# MIT-1612 Post Mortem

December 11, 2016 7:20 AM

Rewrite the bi-algebras box following the GWU-1612 talking points

I wish I had started the talk w/ that story!!!

Dror Bar-Natan: Talks: MIT-1612: (thanks for accepting my invitation!)

ωεβ:=http://drorbn.net/MIT-1612/

## A Poly-Time Knot Polynomial Via Solvable Approximation

Work in Progress! Fluid! Help Needed!

**Abstract.** Rozansky [Ro2] and Overbay [Ov] described a spectacular knot polynomial that failed to attract the attention it deserved as the first poly-time-computable knot polynomial since Alexander's [Al, 1928] and (in my opinion) as the second most likely knot polynomial (after Alexander's) to carry topological information. With Roland van der Veen, I will explain how to compute the Rozansky polynomial using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra $sl_2$.

Rozansky

Overbay

**Theorem** ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of $K$, in the $d$-dimensional representation of $sl_2$. Writing

Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar}=\sum_{j,m\geq 0}a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar)=1$.

"Above diagonal" we have **Rozansky's Theorem** [Ro3, (1.2)]:

$$J_d(K)(q)=\frac{q^d-q^{-d}}{(q-q^{-1})A(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^kR_k(K)(q^d)}{A^{2k}(K)(q^d)}\right).$$

**Why "spectacular"?** Foremost reason: *OBVIOUSLY.* Cf. proving (incomputable $A$)=(incomputable $B$), or categorifying (incomputable $C$).

Also, will bound genus and may disprove {ribbon} = {slice}.

**(v-)Tangles.**

$$\left(T_1,\ T_2\right)\xrightarrow{\sqcup}T_1\ T_2$$

$$T\xrightarrow{m^{ab}_c}T$$

(meta-associativity: $m^{ab}_x /\!\!/ m^{xc}_y=m^{bc}_x /\!\!/ m^{ax}_y$) (tangles are generated by ╳ and ╳)

**Genus.**

example [BN2]

a ribbon singularity a clasp singularity

**A bit about ribbon knots.** A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3=\partial B^4$ which is the boundary of a non-singular disk in $B^4$. Every ribbon knots is clearly slice, yet,

**Conjecture.** Some slice knots are not ribbon.

**Fox-Milnor.** The Alexander polynomial of a ribbon knot is always of the form $A(t)=f(t)f(1/t)$. (also for slice)

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Gompf, Scharlemann, Thompson [GST]

$$\begin{tikzcd} & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n \\ \mathcal{T}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] \arrow[rr, "z"] & & \mathcal{A}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] & \\ & & & \end{tikzcd}$$

with $\mathcal{R}:=\kappa(\tau^{-1}(1))$

ribbon $K\in\mathcal{T}_1$ $z(K)\in\mathcal{R}\subseteq\mathcal{A}_1$

Faster is better, leaner is meaner!

$A^+=-t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$

$\rho_1^+=5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$

**The Gold Standard** is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An $S$-component tangle $T$ has $\Gamma(T)\in R_S\times M_{S\times S}(R_S)=\left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S:=\mathbb{Z}(\{t_a\colon a\in S\})$:

$$\left({}_a\nearrow_b,\ {}_b\nwarrow_a\right)\to\begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-t_a^{\pm1}\\ b & 0 & t_a^{\pm1}\end{array}\qquad T_1\sqcup T_2\to\begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array}\xrightarrow[t_a,t_b\to t_c]{m^{ab}_c}\left(\begin{array}{c|cc}(1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

(Roland: "add to $A$ the product of column $b$ and row $a$, divide by $(1-A_{ab})$, delete column $b$ and row $a$".)

For long knots, $\omega$ is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

(There are also formulas for strand doubling and strand reversal).

**Theorem** [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z\colon\left\{\begin{matrix}S\text{-component}\\ (v/b\text{-})\text{tangles}\end{matrix}\right\}\to\mathcal{A}^v_S:=$$

Etingof Kazhdan Haviv Enriquez Ševera

**Algebras and Invariants.** Given any unital algebra $A$ (even better if $A$ is Hopf; typically, $A\sim\hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R\in A\otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure $S$-component tangles:

$\to Z=\sum$

with ● :$c$ ● :$u$ ● :$w$ — :$b$

**Good News.** In theory, enough to know $R$, the cuaps, and stitching/multiplication $m^{ij}_k\colon A_i\otimes A_j\to A_k$.

