MIT-1612 handout on 161125

November 25, 2016 9:36 AM

Solve the PgUp/PgDn HandoutBrowser bug!

choices may change?

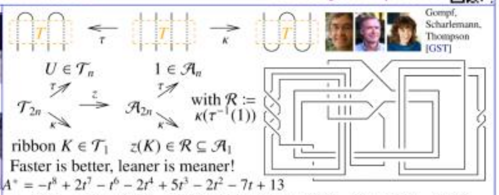
Dror Bar-Natan: Talks: MIT-1612: (thanks for accepting my invitation!) ωεβ:=http://drorbn.net/MIT-1612/

A Poly-Time Knot Polynomial Via Solvable Approximation

Work in Progress! Help Needed!

Abstract. Rozansky [Ro2] and Overbay [Ov] described a spectacular knot polynomial that failed to attract the attention it deserved as the first poly-time-computable knot polynomial since Alexander's [Al, 1928] and (in my opinion) as the second most likely knot polynomial (after Alexander's) to carry topological information. With Roland van der Veen, I will explain how to compute the Rozansky polynomial using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.

Rozansky Overbay



Gompf, Scharlemann, Thompson [GST]

ribbon $K \in \mathcal{T}_1$ $z(K) \in \mathcal{R} \subseteq \mathcal{A}_1$

Faster is better, leaner is meaner!

$A^* = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$

$\rho_1^* = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot A(K)(e^\hbar) = 1$.

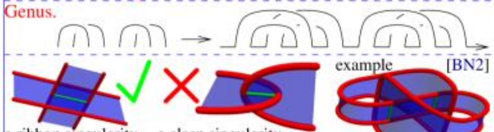
"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})A(K)(q^d)}\left(1 + \sum_{k=1}^\infty \frac{(q-1)^k R_k(K)(q^d)}{A^{2k}(K)(q^d)}\right).$$

Why "spectacular"? Foremost reason: *OBVIOUSLY*. Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Also, will bound genus and may disprove {ribbon} = {slice}.

Genus.



example [BN2]

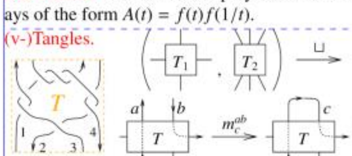
a ribbon singularity a clasp singularity

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

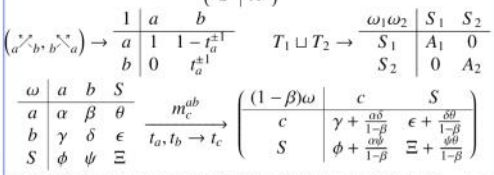
Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

(v-)Tangles.



(meta-associativity: $m^{ab}_x /\!\!/ m^{xc}_y = m^{bc}_x /\!\!/ m^{ax}_y$)

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a : a \in S\})$:



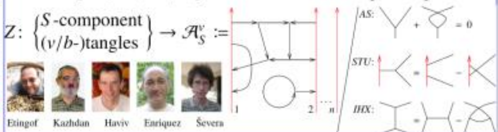
(Roland: "add to A the product of column b and row a, divide by $(1 - A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

(There are also formulas for strand doubling and strand reversal).

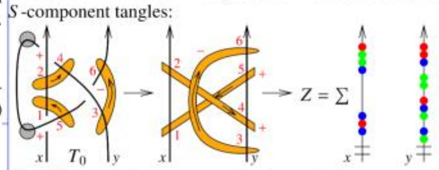
Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$Z: \left\{\begin{array}{c}S\text{-component}\\(v/b\text{-})\text{tangles}\end{array}\right\} \to \mathcal{A}^v_S :=$



Etingof Kazhdan Haviv Enriquez Ševera

Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



with •:c •:u •:w —:b

Good News. In theory, enough to know R, the cuaps, and stitching/multiplication $m^{ij}_k : A_i \otimes A_j \to A_k$.

Problem. Extract information out of Z.

Textbook Solution. Use representation theory …works, slowly.

