Les Diablerets Dogma Handout on 170828

August 28, 2017 2:18 AM

1. Verify links.
2. Partition. →

Dror Bar-Natan: Talks: LesDiablerets-1708:

The Dogma is Wrong

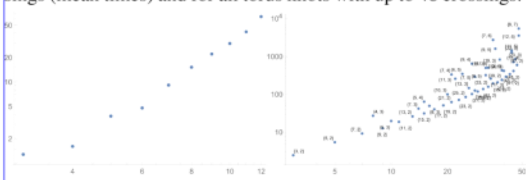
Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. Preliminary writeup [BV1], fuller writeup [BV2]. More at ωεβ/talks.

Happy Birthday Anton! ωεβ:=http://drorbn.net/ld17/

Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: "quantize and use representation theory". We present an alternative and better procedure: "centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra". While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (ωεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

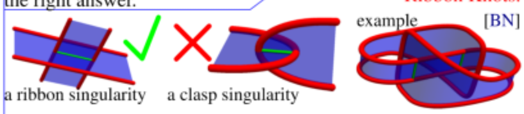
Experimental Analysis (ωεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \leq 2g-1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Ribbon Knots.



Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccccc} & & U \in \mathcal{T}_n & & 1 \in \mathcal{A}_n \\ & \nearrow^{\tau} & & \nearrow^{\tau} & \\ \mathcal{T}_{2n} & \xrightarrow{z} & \mathcal{A}_{2n} & & \\ & \searrow_{\kappa} & & \searrow_{\kappa} & \\ & & \text{ribbon } K \in \mathcal{T}_1 & & z(K) \in \mathcal{R} \subseteq \mathcal{A}_1 \end{array}$$

[Vo]: Works for Alexander! with $\mathcal{R} := \kappa(\tau^{-1}(1))$

$A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$

$\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$

Faster is better, leaner is meaner!

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of $poly$ in $S(\oplus_i \mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,

$$\mathbb{O}\left(a_1^3 y_1 a_2 e^{y_3} x_3^9 \mid x_3 a_1 \otimes y_1 y_3 a_2\right) = x^9 a^3 \otimes y e^y a \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

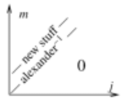
This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

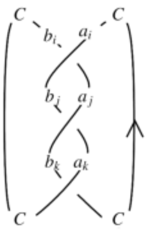
$$\left.\frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m \geq 0} a_{jm}(K) d^j \hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot \omega(K)(e^\hbar) = 1$.



"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})\omega(K)(q^d)}\left(1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)}\right).$$



The Yang-Baxter Technique. Given an algebra A (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in A \otimes A \quad \text{and} \quad C \in A,$$

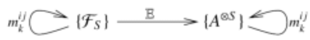
form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

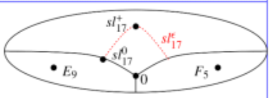
Problem. Extract information from Z.

The Dogma. Use representation theory. In principle finite, but *slow*.

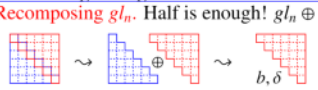
The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional "space of formulas".



The (fake) moduli of Lie algebras on V, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon/(\epsilon^{k+1} = 0)$.

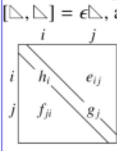


Recomposing gl_n. Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\triangledown, b, \delta)$:



$b(\triangledown) = b: \triangledown \otimes \triangledown \to \triangledown$
$b(\triangle) \rightsquigarrow \delta: \triangledown \to \triangledown \otimes \triangledown$

Now define $gl_n^\epsilon := \mathcal{D}(\triangledown, b, \epsilon\delta)$. Schematically, this is $[\triangledown, \triangledown] = \triangledown$, $[\triangle, \triangle] = \epsilon\triangle$, and $[\triangledown, \triangle] = \triangle + \epsilon\triangledown$. In detail, it is



$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} \qquad [f_{ij}, f_{kl}] = \epsilon\delta_{jk} f_{il} - \epsilon\delta_{li} f_{kj}$$
$$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k} e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l} f_{il}) - \delta_{li}(\epsilon\delta_{k<j} e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j} f_{kj})$$
$$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik}) e_{jk} \qquad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) e_{jk}$$
$$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik}) f_{jk} \qquad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) f_{jk}$$

switch to yax? (see NPSA paper)

