / ✘		10^a_{77} $2t^3-7t^2+14t-17$ $-5t^5+24t^4-71t^3+132t^2-189t+208$	3 / ✘ 2, 3 / ✘
	10^a_{78} $-t^3+7t^2-16t+21$ $2t^5-24t^4+105t^3-244t^2+390t-448$	3 / ✘ 2 / ✘		10^a_{79} $t^4-3t^3+7t^2-12t+15$ 0	4 / ✘ 2, 3 / ✔
	10^a_{80} $3t^3-9t^2+15t-17$ $30t^5-112t^4+260t^3-426t^2+568t-616$	3 / ✘ 3 / ✘		10^a_{81} $-t^3+8t^2-20t+27$ 0	3 / ✘ 2 / ✔
	10^a_{82} $-t^4+4t^3-8t^2+12t-13$ $t^7-6t^6+19t^5-42t^4+64t^3-78t^2+84t-84$	4 / ✘ 1 / ✘		10^a_{83} $2t^3-9t^2+19t-23$ $-5t^5+34t^4-110t^3+214t^2-301t+332$	3 / ✘ 2 / ✘
	10^a_{84} $2t^3-9t^2+20t-25$ $-5t^5+34t^4-116t^3+246t^2-373t+424$	3 / ✘ 1 / ✘		10^a_{85} $t^4-4t^3+8t^2-10t+11$ $2t^7-12t^6+36t^5-68t^4+101t^3-124t^2+138t-140$	4 / ✘ 2 / ✘
	10^a_{86} $-2t^3+9t^2-19t+25$ $-t^5+6t^4-21t^3+58t^2-105t+128$	3 / ✘ 2 / ✘		10^a_{87} $-2t^3+9t^2-18t+23$ $-t^5+6t^4-23t^3+66t^2-125t+152$	3 / ✔ 2 / ✘
	10^a_{88} $-t^3+8t^2-24t+35$ 0	3 / ✘ 1 / ✔		10^a_{89} $t^3-8t^2+24t-33$ $t^5-14t^4+83t^3-264t^2+495t-596$	3 / ✘ 2 / ✘
	10^a_{90} $-2t^3+8t^2-17t+23$ $-t^5+6t^4-21t^3+54t^2-93t+112$	3 / ✘ 2 / ✘		10^a_{91} $t^4-4t^3+9t^2-14t+17$ $t^5-2t^4+2t^3-3t+4$	4 / ✘ 1 / ✘
	10^a_{92} $-2t^3+10t^2-20t+25$ $-9t^5+68t^4-216t^3+428t^2-622t+696$	3 / ✘ 2 / ✘		10^a_{93} $2t^3-8t^2+15t-17$ $3t^5-18t^4+43t^3-58t^2+55t-48$	3 / ✘ 2 / ✘
	10^a_{94} $-t^4+4t^3-9t^2+14t-15$ $-t^7+6t^6-20t^5+46t^4-76t^3+102t^2-115t+120$	4 / ✘ 2 / ✘		10^a_{95} $2t^3-9t^2+21t-27$ $-5t^5+32t^4-114t^3+248t^2-384t+436$	3 / ✘ 1 / ✘
	10^a_{96} $-t^3+7t^2-22t+33$ $-7t^3+50t^2-147t+212$	3 / ✘ 2 / ✘		10^a_{97} $-5t^2+22t-33$ $-37t^3+242t^2-603t+788$	2 / ✘ 2 / ✘
	10^a_{98} $-2t^3+9t^2-18t+23$ $9t^5-60t^4+177t^3-348t^2+501t-564$	3 / ✘ 2 / ✘		10^a_{99} $t^4-4t^3+10t^2-16t+19$ 0	4 / ✔ 2 / ✔
	10^a_{100} $t^4-4t^3+9t^2-12t+13$ $2t^7-12t^6+39t^5-80t^4+128t^3-164t^2+192t-196$	4 / ✘ 2, 3 / ✘		10^a_{101} $7t^2-21t+29$ $-129t^3+480t^2-942t+1148$	2 / ✘ 2, 3 / ✘
	10^a_{102} $-2t^3+8t^2-16t+21$ $-t^5+6t^4-19t^3+50t^2-89t+108$	3 / ✘ 1 / ✘		10^a_{103} $2t^3-8t^2+17t-21$ $5t^5-30t^4+93t^3-178t^2+254t-280$	3 / ✘ 3 / ✘
	10^a_{104} $t^4-4t^3+9t^2-15t+19$ $t^5-2t^4+2t^3-3t+4$	4 / ✘ 1 / ✘		10^a_{105} $t^3-8t^2+22t-29$ $-t^5+14t^4-71t^3+184t^2-292t+332$	3 / ✘ 2 / ✘
	10^a_{106} $-t^4+4t^3-9t^2+15t-17$ $-t^7+6t^6-20t^5+48t^4-82t^3+114t^2-134t+140$	4 / ✘ 2 / ✘		10^a_{107} $-t^3+8t^2-22t+31$ $2t^3-8t^2+13t-16$	3 / ✘ 1 / ✘