Today's Solution (with van der Veen). For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$\left\{\begin{array}{c}\text{ordered perturbed}\\\text{Gaussian formulas}\end{array}\right\} \to \hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ van der Veen

bring to front?

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified) www.katlas.org The Knot Atlas

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and with $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset 0$. (note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon = 0$, or $\mathbb{Q}\langle b, c, u, w\rangle/([b, \cdot] = 0, [c, u] = u, [c, w] = -w, [u, w] = b$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{b}^* \rtimes \mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c, w\rangle$ and (b, u) is the dual basis of (c, w). For topology, it is more valuable than $\mathfrak{g}_1$ / sl_2, but topology already got by other means almost everything $\mathfrak{g}_0$ gives.

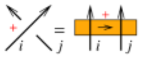
Improve

How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w\rangle/[w, c] = w$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (c, w) \mapsto (0, c \wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get $\mathfrak{g}_0$. At $k = 1$, get $[w, c] = w$, $[w, b'] = -\epsilon w$, $[c, u] = u$, $[b', u] = -\epsilon u$, $[b', c] = 0$, and $[u, w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is $\mathfrak{g}_1$.

Improve

Ordering Symbols. $\mathbb{O}$ (*poly* | *specs*) plants the variables of *poly* in $S(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,
$$\mathbb{O}\left(c_1^3 u_1 c_2 e^{u_3} w_3^9 | x\colon w_3 c_1,\ y\colon u_1 u_3 c_2\right) = w^9 c^3 \otimes u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$$
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$= R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$ solves YB/R3.

Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \mathbb{O}\left(\exp\left(b_i c_j + \frac{e^{b_i}-1}{b_i} u_i w_j\right) | i\colon u_i,\ j\colon c_j w_j\right)$

Example. $Z(T_0) = \qquad = \sum_{m,n} \frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!} u^n \otimes c^m w^n.$
$$\mathbb{O}\Big(\exp\left(b_5 c_1 + \tfrac{e^{b_5}-1}{b_5} u_5 w_1 + b_2 c_4 + \tfrac{e^{b_2}-1}{b_2} u_2 w_4 - b_3 c_6 + \tfrac{e^{-b_3}-1}{b_3} u_3 w_6\right) | x\colon c_1 w_1 u_2,\ y\colon u_3 c_4 w_4 u_5 c_6 w_6\Big) = \mathbb{O}(\zeta | x\colon c_x u_x w_x,\ y\colon c_y u_y w_y)$$
"cuw form"

Goal. Write ζ as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S := \mathbb{Q}(b_i, e^{b_i})$, and $\omega \in R_S$.

expand 2 6

The Big $\mathfrak{g}_0$ Lemma. Under $[c, u] = u$, $[c, w] = -w$, and $[u, w] = b$:

1a. $N^{uc} := \mathbb{O}(e^{\gamma c + \beta u} | uc) \stackrel{\rightarrow}{=} \mathbb{O}(e^{\gamma c + e^{-\gamma}\beta u} | cu)$ (means $e^{\beta u} e^{\gamma c} = e^{\gamma c} e^{e^{-\gamma}\beta u}$

1b. $N^{wc} := \mathbb{O}(e^{\gamma c + \alpha w} | wc) \stackrel{\rightarrow}{=} \mathbb{O}(e^{\gamma c + e^{\gamma}\alpha w} | cw)$ …in the $\{ax + b\}$ group)

2. $\mathbb{O}(e^{\alpha w + \beta u} | wu) = \mathbb{O}(e^{-b\alpha\beta + \alpha w + \beta u} | uw)$ (the Weyl relations)

3. $\mathbb{O}(e^{\delta uw} | wu) e^{\beta u} = e^{\nu\beta u} \mathbb{O}(e^{\delta uw} | wu)$, with $\nu = (1 + b\delta)^{-1}$

(a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)

4. $\mathbb{O}(e^{\delta uw} | wu) = \mathbb{O}(\nu e^{\nu\delta uw} | uw)$ (same techniques)

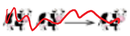
5. $N^{wu} := \mathbb{O}(e^{\beta u + \alpha w + \delta uw} | wu) \stackrel{\rightarrow}{=} \mathbb{O}(\nu e^{-b\nu\alpha\beta + \nu\alpha w + \nu\beta u + \nu\delta uw} | uw)$

