MIT-1612 handout on 161208

December 8, 2016 9:35 AM

"export" is to GWU-1612.

Dror Bar-Natan: Talks: MIT-1612: (thanks for accepting my invitation!) ωεβ:=http://drorbn.net/MIT-1612/

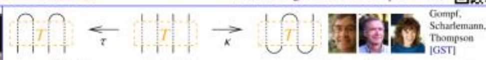
A Poly-Time Knot Polynomial Via Solvable Approximation

Work in Progress! Fluid! Help Needed!

Abstract. Rozansky [Ro2] and Overbay [Ov] described a spectacular knot polynomial that failed to attract the attention it deserved as the first poly-time-computable knot polynomial since Alexander's [Al, 1928] and (in my opinion) as the second most likely knot polynomial (after Alexander's) to carry topological information. With Roland van der Veen, I will explain how to compute the Rozansky polynomial using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.

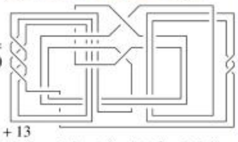
Rozansky

Overbay



Gompf, Scharlemann, Thompson [GST]

$U \in \mathcal{T}_n \xrightarrow{\tau}$, $1 \in \mathcal{A}_n$; $\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$ with $\mathcal{R} := \kappa(\tau^{-1}(1))$



ribbon $K \in \mathcal{T}_1 \quad z(K) \in \mathcal{R} \subseteq \mathcal{A}_1$

Faster is better, leaner is meaner!

$A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$

$\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot A(K)(e^\hbar) = 1$.

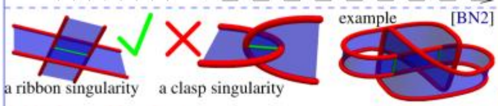
"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})A(K)(q^d)}\left(1 + \sum_{k=1}^\infty \frac{(q-1)^k R_k(K)(q^d)}{A^{2k}(K)(q^d)}\right).$$

Why "spectacular"? Foremost reason: *OBVIOUSLY*. Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Also, will bound genus and may disprove {ribbon} = {slice}.

Genus.



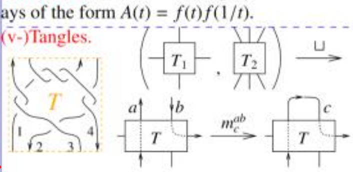
a ribbon singularity a clasp singularity

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

(v-)Tangles.



(meta-associativity: $m^{ab}_x /\!/ m^{xc}_y = m^{bc}_x /\!/ m^{ax}_y$)

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a : a \in S\})$:

$$\left(\nearrow_{a\,b},\ \nwarrow_{b\,a}\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta}\end{array}\right)$$

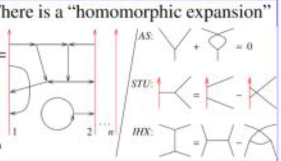
(Roland: "add to A the product of column b and row a, divide by $(1 - A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

(There are also formulas for strand doubling and strand reversal).

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

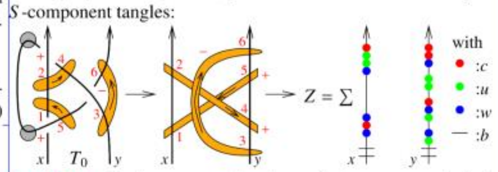
$Z: \left\{\begin{matrix} S\text{-component} \\ (v/b\text{-})\text{tangles}\end{matrix}\right\} \to \mathcal{A}^v_S :=$



Etingof Kazhdan Haviv Enriquez Ševera

Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:

export last



Good News. In theory, enough to know R, the cuaps, and stitching/multiplication $m^{ij}_k : A_i \otimes A_j \to A_k$.

Problem. Extract information out of Z.

Textbook Solution. Use representation theory ...works, slowly.