	10^a_{108} $2t^3-8t^2+14t-15$ $-3t^5+18t^4-41t^3+50t^2-40t+32$	3 / ✘ 2 / ✘		10^a_{109} $t^4-4t^3+10t^2-17t+21$ 0	4 / ✘ 2 / ✔
	10^a_{110} $t^3-8t^2+20t-25$ $t^5-14t^4+69t^3-160t^2+219t-236$	3 / ✘ 2 / ✘		10^a_{111} $-2t^3+9t^2-17t+21$ $-9t^5+60t^4-171t^3+316t^2-436t+480$	3 / ✘ 2 / ✘
	10^a_{112} $-t^4+5t^3-11t^2+17t-19$ $t^7-8t^6+29t^5-68t^4+115t^3-152t^2+175t-180$	4 / ✘ 2 / ✘		10^a_{113} $2t^3-11t^2+26t-33$ $-5t^5+42t^4-167t^3+394t^2-623t+720$	3 / ✘ 1 / ✘
	10^a_{114} $-2t^3+10t^2-21t+27$ $t^5-8t^4+30t^3-78t^2+140t-168$	3 / ✘ 1 / ✘		10^a_{115} $-t^3+9t^2-26t+37$ 0	3 / ✘ 2 / ✔
	10^a_{116} $-t^4+5t^3-12t^2+19t-21$ $t^7-8t^6+30t^5-74t^4+132t^3-184t^2+217t-228$	4 / ✘ 2 / ✘		10^a_{117} $2t^3-10t^2+24t-31$ $-5t^5+38t^4-144t^3+330t^2-522t+600$	3 / ✘ 2 / ✘
	10^a_{118} $t^4-5t^3+12t^2-19t+23$ 0	4 / ✘ 1 / ✔		10^a_{119} $-2t^3+10t^2-23t+31$ $-t^5+6t^4-26t^3+86t^2-175t+220$	3 / ✘ 1 / ✘
	10^a_{120} $8t^2-26t+37$ $166t^3-692t^2+1433t-1788$	2 / ✘ 2, 3 / ✘		10^a_{121} $2t^3-11t^2+27t-35$ $5t^5-42t^4+167t^3-396t^2+634t-732$	3 / ✘ 2 / ✘
	10^a_{122} $-2t^3+11t^2-24t+31$ $-t^5+8t^4-34t^3+104t^2-211t+264$	3 / ✘ 2 / ✘		10^a_{123} $t^4-6t^3+15t^2-24t+29$ 0	4 / ✔ 2 / ✔

diagram	n^t_k	Alexander's ω^+	Today's / Rozansky's ρ^+_1	genus / ribbon	unknotting number / amphicheiral
	10^n_{124}	$t^4 - t^3 + t - 1$	$-4t^7 - 6t^4 - 4t^2 - 6t$	4 / ✘	4 / ✘
	10^n_{125}	$t^3 - 2t^2 + 2t - 1$	$-t^5 + 2t^4 - 2t^3 + 3t - 4$	3 / ✘	2 / ✘
	10^n_{126}	$t^3 - 2t^2 + 4t - 5$	$t^5 - 2t^4 + 10t^3 - 12t^2 + 22t - 20$	3 / ✘	2 / ✘
	10^n_{127}	$-t^3 + 4t^2 - 6t + 7$	$2t^5 - 14t^4 + 32t^3 - 52t^2 + 67t - 72$	3 / ✘	2 / ✘
	10^n_{128}	$2t^3 - 3t^2 + t + 1$	$-13t^5 + 12t^4 - 3t^3 - 10t^2 - 9t + 12$	3 / ✘	3 / ✘
	10^n_{129}	$2t^2 - 6t + 9$	$-t^3 - 2t^2 + 14t - 20$	2 / ✔	1 / ✘
	10^n_{130}	$2t^2 - 4t + 5$	$t^3 - 2t^2 + 19t - 24$	2 / ✘	2 / ✘
	10^n_{131}	$-2t^2 + 8t - 11$	$5t^3 - 38t^2 + 87t - 112$	2 / ✘	1 / ✘
	10^n_{132}	$t^2 - t + 1$	$2t^2 + 5t - 4$	2 / ✘	1 / ✘
	10^n_{133}	$-t^2 + 5t - 7$	$t^3 - 14t^2 + 37t - 48$	2 / ✘	1 / ✘
	10^n_{134}	$2t^3 - 4t^2 + 4t - 3$	$-13t^5 + 24t^4 - 33t^3 + 30t^2 - 41t + 40$	3 / ✘	3 / ✘
	10^n_{135}	$3t^2 - 9t + 13$	$t^3 - 6t^2 + 18t - 24$	2 / ✘	2 / ✘
	10^n_{136}	$-t^2 + 4t - 5$	$-t^3 + 4t^2 - 2t - 4$	2 / ✘	1 / ✘