6. $N_k^{c_i c_j} := \mathbb{O}(\zeta | c_i c_j) \stackrel{\rightarrow}{=} \mathbb{O}(\zeta/(c_i, c_j \to c_k) | c_k)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

change to ucw

Strand Stitching, m_k^{ij}, is defined as the composition
$$c_i u_i \overline{w_i c_j} u_j w_j \xrightarrow{N_k^{w_i c_j}} c_i \overline{u_i c_k}\, \overline{w_k u_j}\, w_j \xrightarrow{N_k^{u_i c_k} /\!/ N_k^{w_k u_j}} \overline{c_i c_k}\, \overline{u_k u_k}\, \overline{w_k w_j} \xrightarrow{N_k^{c_i c_k} /\!/ -- /\!/ N_k^{w_k w_j}} c_k u_k w_k$$



1-Smidgen Invariants. Much is the same:

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 6 are the same, yet

5. $\mathbb{O}\left(e^{\alpha w + \beta u + \delta uw} | wu\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\Lambda) e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} | cuw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:
$$\begin{aligned}\Lambda = &-b\nu\left(\nu^2\alpha^2\beta^2 + 4\delta\nu\alpha\beta + 2\delta^2\right)/2 - \delta\nu^3(3b\delta + 2)\beta^2 u^2/2 \\ &- b\delta^4\nu^3 u^2 w^2/2 - \delta^2\nu^3(2b\delta + 1)\beta u^2 w \\ &- \nu^2(2b\delta + 1)(\nu\alpha\beta + 2\delta)\beta u - 2b\delta^2\nu^2(\nu\alpha\beta + \delta)uw \\ &+ \delta\nu^3(b\delta + 2)\alpha^2 w^2/2 + 2(\nu\alpha\beta + \delta)c + 2\delta\nu\beta cu + 2\delta^2\nu cuw \\ &+ 2\delta\nu\alpha cw + \delta^2\nu^3\alpha uw^2 + \nu^2(\nu\alpha\beta + 2\delta)\alpha w.\end{aligned}$$

include a ref2

Proof. ~~A brutal hell.~~ A lengthy computation

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:
$$\mathbb{O}(\epsilon P(c, u) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + e^{-\gamma}\beta u} | cu),$$
and likewise
$$\mathbb{O}\left(\epsilon P(u, w) e^{\alpha w + \beta u + \delta uw} | wu\right) = \mathbb{O}\left(\epsilon P(\partial_\beta, \partial_\alpha) \nu e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} | cuw\right)$$

Note. Strand stitching requires a tiny extra step.

Finally, the values of the generators ⤫, ⤬, $\overrightarrow{n}$, ~~↺~~, $\underset{\to}{u}$, and ~~↻~~ are set by ~~brutally~~ solving many equations, non-uniquely.

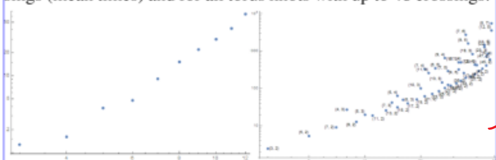
Pragmatic Simplifications. Get rid of $\zeta = (e^b - 1)/b$ factors by rescaling $u \to \bar{u} = \zeta u$. Complement this with $\beta \to \bar{\beta} = \zeta^{-1}\beta$, $\delta \to \bar{\delta} = \zeta^{-1}\delta$, $\epsilon \to \bar{\epsilon} = \zeta^{-1}\epsilon$. Simplify further by naming $e^b \to t$; e.g., $\nu \to \bar{\nu} = (1 + (t - 1)\delta)^{-1}$. Get confused by renaming $(\bar{u}, \bar{\beta}, \bar{\delta}, \bar{\nu}) \to (u, \beta, \delta, \nu)$, and more confused by working with $\mu = \nu^{-1}$ and $\mathbb{E}(\omega, L, Q, P) := \omega^{-1}(1 + \epsilon\omega^{-4}P)e^{L + \omega^{-1}Q}$, where $\omega \in R := \mathbb{Q}(t_k)$, $L = \sum l_{ij} b_i c_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij} u_i w_j$ with $q_{ij} \in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