**Problem.** Extract information out of $Z$.

**Textbook Solution.** Use representation theory …works, slowly.

**Today's Solution** (with van der Veen). For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$$\left\{\begin{matrix}\text{ordered perturbed}\\ \text{Gaussian formulas}\end{matrix}\right\}\to\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$

van der Veen

**1-Smidgen $sl_2$** Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b,c,u,w\rangle$ over the ring $R=\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with $b$ central and with $[w,c]=w$, $[c,u]=u$, and $[u,w]=b-2\epsilon c$, with CYBE $r_{ij}=(b_i-\epsilon c_i)c_j+u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of $sl_2$: $\mathfrak{g}_1\supset\langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset\langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset 0$. (note: $\deg(b,c,u,w,\epsilon)=(1,0,1,0,1)$)

**0-Smidgen $sl_2$ ☺.** Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon=0$, or $\mathbb{Q}\langle b,c,u,w\rangle/([b,\cdot]=0,\ [c,u]=u,\ [c,w]=-w,\ [u,w]=b$ with $r_{ij}=b_ic_j+u_iw_j$. It is $\mathfrak{b}^*\rtimes\mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c,w\rangle$ and $(b,u)$ is the dual basis of $(c,w)$. For topology, it is more valuable than $\mathfrak{g}_1$ / $sl_2$, but topology already got by other means almost everything $\mathfrak{g}_0$ gives.

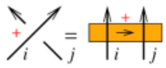
**How did these arise?** $sl_2=\mathfrak{b}^+\oplus\mathfrak{b}^-/\mathfrak{h}=: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+=\langle c,w\rangle/[w,c]=w$ is a Lie bialgebra with $\delta\colon\mathfrak{b}^+\to\mathfrak{b}^+\otimes\mathfrak{b}^+$ by $\delta\colon(c,w)\mapsto(0,c\wedge w)$. Going back, $sl_2^+=\mathcal{D}(\mathfrak{b}^+)=(\mathfrak{b}^+)^*\oplus\mathfrak{b}^+=\langle b,u,c,w\rangle/\cdots$. **Idea.** Replace $\delta\to\epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1}=0)$. At $k=0$, get $\mathfrak{g}_0$. At $k=1$, get $[w,c]=w$, $[w,b']=-\epsilon w$, $[c,u]=u$, $[b',u]=-\epsilon u$, $[b',c]=0$, and $[u,w]=b'-\epsilon c$. Now note that $b'+\epsilon c$ is central, so switch to $b:=b'+\epsilon c$. This is $\mathfrak{g}_1$.

**Ordering Symbols.** $\mathbb{O}(poly\mid specs)$ plants the variables of *poly* in $\mathcal{S}(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(c_1^3u_1c_2e^{u_3}w_3^9|x\colon w_3c_1,\ y\colon u_1u_3c_2\right)=w^9c^3\otimes ue^uc\in\mathcal{U}(\mathfrak{g})_x\otimes\mathcal{U}(\mathfrak{g})_y$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

**0-Smidgen Invariants.** $r=Id\in\mathfrak{b}^-\otimes\mathfrak{b}^+$ solves the CYBE $[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$$=R_{ij}=e^{r_{ij}}=e^{b_ic_j+u_iw_j}\in\mathcal{U}(\mathfrak{g}_{0,i}\oplus\mathfrak{g}_{0,j})$$

solves YB/R3.

**Lemma.** $R_{ij}=e^{b_ic_j+u_iw_j}=\mathbb{O}\left(\exp\left(b_ic_j+\frac{e^{b_i}-1}{b_i}u_iw_j\right)|i\colon u_i,\ j\colon c_jw_j\right)$

$=\sum_{m,n}\frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!}u^n\otimes c^mw^n$.

**Example.** $Z(T_0)=$

$$\mathbb{O}\Bigg(\exp\left(b_5c_1+\frac{e^{b_5}-1}{b_5}u_5w_1+b_2c_4+\frac{e^{b_2}-1}{b_2}u_2w_4-b_3c_6+\frac{e^{-b_3}-1}{b_3}u_3w_6\right)\Big| \; x\colon c_1w_1u_2,\ y\colon u_3c_4w_4u_5c_6w_6\Bigg)=\mathbb{O}\left(\zeta|x\colon u_xc_xw_x,\ y\colon u_yc_yw_y\right)$$

"ucw form"

**Goal.** Write $\zeta$ as a Gaussian: $\omega e^{L+Q}$ where $L$ bilinear in $b_i$ and $c_i$ with integer coefficients, $Q$ a balanced quadratic in $u_i$ and $w_i$ with coefficients in $R_S:=\mathbb{Q}(b_i,e^{b_i})$, and $\omega\in R_S$.

