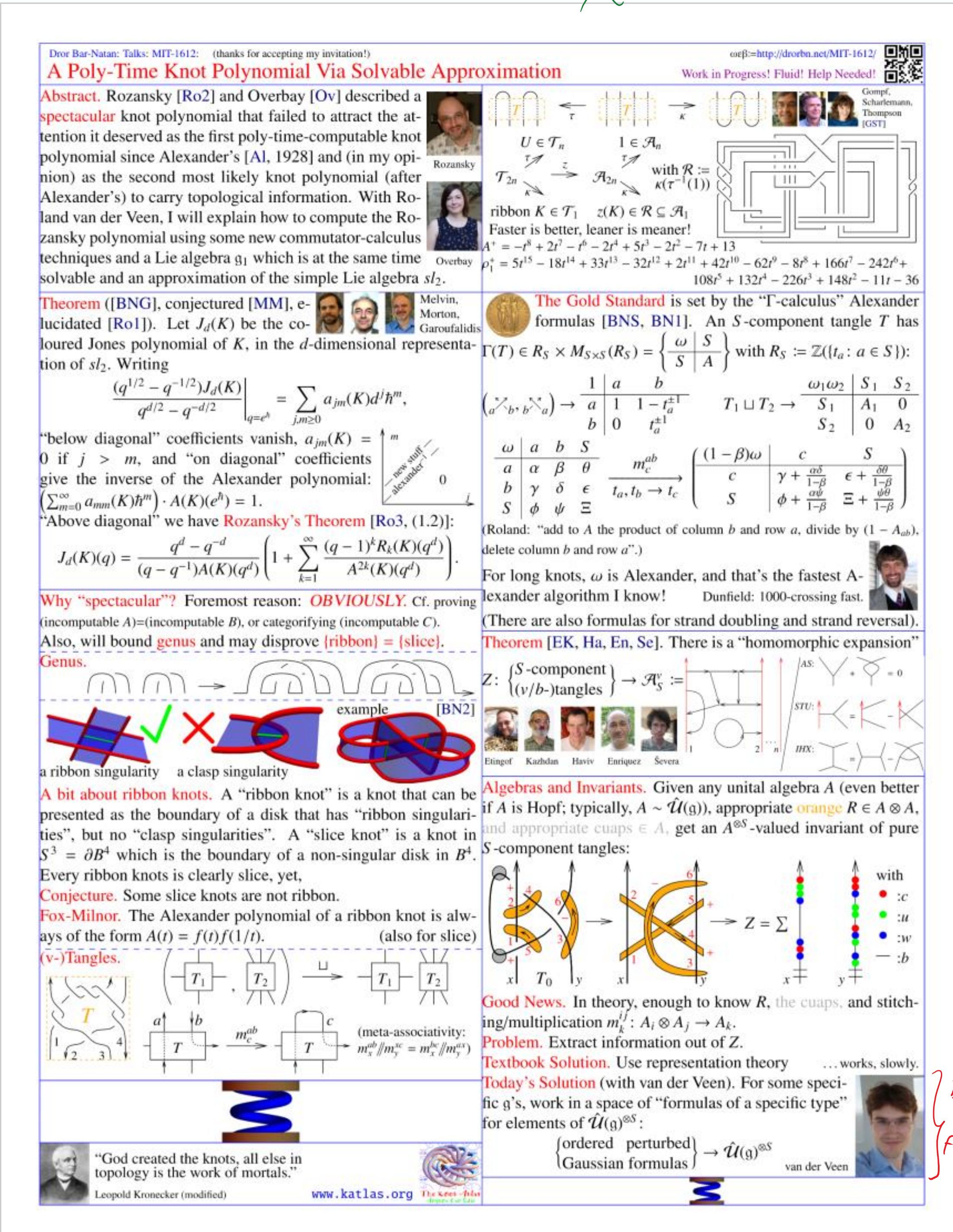
MIT-1612 handout on 161129

November 29, 2016 7:44 PM

Solve the PgUp/PgDn HandoutBrowser bug!



Dror Bar-Natan: Talks: MIT-1612: (thanks for accepting my invitation!) ωεβ:=http://drorbn.net/MIT-1612/

A Poly-Time Knot Polynomial Via Solvable Approximation

Work in Progress! Fluid! Help Needed!