revise

Rough complexity estimate, after $t_k \to t$. n: xing number; w: width, maybe $\sim \sqrt{n}$.
$$\underbrace{n}_{A} \overbrace{\sum_{d=0}^{4} \underbrace{w^{4-d}}_{E} \underbrace{w^d}_{F} \underbrace{n^2}_{G}}^{B} = n^3 w^4 \in [n^5, n^7]$$
A: go over stitchings in order. B: multiplication ops per $N^{u_i w_j}$. d: deg of u_i, w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

ucw

Experimental Analysis (ωεβ/Exp). Log-log plot of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



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Conjecture (checked on the same collections). Given a knot K with Alexander polynomial A, there is a polynomial ρ_1 such that
$$P = A^2\left((t - 2 + t^{-1})\rho_1 + tAA'\left(\frac{(4 + t - t^2)(uw + (t - 1)c)}{2(t - 1)} - 1\right)\right).$$

revise

Furthermore, A and ρ_1 are symmetric under $t \to t^{-1}$, so let A^+ and ρ_1^+ be their "positive parts", so e.g., $\rho_1(t) = \rho_1^+(t) + \rho_1^+(t^{-1}) - \rho_1^+(0)$.

Power. On the 250 knots with at most 10 crossings, the pair (A, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 crossings, always $\deg\rho_1^+ \leq 2g-1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This is bound is often weaker than the Alexander bound, yet for 10 of the 12-crossing Alexander failures it does give the right answer.

Demo Programs for 0-Co. ωεβ/Demo

The R-matrices

```
R⁺0,i_,j_ := 𝔼[bi cj + bi⁻¹ (e^bi − 1) ui wj];
R⁻0,i_,j_ := 𝔼[−bi cj + bi⁻¹ (e^−bi − 1) ui wj];
```

Utilities

```
CF[ω_. 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

Normal Ordering Operators

```
N_{ui_ cj_→k_}[ω_. 𝔼[Q_]] := CF[
   ω 𝔼[e^−γ β uk + γ ck + (Q /. cj | ui → 0)] /. {γ → ∂cj Q, β → ∂ui Q}];
N_{wi_ cj_→k_}[ω_. 𝔼[Q_]] := CF[
   ω 𝔼[e^γ α wk + γ ck + (Q /. cj | wi → 0)] /. {γ → ∂cj Q, α → ∂wi Q}];
N_{wi_ uj_→k_}[ω_. 𝔼[Q_]] := CF[
   ν ω 𝔼[−bk ν α β + ν β uk + ν δ uk wk + ν α wk + (Q /. wi | uj → 0)] /.
    ν → (1 + bk δ)^−1 /.
    {α → ∂ui Q /. uj → 0, β → ∂uj Q /. wi → 0, δ → ∂wi,uj Q}];
```

Stitching

```
m_{i_,j_→k_}[ω_. 𝔼[Q_]] := CF[Module[{x},
   (ω 𝔼[Q] /. bi|j → bk // N_{ui cj→x} // N_{ui cx→x} // N_{wx uj→x}) /.
    {ci → ck, wj → wk, y_x :→ yk}]]
```

Some calculations for T_0

```
T0 = R⁺0,5,1 R⁺0,2,4 R⁻0,3,6
```

$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5}) u_5 w_1}{b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} + \frac{(-1+e^{-b_3}) u_3 w_6}{b_3}\right]$

```
T0 // m1,2→1 // m3,4→3 // m3,5→3 // m3,6→3
```

$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\mathbb{E}\left[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{(-1+e^{b_1})(-1+e^{b_3}) u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_1} - \frac{e^{-b_3}(-1+e^{b_1})(b_3 u_1 - e^{b_3}(-1+e^{b_3}) b_1 u_3) w_3}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_1 b_3} + \frac{e^{-b_1}(-1+e^{b_3}) u_3 (-e^{b_1+b_3} w_1 + (e^{b_1}+e^{b_3}-e^{b_1+b_3}) w_3)}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_3}\right]$

Verifying meta-associativity