**The Big $\mathfrak{g}_0$ Lemma.** Under $[c,u]=u$, $[c,w]=-w$, and $[u,w]=b$:

1a. $N^{cu}:=\mathbb{O}(e^{\gamma c+\beta u}|uc)\overset{\rightarrow}{=}\mathbb{O}(e^{\gamma c+e^{\gamma}\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c}=e^{\gamma c}e^{e^{\gamma}\beta u}$

1b. $N^{wc}:=\mathbb{O}(e^{\gamma c+\alpha w}|wc)\overset{\rightarrow}{=}\mathbb{O}(e^{\gamma c+e^{\gamma}\alpha w}|cw)$ …in the $\{ax+b\}$ group)

2. $\mathbb{O}(e^{\alpha w+\beta u}|wu)=\mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$ (the Weyl relations)

3. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u}=e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu=(1+b\delta)^{-1}$

(a. expand and crunch. b. use $w=b\hat{x}$, $u=\partial_x$. c. use "scatter and glow".)

4. $\mathbb{O}(e^{\delta uw}|wu)=\mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)

5. $N^{wu}:=\mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu)\overset{\rightarrow}{=}\mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$

6. $N_k^{c_ic_j}:=\mathbb{O}(\zeta|c_ic_j)\overset{\rightarrow}{=}\mathbb{O}(\zeta/(c_i,c_j\to c_k)|c_k)$

**Sneaky.** $\alpha$ may contain (other) $u$'s, $\beta$ may contain (other) $w$'s.

**Strand Stitching,** $m_k^{ij}$, is defined as the composition

$$u_ic_i\overline{w_iu_j}c_jw_j\xrightarrow{N_x^{w_iu_j}}u_i\overline{c_iu_x}\,\overline{w_xc_j}w_j\xrightarrow{N_x^{c_iu_x}/\!\!/N_x^{w_xc_j}}\overline{u_iu_x}\,\overline{c_xc_x}\,\overline{w_xw_j}\xrightarrow{i,j,x\to k}u_kc_kw_k$$

**On to 1-smidgen invariants**, where much is the same…

**The Big $\mathfrak{g}_1$ Lemma.** Parts 1 and 6 are the same, yet

5. $\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|ucw\right)$

Here $\Lambda$ is for Λόγος, "a principle of order and knowledge", a balanced quartic in $\alpha$, $\beta$, $u$, $c$, and $w$:

$$\begin{aligned}\Lambda=&-b\nu(\alpha^2\beta^2\nu^2+4\alpha\beta\delta\nu+2\delta^2)/2+\beta^2\delta\nu^3(b\delta+2)u^2/2\\&+\delta^3\nu^3(3b\delta+4)u^2w^2/2+\beta\delta^2\nu^3(2b\delta+3)u^2w\\&+\alpha\delta^2\nu^3(2b\delta+3)uw^2+2\delta\nu^2(b\delta+2)(\alpha\beta\nu+\delta)uw\\&+\alpha^2\delta\nu^3(b\delta+2)w^2/2+2(\alpha\beta\nu+\delta)c+2\beta\delta\nu uc+2\delta^2\nu ucw\\&+2\alpha\delta\nu cw+\beta\nu^2(\alpha\beta\nu+2\delta)u+\alpha\nu^2(\alpha\beta\nu+2\delta)w.\end{aligned}$$

**Proof.** A lengthy computation. (Verification: ωεβ/Big)

**Problem.** We now need to normal-order perturbed Gaussians!

**Solution.** Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c,u)e^{\gamma c+\beta u}|uc)=\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+\beta u}|uc)=\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+e^{-\gamma}\beta u}|cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u,w)e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\epsilon P(\partial_\beta,\partial_\alpha)\nu e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|ucw\right)$$

Switch to ucw
conventions
here too!