The Main sl_2 Theorem. Let $\mathfrak{g}^\epsilon = \langle t, y, a, x \rangle / ([t, \cdot] = 0, [a, x] = x, [a, y] = -y, [x, y] = t - 2\epsilon a)$ and let $\mathfrak{g}_k = \mathfrak{g}^\epsilon/(\epsilon^{k+1} = 0)$. The $\mathfrak{g}_k$-invariant of any S-component tangle K can be written in the form $Z(K) = \mathbb{O}\left(\omega e^{L+Q+P} : \bigotimes_{i \in S} y_i a_i x_i\right)$, where ω is a scalar (a rational function in the variables t_i and their exponentials $T_i := e^{t_i}$), where $L = \sum l_{ij} t_i a_j$ is a quadratic in t_i and a_j with integer coefficients l_{ij}, where $Q = \sum q_{ij} y_i x_j$ is a quadratic in the variables y_i and x_j with scalar coefficients q_{ij}, and where P is a polynomial in $\{\epsilon, y_i, a_i, x_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2d+2$ in $\{y_i, \sqrt{a_i}, x_i\}$. Furthermore, after setting $t_i = t$ and $T_i = T$ for all i, the invariant $Z(K)$ is poly-time computable.

The PBW Problem. In $\mathcal{U}(\mathfrak{g}^\epsilon)$, bring $z = y^3a^2x^2y^2a^2x$ to yax-order. In other words, find $g \in \mathbb{Z}[\epsilon, t, y, a, x]$ such that $\mathbb{O}(f = y_1^3 y_2^2 a_1^2 a_2^2 x_1^2 x_2 : y_1 a_1 x_1 y_2 a_2 x_2) = \mathbb{O}(g : yax)$.

Solution, Part 1. In $\hat{\mathcal{U}}(\mathfrak{g}^\epsilon)$ we have

$$X_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} := e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x} e^{\tau_2 t} e^{\eta_2 y} e^{\alpha_2 a} e^{\xi_2 x} = e^{\tau t} e^{\eta y} e^{\alpha a} e^{\xi x} =: Y_{\tau,\eta,\alpha,\xi},$$

where τ, η, α, ξ are ~~utterly non-interesting~~ ugly functions of $\tau_1, \eta_i, \alpha_i, \xi_i$:

$$\tau = \tau_1 + \tau_2 - \frac{\log(1 - \epsilon\eta_2\xi_1)}{\epsilon} = \tau_1 + \tau_2 + \eta_2\xi_1 + \frac{\epsilon}{2}\eta_2^2\xi_1^2 + \ldots,$$
$$\eta = \eta_1 + \frac{e^{-\alpha_1}\eta_2}{(1 - \epsilon\eta_2\xi_1)} = \eta_1 + e^{-\alpha_1}\eta_2 + \epsilon e^{-\alpha_1}\eta_2^2\xi_1 + \ldots,$$

grey

Note 2. ~~There are two great evils in mathematics:~~ The two great evil of mathematics are non-commutativity and non-linearity. We have traded one for the other.

Note 3. We could have done similarly with $e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x} = e^{\tau t + \eta y + \alpha a + \xi x}$, and with $S(e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x})$, $\Delta(e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x})$, $\prod_{i=1}^5 e^{\tau_i t} e^{\eta_i y} e^{\alpha_i a} e^{\xi_i x}$.

Note 4. Identifying $f \leftrightarrow D$ (and likewise for g), we find that $g = \Phi_* f$. expand?

Fact. $R_{12} \to \exp\left(\partial_{\tau_1}\partial_{\alpha_2} + \partial_{y_1}\partial_{x_2}\right)(1 + \sum_{d \geq 1} \epsilon^d p_d)$, where the p_d are computable polynomials of a-priori bounded degrees.

Moral. We need to understand the pushforwards via maps like Φ of (formally ∞-order) "differential operators at 0", that in them-

$$\tau = \tau_1 + \tau_2 - \frac{\log(1-\epsilon\eta_2\xi_1)}{\epsilon} = \tau_1 + \tau_2 + \eta_2\xi_1 + \frac{\epsilon}{2}\eta_2^2\xi_1^2 + \dots,$$
$$\eta = \eta_1 + \frac{e^{-\alpha_1}\eta_2}{(1-\epsilon\eta_2\xi_1)} = \eta_1 + e^{-\alpha_1}\eta_2 + \epsilon e^{-\alpha_1}\eta_2^2\xi_1 + \dots,$$
$$\alpha = \alpha_1 + \alpha_2 + 2\log(1-\epsilon\eta_2\xi_1) = \alpha_1 + \alpha_2 - 2\epsilon\eta_2\xi_1 + \dots,$$
$$\xi = \frac{e^{-\alpha_2}\xi_1}{(1-\epsilon\eta_2\xi_1)} + \xi_2 = e^{-\alpha_2}\xi_1 + \xi_2 + \epsilon e^{-\alpha_2}\eta_2\xi_1^2 + \dots.$$

Note 1. This defines a mapping $\Phi\colon \mathbb{R}^8_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2} \to \mathbb{R}^4_{\tau,\eta,\alpha,\xi}$.