	10^n_{137}	$t^2 - 6t + 11$	$-4t^2 + 24t - 44$	2 / ✔	1 / ✘
	10^n_{138}	$t^3 - 5t^2 + 8t - 7$	$-t^5 + 8t^4 - 22t^3 + 24t^2 - 11t + 8$	3 / ✘	2 / ✘
	10^n_{139}	$t^4 - t^3 + 2t - 3$	$-4t^7 - 12t^4 + 5t^3 - 4t^2 - 16t + 12$	4 / ✘	4 / ✘
	10^n_{140}	$t^2 - 2t + 3$	$8t - 8$	2 / ✔	2 / ✘
	10^n_{141}	$-t^3 + 3t^2 - 4t + 5$	$t^3 - 8t^2 + 16t - 20$	3 / ✘	1 / ✘
	10^n_{142}	$2t^3 - 3t^2 + 2t - 1$	$-13t^5 + 12t^4 - 13t^3 + 4t^2 - 17t + 12$	3 / ✘	3 / ✘
	10^n_{143}	$t^3 - 3t^2 + 6t - 7$	$t^5 - 4t^4 + 15t^3 - 28t^2 + 45t - 48$	3 / ✘	1 / ✘
	10^n_{144}	$-3t^2 + 10t - 13$	$10t^3 - 44t^2 + 80t - 96$	2 / ✘	2 / ✘
	10^n_{145}	$t^2 + t - 3$	$2t^3 + 8t^2 + 6t - 8$	2 / ✘	2 / ✘
	10^n_{146}	$2t^2 - 8t + 13$	$t^3 - 8t^2 + 21t - 28$	2 / ✘	1 / ✘
	10^n_{147}	$-2t^2 + 7t - 9$	$-3t^3 + 12t^2 - 15t + 12$	2 / ✘	1 / ✘
	10^n_{148}	$t^3 - 3t^2 + 7t - 9$	$t^5 - 4t^4 + 18t^3 - 36t^2 + 62t - 68$	3 / ✘	2 / ✘
	10^n_{149}	$-t^3 + 5t^2 - 9t + 11$	$2t^5 - 18t^4 + 55t^3 - 104t^2 + 149t - 164$	3 / ✘	2 / ✘
	10^n_{150}	$-t^3 + 4t^2 - 6t + 7$	$-2t^5 + 12t^4 - 26t^3 + 38t^2 - 45t + 44$	3 / ✘	2 / ✘
	10^n_{151}	$t^3 - 4t^2 + 10t - 13$	$-t^5 + 6t^4 - 21t^3 + 42t^2 - 66t + 72$	3 / ✘	2 / ✘
	10^n_{152}	$t^4 - t^3 - t^2 + 4t - 5$	$4t^7 - 7t^5 + 18t^4 - 7t^3 - 12t^2 + 45t - 52$	4 / ✘	4 / ✘
	10^n_{153}	$t^3 - t^2 - t + 3$	$t^5 - 2t^4 + t^3 + 2t^2 - t$	3 / ✔	2 / ✘
	10^n_{154}	$t^3 - 4t + 7$	$-3t^5 - 6t^4 + 13t^3 - 47t + 68$	3 / ✘	3 / ✘
	10^n_{155}	$-t^3 + 3t^2 - 5t + 7$	$-2t^3 + 12t^2 - 22t + 28$	3 / ✔	2 / ✘
	10^n_{156}	$t^3 - 4t^2 + 8t - 9$	$t^5 - 6t^4 + 19t^3 - 30t^2 + 33t - 32$	3 / ✘	1 / ✘
	10^n_{157}	$-t^3 + 6t^2 - 11t + 13$	$-2t^5 + 22t^4 - 78t^3 + 148t^2 - 218t + 240$	3 / ✘	2 / ✘
	10^n_{158}	$-t^3 + 4t^2 - 10t + 15$	$2t^2 - 7t + 12$	3 / ✘	2 / ✘
	10^n_{159}	$t^3 - 4t^2 + 9t - 11$	$t^5 - 6t^4 + 26t^3 - 60t^2 + 98t - 112$	3 / ✘	1 / ✘
	10^n_{160}	$-t^3 + 4t^2 - 4t + 3$	$-2t^5 + 12t^4 - 20t^3 + 14t^2 - 16t + 12$	3 / ✘	2 / ✘
	10^n_{161}	$t^3 - 2t + 3$	$3t^5 + 6t^4 - 3t^3 + 4t^2 + 14t - 12$	3 / ✘	3 / ✘
	10^n_{162}	$-3t^2 + 9t - 11$	$10t^3 - 38t^2 + 58t - 68$	2 / ✘	2 / ✘
	10^n_{163}	$t^3 - 5t^2 + 12t - 15$	$-t^5 + 8t^4 - 30t^3 + 62t^2 - 89t + 96$	3 / ✘	1, 2 / ✘
	10^n_{164}	$3t^2 - 11t + 17$	$t^3 - 10t^2 + 29t - 40$	2 / ✘	1 / ✘
	10^n_{165}	$-2t^2 + 10t - 15$	$-5t^3 + 50t^2 - 146t + 196$	2 / ✘	2 / ✘