```
Q0 = 𝔼[Sum[fi ci, {i, 3}] + Sum[fi,j ui wj, {i, 3}, {j, 3}]]
```

$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$

```
(Q0 // m1,2→1 // m1,3→1) ≡ (Q0 // m2,3→2 // m1,2→1)
True
```

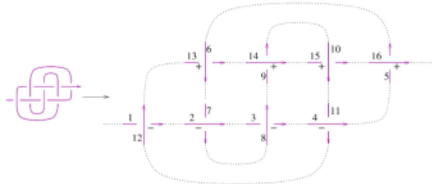
Testing R3

```
t1 = R⁺0,1,2 R⁺0,3,4 R⁺0,5,6 // m3,5→x // m1,6→y // m2,4→z
```

$\mathbb{E}\left[b_x (c_y + c_z) + \frac{(-1+e^{b_x}) u_x (w_y - w_z)}{b_x} + \frac{b_y^2 c_z + (-1+e^{b_y}) u_y w_z}{b_y}\right]$

```
t1 ≡ (R⁺0,1,2 R⁺0,3,4 R⁺0,5,6 // m1,3→x // m2,5→y // m4,6→z)
True
```

8_{17}




```
z1 = R⁻0,12,1 R⁻0,2,7 R⁻0,8,3 R⁻0,4,11 R⁺0,16,5 R⁺0,6,13 R⁺0,14,9 R⁺0,10,15;
Do[z1 = (z1 // m1,n→1) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$

Demo Programs for 1-Co. ωεβ/Demo

The Λόγος

```
Λ[k_] :=
  ((tk − 1) (2 (α β + δ μ)² − α² β²) − 4 νk ck wk δ² μ² −
    δ (1 + μ) (wk² α² + νk² β²) − νk² wk² δ³ (1 + 3 μ) −
    2 (α β + 2 δ μ + νk wk δ² (1 + 2 μ) + 2 ck δ μ²) (wk α + νk β) −
    4 (ck μ² + νk wk δ (1 + μ)) (α β + δ μ)) (1 + tk) / 4;
```

The Generators

```
R⁺i_,j_ := 𝔼[1, Log[ti] cj, vi wj, vi ci wj + ci cj + vi² wj²/4];
R⁻i_,j_ := 𝔼[1, −Log[ti] cj, −ti⁻¹ vi wj,
   −ci cj + ti⁻¹ vi cj wj − ti⁻² vi² wj²/4];
uri_ := 𝔼[ti^−1/2, 0, 0, ci ti⁻²];
nri_ := 𝔼[ti^1/2, 0, 0, −ci ti²];
```

Differential Polynomials

```
DP_{x_→Dα_, y_→Dβ_}[P_][f_] := (* means P[∂α,∂β][f] *)
  Total[CoefficientRules[P, {x, y}] /.
    ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

Utilities

```
CF[ℰ_𝔼] := Expand /@ Together /@ ℰ;
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2⁴ P1 + ω1⁴ P2];
```

Normal Ordering Operators

```
N_{cj_ (x:v|w)i_→k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^γ β xk + γ ck}, CF[
   𝔼[ω, γ ck + (L /. cj → 0), ω e^γ β xk + (Q /. xi → 0),
    e^−q DP_{cj→Dγ, xi→Dβ}[P][e^q]] /. {γ → ∂cj L, β → ω⁻¹ ∂xi Q}]];
N_{wi_ vj_→k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{q = ((1 − tk) α β + β νk + δ νk wk + α wk) / μ}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. wi | vj → 0),
      μ⁴ e^−q DP_{wi→Dα, vj→Dβ}[P][e^q] + ω⁴ Λ[k]] /. μ → 1 + (tk − 1) δ /.
    {α → ω⁻¹ (∂wi Q /. vj → 0), β → ω⁻¹ (∂vj Q /. wi → 0),
     δ → ω⁻¹ ∂wi,vj Q}]];
```

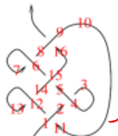
Stitching