Today's Solution (with van der Veen). For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$$\left\{\begin{matrix}\text{ordered} \quad \text{perturbed} \\ \text{Gaussian formulas}\end{matrix}\right\} \to \hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$

van der Veen

export

A: pragmatics: $t := e^b$, $v := \frac{t-1}{b} u$

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and with $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset 0$. A

(note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

re-think (No change)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon = 0$, or $\mathbb{Q}\langle b, c, u, w\rangle/([b, \cdot] = 0, [c, u] = u, [c, w] = -w, [u, w] = b$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{b}^* \rtimes \mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c, w\rangle$ and (b, u) is the dual basis of (c, w). For topology, it is more valuable than $\mathfrak{g}_1$ / sl_2, but topology already got by other means almost everything $\mathfrak{g}_0$ gives.

export

How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w\rangle/[w, c] = w$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (c, w) \mapsto (0, c \wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get $\mathfrak{g}_0$. At $k = 1$, get $[w, c] = w$, $[w, b'] = -\epsilon w$, $[c, u] = u$, $[b', u] = -\epsilon u$, $[b', c] = 0$, and $[u, w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is $\mathfrak{g}_1$.

export

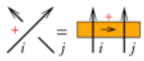
Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of $poly$ in $S(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,

$$\mathbb{O}\left(c_1^3 u_1 c_2 e^{u_3} w_3^9 \middle| x\colon w_3 c_1,\ y\colon u_1 u_3 c_2\right) = w^9 c^3 \otimes u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series. B

B: (In $\mathfrak{g}_1$, no need to specify b/t)

0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$$= R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$$

solves YB/R3.

Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \mathbb{O}\left(\exp\left(b_i c_j + \frac{e^{b_i}-1}{b_i} u_i w_j\right) \middle| i\colon u_i,\ j\colon c_j w_j\right)$

Example. $Z(T_0) =$ $\qquad = \sum_{m,n} \frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!} u^n \otimes c^m w^n$.

$$\mathbb{O}\Bigg(\exp\left(b_5 c_1 + \tfrac{e^{b_5}-1}{b_5} u_5 w_1 + b_2 c_4 + \tfrac{e^{b_2}-1}{b_2} u_2 w_4 - b_3 c_6 + \tfrac{e^{-b_3}-1}{b_3} u_3 w_6\right) \Bigg| \; x\colon c_1 w_1 u_2,\ y\colon u_3 c_4 w_4 u_5 c_6 w_6\Bigg) = \mathbb{O}\left(\zeta \middle| x\colon u_x c_x w_x,\ y\colon u_y c_y w_y\right)$$

"ucw form"

Goal. Write ζ as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S := \mathbb{Q}(b_i, e^{b_i})$, and $\omega \in R_S$.

The Big $\mathfrak{g}_0$ Lemma. Under $[c, u] = u$, $[c, w] = -w$, and $[u, w] = b$:

1a. $N^{cu} := \mathbb{O}(e^{\gamma c + \beta u} | uc) \overset{\to}{=} \mathbb{O}(e^{\gamma c + e^{\gamma}\beta u} | cu)$ (means $e^{\beta u} e^{\gamma c} = e^{\gamma c} e^{e^{\gamma}\beta u}$

1b. $N^{wc} := \mathbb{O}(e^{\gamma c + \alpha w} | wc) \overset{\to}{=} \mathbb{O}(e^{\gamma c + e^{\gamma}\alpha w} | cw)$ …in the $\{ax + b\}$ group)

2. $\mathbb{O}(e^{\alpha w + \beta u} | wu) = \mathbb{O}(e^{-b\alpha\beta + \alpha w + \beta u} | uw)$ (the Weyl relations)

3. $\mathbb{O}(e^{\delta uw} | wu) e^{\beta u} = e^{\nu\beta u} \mathbb{O}(e^{\delta uw} | wu)$, with $\nu = (1 + b\delta)^{-1}$

(a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)

4. $\mathbb{O}(e^{\delta uw} | wu) = \mathbb{O}(\nu e^{\nu\delta uw} | uw)$ (same techniques)

5. $N^{wu} := \mathbb{O}(e^{\beta u + \alpha w + \delta uw} | wu) \overset{\to}{=} \mathbb{O}(\nu e^{-b\nu\alpha\beta + \nu\alpha w + \nu\beta u + \nu\delta uw} | uw)$

6. $N_k^{c_i c_j} := \mathbb{O}(\zeta | c_i c_j) \overset{\to}{=} \mathbb{O}(\zeta/(c_i, c_j \to c_k) | c_k)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition

$$u_i c_i \overline{w_i u_j} c_j w_j \xrightarrow{N_x^{w_i u_j}} u_i \overline{c_i u_x}\, \overline{w_x c_j} w_j \xrightarrow{N_x^{c_i u_x} /\!\!/ N_x^{w_x c_j}} \overline{u_i u_x}\, \overline{c_x c_x}\, \overline{w_x w_j} \xrightarrow{i,j,x \to k} u_k c_k w_k$$