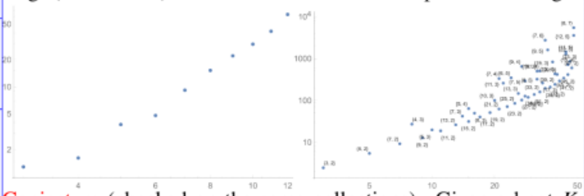
**Finally,** the values of the generators ⤧, ⤨, $\overrightarrow{n}$, and $\underset{\rightarrow}{u}$, are set by solving many equations, non-uniquely.

**Pragmatic Simplifications.** Set $t:=e^b$, work with $v:=(t-1)u/b$, and set $\mathbb{E}(\omega,L,Q,P):=\mathbb{O}\left(\omega^{-1}e^{L+Q/\omega}(1+\epsilon\omega^{-4}P)\colon(i\colon v_ic_iw_i)\right)$. Now $\omega\in R_S:=\mathbb{Z}[t_i,t_i^{-1}]$ is Laurent, $L=\sum l_{ij}\log(t_i)c_j$ with $l_{ij}\in\mathbb{Z}$, $Q=\sum q_{ij}v_iw_j$ with $q_{ij}\in R_S$, and $P$ is a quartic polynomial in $v_i$, $c_j$, $w_k$ with coefficients in $R_S$. The operations are lightly modified, and the Λόγος and the values of the generators become somewhat simpler, as in the implementation below.

**Rough complexity estimate,** after $t_k\to t$. $n$: xing number; $w$: width, maybe $\sim\sqrt{n}$. $A$: go over stitchings in order. $B$: multiplication ops per $N^{u_iw_j}$. $d$: deg of $u_i,w_j$ in $P$. $E$: #terms of deg $d$ in $P$. $F$: ops per term. $G$: cost per polynomial multiplication op.

$$\underbrace{n}_{A}\overbrace{\sum_{d=0}^{4}\underbrace{w^{4-d}}_{E}\underbrace{w^d}_{F}}^{B}\underbrace{n^2}_{G}=n^3w^4\in[n^5,n^7]$$

**Experimental Analysis** (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



**Conjecture** (checked on the same collections). Given a knot $K$ with Alexander polynomial $A$, there is a polynomial $\rho_1$ such that

$$P=A^2\frac{(t-1)^3\rho_1+t^2(2vw+(1-t)(1-2c))AA'}{(1-t)t}.$$

Furthermore, $A$ and $\rho_1$ are symmetric under $t\to t^{-1}$, so let $A^+$ and $\rho_1^+$ be their "positive parts", so e.g., $\rho_1(t)=\rho_1^+(t)+\rho_1^+(t^{-1})-\rho_1^+(0)$.

**Power.** On the 250 knots with at most 10 crossings, the pair $(A,\rho_1)$ attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

**Genus.** Up to 12 xings, always $\deg\rho_1^+\leq 2g-1$, where $g$ is the 3-genus of $K$ (equallity for 2530 knots). This gives a lower bound on $g$ in terms of $\rho_1$ (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

## Demo Programs for 0-Co.

ωεβ/Demo

### The *R*-matrices

```
R⁺_{0,i_,j_} := 𝔼[b_i c_j + b_i^-1 (e^(b_i) - 1) u_i w_j];
R⁻_{0,i_,j_} := 𝔼[-b_i c_j + b_i^-1 (e^(-b_i) - 1) u_i w_j];
```

### Utilities

```
CF[ω_. 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

### Normal Ordering Operators

```
N_{(x:w|u)_i_ c_j_→k_}[ω_. 𝔼[Q_]] := CF[
    ω 𝔼[e^γ α x_k + γ c_k + (Q /. c_j | x_i → 0)] /. {γ → ∂_{c_j} Q, α → ∂_{x_i} Q}];
N_{w_i_ u_j_→k_}[ω_. 𝔼[Q_]] := CF[
    ν ω 𝔼[-b_k ν α β + ν β u_k + ν α w_k + ν δ u_k w_k + (Q /. w_i | u_j → 0)] /.
      ν → (1 + b_k δ)^-1 /.
     {α → ∂_{w_i} Q /. u_j → 0, β → ∂_{u_j} Q /. w_i → 0, δ → ∂_{w_i,u_j} Q}];
```

### Stitching

```
m_{i_,j_→k_}[Z_] := Module[{x, z},
   CF[(Z // N_{w_i u_j→x} // N_{c_i u_x→x} // N_{w_x c_j→x}) /. z_{_i|j|x} → z_k]]
```

### Some calculations for $T_0$