Abstract. Rozansky [Ro2] and Overbay [Ov] described a spectacular knot polynomial that failed to attract the attention it deserved as the first poly-time-computable knot polynomial since Alexander's [Al, 1928] and (in my opinion) as the second most likely knot polynomial (after Alexander's) to carry topological information. With Roland van der Veen, I will explain how to compute the Rozansky polynomial using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.

Rozansky

Overbay

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar)=1$.

m, new stuff, alexander, 0, j

"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d-q^{-d}}{(q-q^{-1})A(K)(q^d)}\left(1+\sum_{k=1}^\infty \frac{(q-1)^k R_k(K)(q^d)}{A^{2k}(K)(q^d)}\right).$$

Why "spectacular"? Foremost reason: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).
Also, will bound genus and may disprove {ribbon} = {slice}.

Genus.

example [BN2]

a ribbon singularity a clasp singularity

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3=\partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t)=f(t)f(1/t)$. (also for slice)

(v-)Tangles.

T, 1, 2, 3, 4

$(T_1, T_2) \xrightarrow{\sqcup} T_1\ T_2$

$T \xrightarrow{m^{ab}_c} T$ (meta-associativity: $m^{ab}_x /\!\!/ m^{xc}_y = m^{bc}_x /\!\!/ m^{ax}_y$)

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

$T \xleftarrow{\tau} T \xrightarrow{\kappa} T$

Gompf, Scharlemann, Thompson [GST]

$U\in\mathcal{T}_n \quad 1\in\mathcal{A}_n$

$\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$ with $\mathcal{R} := \kappa(\tau^{-1}(1))$

ribbon $K\in\mathcal{T}_1 \quad z(K)\in\mathcal{R}\subseteq\mathcal{A}_1$

Faster is better, leaner is meaner!

$A^+ = -t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$

$\rho_1^+ = 5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T)\in R_S\times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a\colon a\in S\})$:

$$\left({}_a\!\nearrow_b, {}_b\!\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm1} \\ b & 0 & t_a^{\pm1}\end{array} \qquad T_1\sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a,t_b\to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta} \\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

(Roland: "add to A the product of column b and row a, divide by $(1-A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

(There are also formulas for strand doubling and strand reversal).

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z\colon \left\{\begin{matrix} S\text{-component} \\ (v/b\text{-})\text{tangles}\end{matrix}\right\} \to \mathcal{A}^v_S :=$$

1, 2, …, n

AS: = 0
STU: = −
IHX: = −

Etingof Kazhdan Haviv Enriquez Ševera

Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A\sim\hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R\in A\otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:

$x\ T_0\ y \to x\ y \to Z=\sum$ x y with ● $:c$ ● $:u$ ● $:w$ — $:b$

Good News. In theory, enough to know R, the cuaps, and stitching/multiplication $m^{ij}_k\colon A_i\otimes A_j\to A_k$.

Problem. Extract information out of Z.

Textbook Solution. Use representation theory …works, slowly.

Today's Solution (with van der Veen). For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$$\left\{\begin{matrix}\text{ordered perturbed} \\ \text{Gaussian formulas}\end{matrix}\right\} \to \hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$

van der Veen

bring to front?

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b,c,u,w\rangle$ over the ring $R=\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with b central and with $[w,c]=w$, $[c,u]=u$, and $[u,w]=b-2\epsilon c$, with CYBE $r_{ij}=(b_i-\epsilon c_i)c_j+u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1\supset\langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset\langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset 0$. (note: $\deg(b,c,u,w,\epsilon)=(1,0,1,0,1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon=0$, or $\mathbb{Q}\langle b,c,u,w\rangle/([b,\cdot]=0,\ [c,u]=u,\ [c,w]=-w,\ [u,w]=b$ with $r_{ij}=b_ic_j+u_iw_j$. It is $\mathfrak{b}^*\rtimes\mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c,w\rangle$ and (b,u) is the dual basis of (c,w). For topology, it is more valuable than $\mathfrak{g}_1$ / sl_2, but topology already got by other means almost everything $\mathfrak{g}_0$ gives.