On to 1-smidgen invariants, where much is the same…

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 6 are the same, yet

5. $\mathbb{O}\left(e^{\alpha w + \beta u + \delta uw} \middle| wu\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\Lambda) e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} \middle| ucw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, u, c, and w:

$$\begin{aligned}\Lambda = &-b\nu(\alpha^2\beta^2\nu^2 + 4\alpha\beta\delta\nu + 2\delta^2)/2 + \beta^2\delta\nu^3(b\delta + 2)u^2/2 \\ &+ \delta^3\nu^3(3b\delta + 4)u^2w^2/2 + \beta\delta^2\nu^3(2b\delta + 3)u^2 w \\ &+ \alpha\delta^2\nu^3(2b\delta + 3)uw^2 + 2\delta\nu^2(b\delta + 2)(\alpha\beta\nu + \delta)uw \\ &+ \alpha^2\delta\nu^3(b\delta + 2)w^2/2 + 2(\alpha\beta\nu + \delta)c + 2\beta\delta\nu uc + 2\delta^2\nu ucw \\ &+ 2\alpha\delta\nu cw + \beta\nu^2(\alpha\beta\nu + 2\delta)u + \alpha\nu^2(\alpha\beta\nu + 2\delta)w.\end{aligned}$$

Proof. A lengthy computation. (Verification: ωεβ/Big)

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c, u) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + e^{-\gamma}\beta u} | cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u, w) e^{\alpha w + \beta u + \delta uw} \middle| wu\right) = \mathbb{O}\left(\epsilon P(\partial_\beta, \partial_\alpha) \nu e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} \middle| ucw\right)$$

Finally, the values of the generators ╳, ╳, $\overrightarrow{n}$, and $\underset{\to}{u}$, are set by solving many equations, non-uniquely.

Pragmatic Simplifications. Set $t := e^b$, work with $v := (t - 1)u/b$, and set $\mathbb{E}(\omega, L, Q, P) := \mathbb{O}\left(\omega^{-1} e^{L + Q/\omega}(1 + \epsilon\omega^{-4}P)\colon (i\colon v_i c_i w_i)\right)$. Now $\omega \in R_S := \mathbb{Z}[t_i, t_i^{-1}]$ is Laurent, $L = \sum l_{ij} \log(t_i) c_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij} v_i w_j$ with $q_{ij} \in R_S$, and P is a quartic polynomial in v_i, c_j, w_k with coefficients in R_S. The operations are lightly modified, and the Λόγος and the values of the generators become somewhat simpler, as in the implementation below.

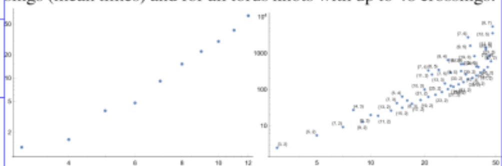
confirm.

Rough complexity estimate, after $t_k \to t$. n: xing number; w: width, maybe $\sim \sqrt{n}$.

$$\underset{A}{n} \overbrace{\sum_{d=0}^{4} \underset{E}{w^{4-d}}\, \underset{F}{w^d}\, \underset{G}{n^2}}^{B} = n^3 w^4 \in [n^5, n^7]$$

A: go over stitchings in order. B: multiplication ops per $N^{u_i w_j}$. d: deg of u_i, w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Experimental Analysis (ωεβ/Exp). Log-log plot of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial A, there is a polynomial ρ_1 such that

$$P = A^2 \frac{(t-1)^3\rho_1 + t^2(2vw + (1-t)(1-2c))AA'}{(1-t)t}.$$

Furthermore, A and ρ_1 are symmetric under $t \to t^{-1}$, so let A^+ and ρ_1^+ be their "positive parts", so e.g., $\rho_1(t) = \rho_1^+(t) + \rho_1^+(t^{-1}) - \rho_1^+(0)$.