```
T0 = R⁺_{0,5,1} R⁺_{0,2,4} R⁻_{0,3,6}
```

$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{\left(-1+e^{b_5}\right) u_5 w_1}{b_5} + \frac{\left(-1+e^{b_2}\right) u_2 w_4}{b_2} + \frac{\left(-1+e^{-b_3}\right) u_3 w_6}{b_3}\right]$$

```
T0 // m_{1,2→1} // m_{3,4→3} // m_{3,5→3} // m_{3,6→3}
```

$$\frac{1}{1-\left(-1+e^{b_1}\right)\left(-1+e^{b_3}\right)} \mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{e^{b_3}\left(-1+e^{b_1}\right)\left(-1+e^{b_3}\right) u_1 w_1}{\left(-e^{b_1}-e^{b_3}+e^{b_1+b_3}\right) b_1} - \frac{e^{b_1}\left(-1+e^{b_3}\right) u_3 w_1}{\left(-1+\left(-1+e^{b_1}\right)\left(-1+e^{b_3}\right)\right) b_3} - \frac{e^{-b_3}\left(-1+e^{b_3}\right) u_3 w_3}{b_3} - \frac{e^{-b_3}\left(-1+e^{b_1}\right)\left(-e^{b_3} b_3 u_1 + e^{b_1}\left(-1+e^{b_3}\right) b_1 u_3\right) w_3}{b_1\left(b_3-\left(-1+e^{b_1}\right)\left(-1+e^{b_3}\right) b_3\right)}\Big]$$

### Verifying meta-associativity

```
Q0 = 𝔼[Sum[f_i c_i, {i, 3}] + Sum[f_{i,j} u_i w_j, {i, 3}, {j, 3}]]
```

$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

```
(Q0 // m_{1,2→1} // m_{1,3→1}) ≡ (Q0 // m_{2,3→2} // m_{1,2→1})
True
```

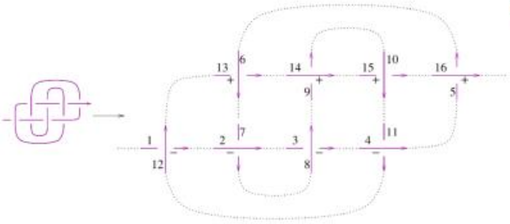
### Testing R3

```
t1 = R⁺_{0,1,2} R⁺_{0,3,4} R⁺_{0,5,6} // m_{3,5→x} // m_{1,6→y} // m_{2,4→z}
```

$$\mathbb{E}\left[b_x c_y + b_x c_z + b_y c_z + \frac{e^{b_x}\left(-1+e^{b_y}\right) u_y w_z}{b_y} + \frac{\left(-1+e^{b_x}\right) u_x \left(w_y+w_z\right)}{b_x}\right]$$

```
t1 ≡ (R⁺_{0,1,2} R⁺_{0,3,4} R⁺_{0,5,6} // m_{1,3→x} // m_{2,5→y} // m_{4,6→z})
True
```

### $8_{17}$



```
z1 = R⁻_{0,12,1} R⁻_{0,2,7} R⁻_{0,8,3} R⁻_{0,4,11} R⁺_{0,16,5} R⁺_{0,6,13} R⁺_{0,14,9} R⁺_{0,10,15};
Do[z1 = (z1 // m_{1,n→1}) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$$\left\{-\frac{e^{3 b}\,\mathbb{E}[0]}{1-4 e^{b}+8 e^{2 b}-11 e^{3 b}+8 e^{4 b}-4 e^{5 b}+e^{6 b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8 t+4 t^2-t^3\right\}$$

## Demo Programs for 1-Co.

ωεβ/Demo

### The Λόγος

```
Λ[k_] := ((t_k - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 v_k c_k w_k δ^2 μ^2 -
      δ (1 + μ) (w_k^2 α^2 + v_k^2 β^2) - v_k^2 w_k^2 δ^3 (1 + 3 μ) -
      2 (α β + 2 δ μ + v_k w_k δ^2 (1 + 2 μ) + 2 c_k δ μ^2) (w_k α + v_k β) -
      4 (c_k μ^2 + v_k w_k δ (1 + μ)) (α β + δ μ)) (1 + t_k) / 4;
```