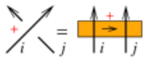
improved

How did these arise? $sl_2=\mathfrak{b}^+\oplus\mathfrak{b}^-/\mathfrak{h}=:sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+=\langle c,w\rangle/[w,c]=w$ is a Lie bialgebra with $\delta\colon\mathfrak{b}^+\to\mathfrak{b}^+\otimes\mathfrak{b}^+$ by $\delta\colon(c,w)\mapsto(0,c\wedge w)$. Going back, $sl_2^+=\mathcal{D}(\mathfrak{b}^+)=(\mathfrak{b}^+)^*\oplus\mathfrak{b}^+=\langle b,u,c,w\rangle/\cdots$. **Idea.** Replace $\delta\to\epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1}=0)$. At $k=0$, get $\mathfrak{g}_0$. At $k=1$, get $[w,c]=w$, $[w,b']=-\epsilon w$, $[c,u]=u$, $[b',u]=-\epsilon u$, $[b',c]=0$, and $[u,w]=b'-\epsilon c$. Now note that $b'+\epsilon c$ is central, so switch to $b:=b'+\epsilon c$. This is $\mathfrak{g}_1$.

improved

Ordering Symbols. $\mathbb{O}(poly\mid specs)$ plants the variables of *poly* in $S(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,
$$\mathbb{O}\left(c_1^3u_1c_2e^{u_3}w_3^9\middle|x\colon w_3c_1,\ y\colon u_1u_3c_2\right)=w^9c^3\otimes ue^uc\in\mathcal{U}(\mathfrak{g})_x\otimes\mathcal{U}(\mathfrak{g})_y$$
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r=Id\in\mathfrak{b}^-\otimes\mathfrak{b}^+$ solves the CYBE $[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes3}$ and, by luck,



$=R_{ij}=e^{r_{ij}}=e^{b_ic_j+u_iw_j}\in\mathcal{U}(\mathfrak{g}_{0,i}\oplus\mathfrak{g}_{0,j})$ solves YB/R3.

Lemma. $R_{ij}=e^{b_ic_j+u_iw_j}=\mathbb{O}\left(\exp\left(b_ic_j+\frac{e^{b_i}-1}{b_i}u_iw_j\right)\middle|i\colon u_i,\ j\colon c_jw_j\right)$

Example. $Z(T_0)=$ $\quad=\sum_{m,n}\frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!}u^n\otimes c^mw^n$.
$$\mathbb{O}\Bigg(\exp\left(b_5c_1+\frac{e^{b_5}-1}{b_5}u_5w_1+b_2c_4+\frac{e^{b_2}-1}{b_2}u_2w_4-b_3c_6+\frac{e^{-b_3}-1}{b_3}u_3w_6\right)\Bigg|\ x\colon c_1w_1u_2,\ y\colon u_3c_4w_4u_5c_6w_6\Bigg)=\mathbb{O}\left(\zeta\middle|x\colon u_xc_xw_x,\ y\colon u_yc_yw_y\right)$$
"ucw form"

Goal. Write ζ as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S:=\mathbb{Q}(b_i,e^{b_i})$, and $\omega\in R_S$.

new

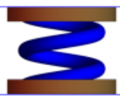
expand?

The Big $\mathfrak{g}_0$ Lemma. Under $[c,u]=u$, $[c,w]=-w$, and $[u,w]=b$:

1a. $N^{uc}:=\mathbb{O}(e^{\gamma c+\beta u}|uc)\overset{\rightarrow}{\equiv}\mathbb{O}(e^{\gamma c+e^{-\gamma}\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c}=e^{\gamma c}e^{e^{-\gamma}\beta u}$)
1b. $N^{wc}:=\mathbb{O}(e^{\gamma c+\alpha w}|wc)\overset{\rightarrow}{\equiv}\mathbb{O}(e^{\gamma c+e^{\gamma}\alpha w}|cw)$ …in the $\{ax+b\}$ group)
2. $\mathbb{O}(e^{\alpha w+\beta u}|wu)=\mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$ (the Weyl relations)
3. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u}=e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu=(1+b\delta)^{-1}$
(a. expand and crunch. b. use $w=b\hat{x}$, $u=\partial_x$. c. use "scatter and glow".)
4. $\mathbb{O}(e^{\delta uw}|wu)=\mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)
5. $N^{wu}:=\mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu)\overset{\rightarrow}{\equiv}\mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$
6. $N_k^{c_ic_j}:=\mathbb{O}(\zeta|c_ic_j)\overset{\rightarrow}{\equiv}\mathbb{O}(\zeta/(c_i,c_j\to c_k)|c_k)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition
$$u_ic_i\overline{w_iu_j}c_jw_j\xrightarrow{N_x^{w_iu_j}}u_i\overline{c_iu_x}\,\overline{w_xc_j}w_j\xrightarrow{N_x^{c_iu_x}/\!/N_x^{w_xc_j}}\overline{u_iu_x}\,\overline{c_xc_x}\,\overline{w_xw_j}\xrightarrow{i,j,x\to k}u_kc_kw_k$$



1-Smidgen Invariants. Much is the same:
The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 6 are the same, yet

ucw

5. $\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:
$$\begin{aligned}\Lambda=&-b\nu\left(\nu^2\alpha^2\beta^2+4\delta\nu\alpha\beta+2\delta^2\right)/2-\delta\nu^3(3b\delta+2)\beta^2u^2/2\\&-b\delta^4\nu^3u^2w^2/2-\delta^2\nu^3(2b\delta+1)\beta u^2w\\&-\nu^2(2b\delta+1)(\nu\alpha\beta+2\delta)\beta u-2b\delta^2\nu^2(\nu\alpha\beta+\delta)uw\\&+\delta\nu^3(b\delta+2)\alpha^2w^2/2+2(\nu\alpha\beta+\delta)c+2\delta\nu\beta cu+2\delta^2\nu cuw\\&+2\delta\nu\alpha cw+\delta^2\nu^3\alpha uw^2+\nu^2(\nu\alpha\beta+2\delta)\alpha w.\end{aligned}$$

changed

Proof. A lengthy computation. include a ref?
Problem. We now need to normal-order perturbed Gaussians!
Solution. Borrow some tactics from QFT:
$$\mathbb{O}(\epsilon P(c,u)e^{\gamma c+\beta u}|uc)=\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+\beta u}|uc)=\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+e^{-\gamma}\beta u}|cu),$$
and likewise

ucw

$$\mathbb{O}\left(\epsilon P(u,w)e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\epsilon P(\partial_\beta,\partial_\alpha)\nu e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$$

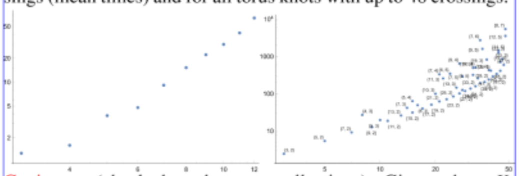
Note. Strand stitching requires a tiny extra step.
Finally, the values of the generators ⤡, ⤢, $\overrightarrow{n}$, and $\underset{\rightarrow}{u}$, are set by solving many equations, non-uniquely.

revised

Pragmatic Simplifications. Get rid of $\zeta=(e^b-1)/b$ factors by rescaling $u\to\bar{u}=\zeta u$. Complement this with $\beta\to\bar{\beta}=\zeta^{-1}\beta$, $\delta\to\bar{\delta}=\zeta^{-1}\delta$, $\epsilon\to\bar{\epsilon}=\zeta^{-1}\epsilon$. Simplify further by naming $e^b\to t$; e.g., $\nu\to\bar{\nu}=(1+(t-1)\delta)^{-1}$. Get confused by renaming $(\bar{u},\bar{\beta},\bar{\delta},\bar{\nu})\to(u,\beta,\delta,\nu)$, and more confused by working with $\mu=\nu^{-1}$ and $\mathbb{E}(\omega,L,Q,P):=\omega^{-1}(1+\epsilon\omega^{-4}P)e^{L+\omega^{-1}Q}$, where $\omega\in R:=\mathbb{Q}(t_k)$, $L=\sum l_{ij}b_ic_j$ with $l_{ij}\in\mathbb{Z}$, $Q=\sum q_{ij}u_iw_j$ with $q_{ij}\in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