Power. On the 250 knots with at most 10 crossings, the pair (A, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always $\deg \rho_1^+ \leq 2g - 1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Demo Programs for 0-Co. ωεβ/Demo

The R-matrices

```
R⁺_{0,i_,j_} := 𝔼[b_i c_j + b_i^-1 (e^{b_i} - 1) u_i w_j];
R⁻_{0,i_,j_} := 𝔼[-b_i c_j + b_i^-1 (e^{-b_i} - 1) u_i w_j];
```

Utilities

```
CF[ω_ . 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_ . 𝔼[Q1_] ≡ ω2_ . 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

Normal Ordering Operators

```
N_{(x:w|u)_i_ c_j_ → k_}[ω_ . 𝔼[Q_]] := CF[
   ω 𝔼[e^γ α x_k + γ c_k + (Q /. c_j | x_i → 0)] /. {γ → ∂_{c_j} Q, α → ∂_{x_i} Q}];
N_{w_i_ u_j_ → k_}[ω_ . 𝔼[Q_]] := CF[
   ν ω 𝔼[-b_k ν α β + ν β u_k + ν α w_k + ν δ u_k w_k + (Q /. w_i | u_j → 0)] /.
    ν → (1 + b_k δ)^-1 /.
   {α → ∂_{w_i} Q /. u_j → 0, β → ∂_{u_j} Q /. w_i → 0, δ → ∂_{w_i,u_j} Q}];
```

Stitching

```
m_{i_,j_→k_}[Z_] := Module[{x, z},
  CF[(Z // N_{w_i u_j→x} // N_{c_i u_x→x} // N_{w_x c_j→x}) /. z_{-(i|j|x)} → z_k]]
```

Some calculations for T_0

```
T0 = R⁺_{0,5,1} R⁺_{0,2,4} R⁻_{0,3,6}
```

$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5})\, u_5 w_1}{b_5} + \frac{(-1+e^{b_2})\, u_2 w_4}{b_2} + \frac{(-1+e^{-b_3})\, u_3 w_6}{b_3}\right]$$

```
T0 // m_{1,2→1} // m_{3,4→3} // m_{3,5→3} // m_{3,6→3}
```

$$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\,\mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{e^{b_3}(-1+e^{b_1})(-1+e^{b_3})\, u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\, b_1} - \frac{e^{b_1}(-1+e^{b_3})\, u_3 w_1}{(-1+(-1+e^{b_1})(-1+e^{b_3}))\, b_3} - \frac{e^{-b_3}(-1+e^{b_3})\, u_3 w_3}{b_3} - \frac{e^{-b_3}(-1+e^{b_1})(-e^{b_3} b_3 u_1 + e^{b_1}(-1+e^{b_3})\, b_1 u_3)\, w_3}{b_1 (b_3 - (-1+e^{b_1})(-1+e^{b_3})\, b_3)}\Big]$$

Verifying meta-associativity

```
Q0 = 𝔼[Sum[f_i c_i, {i, 3}] + Sum[f_{i,j} u_i w_j, {i, 3}, {j, 3}]]
```

$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

```
(Q0 // m_{1,2→1} // m_{1,3→1}) ≡ (Q0 // m_{2,3→2} // m_{1,2→1})
True
```

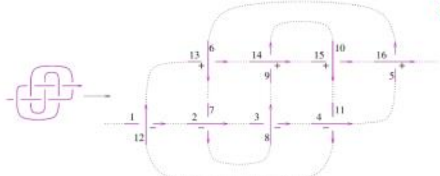
Testing R3

```
t1 = R⁺_{0,1,2} R⁺_{0,3,4} R⁺_{0,5,6} // m_{3,5→x} // m_{1,6→y} // m_{2,4→z}
```

$$\mathbb{E}\left[b_x c_y + b_x c_z + b_y c_z + \frac{e^{b_x}(-1+e^{b_y})\, u_y w_z}{b_y} + \frac{(-1+e^{b_x})\, u_x (w_y+w_z)}{b_x}\right]$$

```
t1 ≡ (R⁺_{0,1,2} R⁺_{0,3,4} R⁺_{0,5,6} // m_{1,3→x} // m_{2,5→y} // m_{4,6→z})
True
```

8_{17}




```
z1 = R⁻_{0,12,1} R⁻_{0,2,7} R⁻_{0,8,3} R⁻_{0,4,11} R⁺_{0,16,5} R⁺_{0,6,13} R⁺_{0,14,9} R⁺_{0,10,15};
Do[z1 = (z1 // m_{1,n→1}) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$$

better if no column break (not this time)