### The Generators

```
R⁺_{i_,j_} := 𝔼[1, Log[t_i] c_j, v_i w_j, v_i c_i w_j + c_i c_j + v_i^2 w_j^2/4];
R⁻_{i_,j_} := 𝔼[1, -Log[t_i] c_j, -t_i^-1 v_i w_j,
    t_i^-1 v_i c_j w_j - c_i c_j - t_i^-2 v_i^2 w_j^2/4];
(ur_i_ := 𝔼[t_i^(-1/2), 0, 0, c_i t_i^-2]; nr_i_ := 𝔼[t_i^(1/2), 0, 0, -c_i t_i^2];)
```

### Differential Polynomials

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := (* means P[∂_α,∂_β][f] *)
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

### Utilities

```
CF[ℰ_𝔼] := Expand /@ Together /@ ℰ;
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

### Normal Ordering Operators

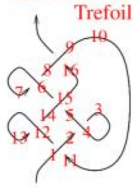
```
N_{c_j_ (x:v|w)_i_→k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^γ β x_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω e^γ β x_k + (Q /. x_i → 0),
      e^-q DP_{c_j→D_γ, x_i→D_β}[P][e^q]] /. {γ → ∂_{c_j} L, β → ω^-1 ∂_{x_i} Q}]];
N_{w_i_ v_j_→k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{q = ((1 - t_k) α β + β v_k + α w_k + δ v_k w_k) / μ}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. w_i | v_j → 0),
       μ^4 e^-q DP_{w_i→D_α, v_j→D_β}[P][e^q] + ω^4 Λ[k]] /. μ → 1 + (t_k - 1) δ /.
     {α → ω^-1 (∂_{w_i} Q /. v_j → 0), β → ω^-1 (∂_{v_j} Q /. w_i → 0),
      δ → ω^-1 ∂_{w_i,v_j} Q}]];
```

### Stitching

```
m_{i_,j_→k_}[Z_𝔼] := Module[{x, z},
   CF[(Z // N_{w_i v_j→x} // N_{c_i v_x→x} // N_{w_x c_j→x}) /. z_{_i|j|x} → z_k]]
```

### The 0-Framed Trefoil