Rough complexity estimate, after $t_k\to t$. n: xing number; w: width, maybe $\sim\sqrt{n}$.
$$\underset{A}{n}\overbrace{\sum_{d=0}^4\underset{E}{w^{4-d}}\,\underset{F}{w^d}\,\underset{G}{n^2}}^{B}=n^3w^4\in[n^5,n^7]$$
A: go over stitchings in order. B: multiplication ops per $N^{u_iw_j}$. d: deg of u_i,w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Experimental Analysis (ωεβ/Exp). Log-log plot of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial A, there is a polynomial ρ_1 such that
$$P=A^2\frac{(t-1)^3\rho_1+t^2(2\nu w+(1-t)(1-2c))AA'}{(1-t)t}.$$
Furthermore, A and ρ_1 are symmetric under $t\to t^{-1}$, so let A^+ and ρ_1^+ be their "positive parts", so e.g., $\rho_1(t)=\rho_1^+(t)+\rho_1^+(t^{-1})-\rho_1^+(0)$.
Power. On the 250 knots with at most 10 crossings, the pair (A,ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 crossings, always $\deg \rho_1^+ \leq 2g-1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-crossing Alexander failures it does give the right answer.

Demo Programs for 0-Co. ωεβ/Demo

The R-matrices

```
R⁺_{0,i_,j_} := E[b_i c_j + b_i^-1 (e^{b_i} - 1) u_i w_j];
R⁻_{0,i_,j_} := E[-b_i c_j + b_i^-1 (e^{-b_i} - 1) u_i w_j];
```

Utilities

```
CF[ω_. E[Q_]] := Simplify[ω] E[Simplify[Q]];
E /: E[Q1_] E[Q2_] := CF@E[Q1 + Q2];
ω1_. E[Q1_] ≡ ω2_. E[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

Normal Ordering Operators

```
N_{(x:w|u)_i_ c_j_→k_}[ω_. E[Q_]] := CF[
   ω E[e^γ α x_k + γ c_k + (Q /. c_j | x_i → 0)] /. {γ → ∂_{c_j} Q, α → ∂_{x_i} Q}];
N_{w_i_ u_j_→k_}[ω_. E[Q_]] := CF[        interchange
   ν ω E[-b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k + (Q /. w_i | u_j → 0)] /.
    ν → (1 + b_k δ)^-1 /.
   {α → ∂_{w_i} Q /. u_j → 0, β → ∂_{u_j} Q /. w_i → 0, δ → ∂_{w_i,u_j} Q}];
```

Stitching

```
m_{i_,j_→k_}[Z_] := Module[{x, z},
  CF[(Z // N_{w_i u_j→x} // N_{c_i u_x→x} // N_{w_x c_j→x}) /. z_{i|j|x} → z_k]]
```

Some calculations for T_0

```
T0 = R⁺_{0,5,1} R⁺_{0,2,4} R⁻_{0,3,6}
```

$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5}) u_5 w_1}{b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} + \frac{(-1+e^{-b_3}) u_3 w_6}{b_3}\right]$

```
T0 // m_{1,2→1} // m_{3,4→3} // m_{3,5→3} // m_{3,6→3}
```

$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})} \mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{e^{b_3}(-1+e^{b_1})(-1+e^{b_3}) u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_1} - \frac{e^{b_1}(-1+e^{b_3}) u_3 w_1}{(-1+(-1+e^{b_1})(-1+e^{b_3})) b_3} - \frac{e^{-b_3}(-1+e^{b_3}) u_3 w_3}{b_3} - \frac{e^{-b_3}(-1+e^{b_1})(-e^{b_3} b_3 u_1 + e^{b_1}(-1+e^{b_3}) b_1 u_3) w_3}{b_1 (b_3 - (-1+e^{b_1})(-1+e^{b_3}) b_3)}\Big]$

Verifying meta-associativity

```
Q0 = E[Sum[f_i c_i, {i, 3}] + Sum[f_{i,j} u_i w_j, {i, 3}, {j, 3}]]
```

$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$