Demo Programs for 1-Co. ωεβ/Demo

The Λόγος

```
Λ[k_] := ((t_k - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 v_k c_k w_k δ^2 μ^2 -
     δ (1 + μ) (w_k^2 α^2 + v_k^2 β^2) - v_k^2 w_k^2 δ^3 (1 + 3 μ) -
     2 (α β + 2 δ μ + v_k w_k δ^2 (1 + 2 μ) + 2 c_k δ μ^2) (w_k α + v_k β) -
     4 (c_k μ^2 + v_k w_k δ (1 + μ)) (α β + δ μ)) (1 + t_k) / 4;
```

The Generators

```
R⁺_{i_,j_} := 𝔼[1, Log[t_i] c_j, v_i w_j, v_i c_i w_j + c_i c_j + v_i^2 w_j^2/4];
R⁻_{i_,j_} := 𝔼[1, -Log[t_i] c_j, -t_i^-1 v_i w_j,
    t_i^-1 v_i c_j w_j - c_i c_j - t_i^-2 v_i^2 w_j^2/4];
(ur_{i_} := 𝔼[t_i^{-1/2}, 0, 0, c_i t_i^-2]; nr_{i_} := 𝔼[t_i^{1/2}, 0, 0, -c_i t_i^2];)
```

Differential Polynomials

Not now

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := (* means P[∂_α, ∂_β][f] *)
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) ↦ c D[f, {α, m}, {β, n}]]
```

Utilities

```
CF[ℰ_𝔼] := Expand /@ Together /@ ℰ;
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

Normal Ordering Operators

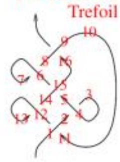
```
N_{c_j_ (x:v|w)_i_ → k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^γ β x_k + γ c_k}, CF[
   𝔼[ω, γ c_k + (L /. c_j → 0), ω e^γ β x_k + (Q /. x_i → 0),
     e^-q DP_{c_j→D_γ, x_i→D_β}[P][e^q]] /. {γ → ∂_{c_j} L, β → ω^-1 ∂_{x_i} Q}]];
N_{w_i_ v_j_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{q = ((1 - t_k) α β + β v_k + α w_k + δ v_k w_k) / μ}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. w_i | v_j → 0),
      μ^4 e^-q DP_{w_i→D_α, v_j→D_β}[P][e^q] + ω^4 Λ[k]] /. μ → 1 + (t_k - 1) δ /.
    {α → ω^-1 (∂_{w_i} Q /. v_j → 0), β → ω^-1 (∂_{v_j} Q /. w_i → 0),
     δ → ω^-1 ∂_{w_i,v_j} Q}]];
```

optimize! as in 2016-09/..optimization-s.nb and beyond. Not now

Stitching

```
m_{i_,j_→k_}[Z_𝔼] := Module[{x, z},
  CF[(Z // N_{w_i v_j→x} // N_{c_i v_x→x} // N_{w_x c_j→x}) /. z_{-(i|j|x)} → z_k]]
```

The 0-Framed Trefoil




```
z2 = R⁺_{1,11} R⁻_{4,2} nr_3 R⁺_{15,5} R⁻_{6,8} ur_7 R⁺_{9,16} nr_10 R⁻_{12,14} ur_13;
(Do[z2 = z2 // m_{1,k→1}, {k, 2, 16}];
 z2 = z2 /. a_1 ↦ a)
```

$$\mathbb{E}\Big[-1+\frac{1}{t}+t,\ 0,\ 0,\ 16+\frac{2c}{t^4}-\frac{1}{t^3}-\frac{6c}{t^3}+\frac{4}{t^2}+\frac{10c}{t^2}-\frac{10}{t}-\frac{8c}{t}-18t+8ct+14t^2-10ct^2-7t^3+6ct^3+2t^4-2ct^4+2vw-\frac{2vw}{t^4}+\frac{4vw}{t^3}-\frac{6vw}{t^2}+\frac{2vw}{t}-6tvw+4t^2vw-2t^3vw\Big]$$

Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $b^+ \leftrightarrow b^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