```
z2 = R⁺_{1,11} R⁻_{4,2} nr_3 R⁺_{15,5} R⁻_{6,8} ur_7 R⁺_{9,16} nr_10 R⁻_{12,14} ur_13;
(Do[z2 = z2 // m_{1,k→1}, {k, 2, 16}];
 z2 = z2 /. a_1 :→ a)
```



$$\mathbb{E}\Big[-1+\frac{1}{t}+t,\ 0,\ 0,\ 16+\frac{2 c}{t^4}-\frac{1}{t^3}-\frac{6 c}{t^3}+\frac{4}{t^2}+\frac{10 c}{t^2}-\frac{10}{t}-\frac{8 c}{t}-18 t+8 c t+14 t^2-10 c t^2-7 t^3+6 c t^3+2 t^4-2 c t^4+2 v w-\frac{2 v w}{t^4}+\frac{4 v w}{t^3}-\frac{6 v w}{t^2}+\frac{2 v w}{t}-6 t v w+4 t^2 v w-2 t^3 v w\Big]$$

**Questions and To Do List.** • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality (ℤ) properties? • Can everything be re-stated using integrals (∫)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on *R*. • Is there a canonical *R*? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $b^+ \leftrightarrow b^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" $q$-algebra. • $k$-smidgen $sl_n$, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

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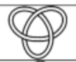
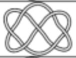
| diagram | $n_k^t$ Alexander's $A_+$<br>Today's / Rozansky's $\rho_1^+$ | genus / ribbon<br>unknotting number / amphicheiral | diagram | $n_k^t$ Alexander's $A_+$<br>Today's / Rozansky's $\rho_1^+$ | genus / ribbon<br>unknotting number / amphicheiral |
|---|---|---|---|---|---|
|  | $0_1^a$ $1$<br>$0$ | 0 / ✓<br>0 / ✓ |  | $3_1^a$ $t-1$<br>$t$ | 1 / ✗<br>1 / ✗ |
|  | $4_1^a$ $3-t$<br>$0$ | 1 / ✗<br>1 / ✓ |  | $5_1^a$ $t^2-t+1$<br>$2t^3+3t$ | 2 / ✗<br>2 / ✗ |
|  | $5_2^a$ $2t-3$<br>$5t-4$ | 1 / ✗<br>1 / ✗ |  | $6_1^a$ $5-2t$<br>$t-4$ | 1 / ✓<br>1 / ✗ |
|  | $6_2^a$ $-t^2+3t-3$<br>$t^3-4t^2+4t-4$ | 2 / ✗<br>1 / ✗ |  | $6_3^a$ $t^2-3t+5$<br>$0$ | 2 / ✗<br>1 / ✓ |
|  | $7_1^a$ $t^3-t^2+t-1$<br>$3t^5+5t^3+6t$ | 3 / ✗<br>3 / ✗ |  | $7_2^a$ $3t-5$<br>$14t-16$ | 1 / ✗<br>1 / ✗ |
|  | $7_3^a$ $2t^2-3t+3$<br>$-9t^3+8t^2-16t+12$ | 2 / ✗<br>2 / ✗ |  | $7_4^a$ $4t-7$<br>$32-24t$ | 1 / ✗<br>2 / ✗ |
|  | $7_5^a$ $2t^2-4t+5$<br>$9t^3-16t^2+29t-28$ | 2 / ✗<br>2 / ✗ |  | $7_6^a$ $-t^2+5t-7$<br>$t^3-8t^2+19t-20$ | 2 / ✗<br>1 / ✗ |
|  | $7_7^a$ $t^2-5t+9$<br>$8-3t$ | 2 / ✗<br>1 / ✗ |  | $8_1^a$ $7-3t$<br>$5t-16$ | 1 / ✗<br>1 / ✗ |
|  | $8_2^a$ $-t^3+3t^2-3t+3$<br>$2t^5-8t^4+10t^3-12t^2+13t-12$ | 3 / ✗<br>2 / ✗ |  | $8_3^a$ $9-4t$<br>$0$ | 1 / ✗<br>2 / ✓ |
|  | $8_4^a$ $-2t^2+5t-5$<br>$3t^3-8t^2+6t-4$ | 2 / ✗<br>2 / ✗ |  | $8_5^a$ $-t^3+3t^2-4t+5$<br>$-2t^5+8t^4-13t^3+20t^2-22t+24$ | 3 / ✗<br>2 / ✗ |
|  | $8_6^a$ $-2t^2+6t-7$<br>$5t^3-20t^2+28t-32$ | 2 / ✗<br>2 / ✗ |  | $8_7^a$ $t^3-3t^2+5t-5$<br>$-t^5+4t^4-10t^3+12t^2-13t+12$ | 3 / ✗<br>1 / ✗ |
|  | $8_8^a$ $2t^2-6t+9$<br>$-t^3+4t^2-12t+16$ | 2 / ✓<br>2 / ✗ |  | $8_9^a$ $-t^3+3t^2-5t+7$<br>$0$ | 3 / ✓<br>1 / ✓ |
|  | $8_{10}^a$ $t^3-3t^2+6t-7$<br>$-t^5+4t^4-11t^3+16t^2-21t+20$ | 3 / ✗<br>2 / ✗ |  | $8_{11}^a$ $-2t^2+7t-9$<br>$5t^3-24t^2+39t-44$ | 2 / ✗<br>1 / ✗ |
|  | $8_{12}^a$ $t^2-7t+13$<br>$0$ | 2 / ✗<br>2 / ✓ |  | $8_{13}^a$ $2t^2-7t+11$<br>$-t^3+4t^2-14t+20$ | 2 / ✗<br>1 / ✗ |
|  | $8_{14}^a$ $-2t^2+8t-11$<br>$5t^3-28t^2+57t-68$ | 2 / ✗<br>1 / ✗ |  | $8_{15}^a$ $3t^2-8t+11$<br>$21t^3-64t^2+120t-140$ | 2 / ✗<br>2 / ✗ |
|  | $8_{16}^a$ $t^3-4t^2+8t-9$<br>$t^5-6t^4+17t^3-28t^2+35t-36$ | 3 / ✗<br>2 / ✗ |  | $8_{17}^a$ $-t^3+4t^2-8t+11$<br>$0$ | 3 / ✗<br>1 / ✓ |
|  | $8_{18}^a$ $-t^3+5t^2-10t+13$<br>$0$ | 3 / ✗<br>2 / ✓ |  | $8_{19}^n$ $t^3-t^2+1$<br>$-3t^5-4t^2-3t$ | 3 / ✗<br>3 / ✗ |
|  | $8_{20}^n$ $t^2-2t+3$<br>$4t-4$ | 2 / ✓<br>1 / ✗ |  | $8_{21}^n$ $-t^2+4t-5$<br>$t^3-8t^2+16t-20$ | 2 / ✗<br>1 / ✗ |