```
(Q0 // m_{1,2→1} // m_{1,3→1}) ≡ (Q0 // m_{2,3→2} // m_{1,2→1})
True
```

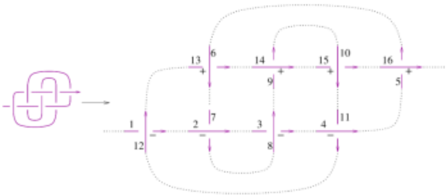
Testing R3

```
t1 = R⁺_{0,1,2} R⁺_{0,3,4} R⁺_{0,5,6} // m_{3,5→x} // m_{1,6→y} // m_{2,4→z}
```

$\mathbb{E}\left[b_x c_y + b_x c_z + b_y c_z + \frac{e^{b_x}(-1+e^{b_y}) u_y w_z}{b_y} + \frac{(-1+e^{b_x}) u_x (w_y+w_z)}{b_x}\right]$

```
t1 ≡ (R⁺_{0,1,2} R⁺_{0,3,4} R⁺_{0,5,6} // m_{1,3→x} // m_{2,5→y} // m_{4,6→z})
True
```

8_{17}




```
z1 = R⁻_{0,12,1} R⁻_{0,2,7} R⁻_{0,8,3} R⁻_{0,4,11} R⁺_{0,16,5} R⁺_{0,6,13} R⁺_{0,14,9} R⁺_{0,10,15};
Do[z1 = (z1 // m_{1,n→1}) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11 - \frac{1}{t^3} + \frac{4}{t^2} - \frac{8}{t} - 8t + 4t^2 - t^3\right\}$

Demo Programs for 1-Co. ωεβ/Demo

The Λόγος

```
Λ[k_] := ((t_k - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 v_k c_k w_k δ^2 μ^2 -
    δ (1 + μ) (w_k^2 α^2 + v_k^2 β^2) - v_k^2 w_k^2 δ^3 (1 + 3 μ) -
    2 (α β + 2 δ μ + v_k w_k δ^2 (1 + 2 μ) + 2 c_k δ μ^2) (w_k α + v_k β) -
    4 (c_k μ^2 + v_k w_k δ (1 + μ)) (α β + δ μ)) (1 + t_k) / 4;
```

The Generators

```
R⁺_{i_,j_} := E[1, Log[t_i] c_j, v_i w_j, v_i c_i w_j + c_i c_j + v_i^2 w_j^2 / 4];
R⁻_{i_,j_} := E[1, -Log[t_i] c_j, -t_i^-1 v_i w_j,
   -c_i c_j + t_i^-1 v_i c_j w_j - t_i^-2 v_i^2 w_j^2 / 4];
(ur_{i_} := E[t_i^{1/2}, 0, 0, c_i t_i^-2]; nr_{i_} := E[t_i^{1/2}, 0, 0, -c_i t_i^2];)
```

Differential Polynomials

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := (* means P[∂_α, ∂_β][f] *)
  Total[CoefficientRules[P, {x, y}] /.
    ({m_, n_} → c_) :> c D[f, {α, m}, {β, n}]]
```

Utilities

```
CF[ℰ_E] := Expand /@ Together /@ ℰ;
E /: E[ω1_, L1_, Q1_, P1_] E[ω2_, L2_, Q2_, P2_] :=
  CF@E[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

Normal Ordering Operators

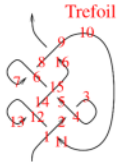
```
N_{c_j_ (x:v|w)_i_→k_}[E[ω_, L_, Q_, P_]] := With[{q = e^γ β x_k + γ c_k}, CF[
    E[ω, γ c_k + (L /. c_j → 0), ω e^γ β x_k + (Q /. x_i → 0),
      e^-q DP_{c_j→D_γ, x_i→D_β}[P][e^q]] /. {γ → ∂_{c_j} L, β → ω^-1 ∂_{x_i} Q}]];
N_{w_i_ v_j_→k_}[E[ω_, L_, Q_, P_]] :=        interchange
  With[{q = ((1 - t_k) α β + β v_k + δ v_k w_k + α w_k) / μ}, CF[
    E[μ ω, L, μ ω q + μ (Q /. w_i | v_j → 0),
       μ^4 e^-q DP_{w_i→D_α, v_j→D_β}[P][e^q] + ω^4 Λ[k]] /. μ → 1 + (t_k - 1) δ /.
     {α → ω^-1 (∂_{w_i} Q /. v_j → 0), β → ω^-1 (∂_{v_j} Q /. w_i → 0),
      δ → ω^-1 ∂_{w_i,v_j} Q}]];
```

optimize!
as in
2016-09/...optimization-s.nb
and beyond.