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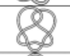
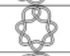
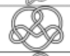
diagram	n^t_k	Alexander's A_+ / Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral	diagram	n^t_k	Alexander's A_+ / Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral
	0^a_1	1 0	0 / ✓ 0 / ✓		3^a_1	$t-1$ t	1 / ✗ 1 / ✗
	4^a_1	$3-t$ 0	1 / ✗ 1 / ✓		5^a_1	t^2-t+1 $2t^3+3t$	2 / ✗ 2 / ✗
	5^a_2	$2t-3$ $5t-4$	1 / ✗ 1 / ✗		6^a_1	$5-2t$ $t-4$	1 / ✓ 1 / ✗
	6^a_2	$-t^2+3t-3$ t^3-4t^2+4t-4	2 / ✗ 1 / ✗		6^a_3	t^2-3t+5 0	2 / ✗ 1 / ✓
	7^a_1	t^3-t^2+t-1 $3t^5+5t^3+6t$	3 / ✗ 3 / ✗		7^a_2	$3t-5$ $14t-16$	1 / ✗ 1 / ✗
	7^a_3	$2t^2-3t+3$ $-9t^3+8t^2-16t+12$	2 / ✗ 2 / ✗		7^a_4	$4t-7$ $32-24t$	1 / ✗ 2 / ✗
	7^a_5	$2t^2-4t+5$ $9t^3-16t^2+29t-28$	2 / ✗ 2 / ✗		7^a_6	$-t^2+5t-7$ $t^3-8t^2+19t-20$	2 / ✗ 1 / ✗
	7^a_7	t^2-5t+9 $8-3t$	2 / ✗ 1 / ✗		8^a_1	$7-3t$ $5t-16$	1 / ✗ 1 / ✗
	8^a_2	$-t^3+3t^2-3t+3$ $2t^5-8t^4+10t^3-12t^2+13t-12$	3 / ✗ 2 / ✗		8^a_3	$9-4t$ 0	1 / ✗ 2 / ✓
	8^a_4	$-2t^2+5t-5$ $3t^3-8t^2+6t-4$	2 / ✗ 2 / ✗		8^a_5	$-t^3+3t^2-4t+5$ $-2t^5+8t^4-13t^3+20t^2-22t+24$	3 / ✗ 2 / ✗
	8^a_6	$-2t^2+6t-7$ $5t^3-20t^2+28t-32$	2 / ✗ 2 / ✗		8^a_7	t^3-3t^2+5t-5 $-t^5+4t^4-10t^3+12t^2-13t+12$	3 / ✗ 1 / ✗
	8^a_8	$2t^2-6t+9$ $-t^3+4t^2-12t+16$	2 / ✓ 2 / ✗		8^a_9	$-t^3+3t^2-5t+7$ 0	3 / ✓ 1 / ✓
	8^a_{10}	t^3-3t^2+6t-7 $-t^5+4t^4-11t^3+16t^2-21t+20$	3 / ✗ 2 / ✗		8^a_{11}	$-2t^2+7t-9$ $5t^3-24t^2+39t-44$	2 / ✗ 1 / ✗
	8^a_{12}	$t^2-7t+13$ 0	2 / ✗ 2 / ✓		8^a_{13}	$2t^2-7t+11$ $-t^3+4t^2-14t+20$	2 / ✗ 1 / ✗
	8^a_{14}	$-2t^2+8t-11$ $5t^3-28t^2+57t-68$	2 / ✗ 1 / ✗		8^a_{15}	$3t^2-8t+11$ $21t^3-64t^2+120t-140$	2 / ✗ 2 / ✗
	8^a_{16}	t^3-4t^2+8t-9 $t^5-6t^4+17t^3-28t^2+35t-36$	3 / ✗ 2 / ✗		8^a_{17}	$-t^3+4t^2-8t+11$ 0	3 / ✗ 1 / ✓
	8^a_{18}	$-t^3+5t^2-10t+13$ 0	3 / ✗ 2 / ✓		8^n_{19}	t^3-t^2+1 $-3t^5-4t^2-3t$	3 / ✗ 3 / ✗
	8^n_{20}	t^2-2t+3 $4t-4$	2 / ✓ 1 / ✗		8^n_{21}	$-t^2+4t-5$ $t^3-8t^2+16t-20$	2 / ✗ 1 / ✗