```
m_{i_,j_→k_}[Z_] := Module[{x, z}, CF[
   Z // N_{wi vj→x} // N_{ci vx→x} // N_{wx cj→x} // ReplaceAll[z_{i|j|x} → zk]]]
```

The 0-Framed Trefoil

```
z2 = R⁺1,11 R⁺4,2 nr3 R⁺15,5 R⁺6,8 ur7 R⁺9,16 nr10 R⁺12,14 ur13;
(Do[z2 = z2 // m1,k→1, {k, 2, 16}];
 z2 = z2 /. a_1 :→ a)
```

$\mathbb{E}\Big[-1+\frac{1}{t}+t,\ 0,\ 0,\ 16+\frac{2c}{t^4}-\frac{1}{t^3}-\frac{6c}{t^3}+\frac{4}{t^2}+\frac{10c}{t^2}-\frac{10}{t}-\frac{8c}{t}-18t+8ct+14t^2-10ct^2-7t^3+6ct^3+2t^4-2ct^4+2vw-\frac{2vw}{t^4}+\frac{4vw}{t^3}-\frac{6vw}{t^2}+\frac{2vw}{t}-6tvw+4t^2vw-2t^3vw\Big]$



Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand

merge
unify w/ 1-Co.
try to cut one line
Try to cut one line
break green background by units.
Include optimization from
2016-09/nb/OneSmidgenOptimization-5.pdf
denser.
overlap.

doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods. Help Needed!

~~Disclaimer. This is all quite new. The overall picture is correct, but many pieces are certainly not in their final form yet.~~

balance columns!!

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diagram	n^t_k Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral	diagram	n^t_k Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral
	0^a_1 1 0	0 / ✔ 0 / ✔		3^a_1 $t-1$ t	1 / ✘ 1 / ✘
	4^a_1 $3-t$ 0	1 / ✘ 1 / ✔		5^a_1 t^2-t+1 $2t^3+3t$	2 / ✘ 2 / ✘
	5^a_2 $2t-3$ $5t-4$	1 / ✘ 1 / ✘		6^a_1 $5-2t$ $t-4$	1 / ✔ 1 / ✘
	6^a_2 $-t^2+3t-3$ t^3-4t^2+4t-4	2 / ✘ 1 / ✘		6^a_3 t^2-3t+5 0	2 / ✘ 1 / ✔
	7^a_1 t^3-t^2+t-1 $3t^5+5t^3+6t$	3 / ✘ 3 / ✘		7^a_2 $3t-5$ $14t-16$	1 / ✘ 1 / ✘
	7^a_3 $2t^2-3t+3$ $-9t^3+8t^2-16t+12$	2 / ✘ 2 / ✘		7^a_4 $4t-7$ $32-24t$	1 / ✘ 2 / ✘
	7^a_5 $2t^2-4t+5$ $9t^3-16t^2+29t-28$	2 / ✘ 2 / ✘		7^a_6 $-t^2+5t-7$ $t^3-8t^2+19t-20$	2 / ✘ 1 / ✘
	7^a_7 t^2-5t+9 $8-3t$	2 / ✘ 1 / ✘		8^a_1 $7-3t$ $5t-16$	1 / ✘ 1 / ✘
	8^a_2 $-t^3+3t^2-3t+3$ $2t^5-8t^4+10t^3-12t^2+13t-12$	3 / ✘ 2 / ✘		8^a_3 $9-4t$ 0	1 / ✘ 2 / ✔
	8^a_4 $-2t^2+5t-5$ $3t^3-8t^2+6t-4$	2 / ✘ 2 / ✘		8^a_5 $-t^3+3t^2-4t+5$ $-2t^5+8t^4-13t^3+20t^2-22t+24$	3 / ✘ 2 / ✘
	8^a_6 $-2t^2+6t-7$ $5t^3-20t^2+28t-32$	2 / ✘ 2 / ✘		8^a_7 t^3-3t^2+5t-5 $-t^5+4t^4-10t^3+12t^2-13t+12$	3 / ✘ 1 / ✘