Stitching

```
m_{i_,j_→k_}[Z_E] := Module[{x, z},
  CF[(Z // N_{w_i v_j→x} // N_{c_i v_x→x} // N_{w_x c_j→x}) /. z_{i|j|x} → z_k]]
```

The 0-Framed Trefoil




```
z2 = R⁺_{1,11} R⁻_{4,2} nr_3 R⁺_{15,5} R⁻_{6,8} ur_7 R⁺_{9,16} nr_10 R⁻_{12,14} ur_13;
(Do[z2 = z2 // m_{1,k→1}, {k, 2, 16}];
 z2 = z2 /. a_1 :> a)
```

$\mathbb{E}\Big[-1 + \frac{1}{t} + t,\ 0,\ 0,\ 16 + \frac{2c}{t^4} - \frac{1}{t^3} - \frac{6c}{t^3} + \frac{4}{t^2} + \frac{10c}{t^2} - \frac{10}{t} - \frac{8c}{t} - 18t + 8ct + 14t^2 - 10ct^2 - 7t^3 + 6ct^3 + 2t^4 - 2ct^4 + 2vw - \frac{2vw}{t^4} + \frac{4vw}{t^3} - \frac{6vw}{t^2} + \frac{2vw}{t} - 6tvw + 4t^2vw - 2t^3vw\Big]$

Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $b^+ \leftrightarrow b^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra.

• k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods. Help Needed!

balance columns?

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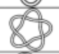
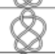
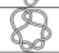
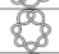
diagram	n_k^t	Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t	Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral
	0_1^a	1 0	0 / ✔ 0 / ✔		3_1^a	$t-1$ t	1 / ✘ 1 / ✘
	4_1^a	$3-t$ 0	1 / ✘ 1 / ✔		5_1^a	t^2-t+1 $2t^3+3t$	2 / ✘ 2 / ✘
	5_2^a	$2t-3$ $5t-4$	1 / ✘ 1 / ✘		6_1^a	$5-2t$ $t-4$	1 / ✔ 1 / ✘
	6_2^a	$-t^2+3t-3$ t^3-4t^2+4t-4	2 / ✘ 1 / ✘		6_3^a	t^2-3t+5 0	2 / ✘ 1 / ✔
	7_1^a	t^3-t^2+t-1 $3t^5+5t^3+6t$	3 / ✘ 3 / ✘		7_2^a	$3t-5$ $14t-16$	1 / ✘ 1 / ✘
	7_3^a	$2t^2-3t+3$ $-9t^3+8t^2-16t+12$	2 / ✘ 2 / ✘		7_4^a	$4t-7$ $32-24t$	1 / ✘ 2 / ✘
	7_5^a	$2t^2-4t+5$ $9t^3-16t^2+29t-28$	2 / ✘ 2 / ✘		7_6^a	$-t^2+5t-7$ $t^3-8t^2+19t-20$	2 / ✘ 1 / ✘
	7_7^a	t^2-5t+9 $8-3t$	2 / ✘ 1 / ✘		8_1^a	$7-3t$ $5t-16$	1 / ✘ 1 / ✘
	8_2^a	$-t^3+3t^2-3t+3$ $2t^5-8t^4+10t^3-12t^2+13t-12$	3 / ✘ 2 / ✘		8_3^a	$9-4t$ 0	1 / ✘ 2 / ✔
	8_4^a	$-2t^2+5t-5$ $3t^3-8t^2+6t-4$	2 / ✘ 2 / ✘		8_5^a	$-t^3+3t^2-4t+5$ $-2t^5+8t^4-13t^3+20t^2-22t+24$	3 / ✘ 2 / ✘
	8_6^a	$-2t^2+6t-7$ $5t^3-20t^2+28t-32$	2 / ✘ 2 / ✘		8_7^a	t^3-3t^2+5t-5 $-t^5+4t^4-10t^3+12t^2-13t+12$	3 / ✘ 1 / ✘