Proof. $\mathfrak{g}^\epsilon$ has a 2D representation ρ:

```
ρt = (1 0; 0 1); ρy = (0 0; -ϵ 0);
ρa = ((1 + 1/ϵ)/2  0; 0  -(1 - 1/ϵ)/2); ρx = (0 1; 0 0);
Simplify@{ρa.ρx - ρx.ρa == ρx, ρa.ρy - ρy.ρa == -ρy,
  ρx.ρy - ρy.ρx == ρt - 2 ϵ ρa}

{True, True, True}
```

It is enough to verify the desired identity in ρ:

```
ME = MatrixExp;
Simplify[
 ME[τ1 ρt].ME[η1 ρy].ME[α1 ρa].ME[ξ1 ρx].ME[τ2 ρt].
    ME[η2 ρy].ME[α2 ρa].ME[ξ2 ρx] ==
   ME[τ0 ρt].ME[η0 ρy].ME[α0 ρa].ME[ξ0 ρx] /.
  {τ0 → -Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2, η0 → η1 + e^-α1 η2/(1-ϵ η2 ξ1),
   α0 → 2 Log[1 - ϵ η2 ξ1] + α1 + α2, ξ0 → e^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2}]

True
```

Solution, Part 2. But now, with $D = \frac{\partial^{12}}{\partial_{y_1}^3\partial_{\alpha_1}^2\partial_{x_1}^2\partial_{y_2}^2\partial_{\alpha_2}^2\partial_{x_2}}$,

$$Z = DX_{\tau_1,\eta_1,\alpha_1,\xi_1,\tau_2,\eta_2,\alpha_2,\xi_2}\big|_{vs=0} = DY_{\tau,\eta,\alpha,\xi}\big|_{vs=0}$$
$$= \mathbb{O}\left(De^{\tau t}e^{\eta y}e^{\alpha a}e^{\xi x}\big|_{vs=0} : yax\right) = \mathbb{O}(g: yax):$$

```
Expand[∂{η1,3} ∂{α1,2} ∂{ξ1,2} ∂{η2,2} ∂{α2,2} ∂{ξ2,1} Exp[
     (-Log[1-ϵ η2 ξ1]/ϵ + τ1 + τ2) t + (η1 + e^-α1 η2/(1-ϵ η2 ξ1)) y +
     (2 Log[1 - ϵ η2 ξ1] + α1 + α2) a + (e^-α2 ξ1/(1-ϵ η2 ξ1) + ξ2) x
   ] /. (τ | η | α | ξ)1|2 → 0]

2 a^4 t^2 x y^3 + 4 t x^2 y^4 - 16 a t x^2 y^4 + 24 a^2 t x^2 y^4 - 16 a^3 t x^2 y^4 +
 4 a^4 t x^2 y^4 + 16 x^3 y^5 - 32 a x^3 y^5 + 24 a^2 x^3 y^5 - 8 a^3 x^3 y^5 + a^4 x^3 y^5 +
 2 a^4 t x y^3 ϵ - 8 a^5 t x y^3 ϵ + 8 x^2 y^4 ϵ - 40 a x^2 y^4 ϵ + 80 a^2 x^2 y^4 ϵ -
 80 a^3 x^2 y^4 ϵ + 40 a^4 x^2 y^4 ϵ - 8 a^5 x^2 y^4 ϵ - 4 a^5 x y^3 ϵ^2 + 8 a^6 x y^3 ϵ^2
```

Fact. $K_{12} \to \exp(\partial_{\tau_1}\partial_{\alpha_2} + \partial_{y_1}\partial_{x_2})(1 + \sum_{d\geq 1}\epsilon^d p_d)$, where the p_d are computable polynomials of a-priori bounded degrees.

Moral. We need to understand the pushforwards via maps like Φ of (formally ∞-order) "differential operators at 0", that in themselves are perturbed Gaussians. This turns out to be the same problem as "0-dimensional QFT" (except no integration is ever needed), and if $\epsilon^{k+1} = 0$, it is explicitly soluble.

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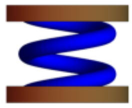

dog·ma (dôg′mə, dŏg′-) — The Free Dictionary, ωεβ/TFD

n. pl. **dog·mas** or **dog·ma·ta** (-mə-tə)

1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.

2. A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: *"Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry" (Bertrand Russell).*

diagram	n^t_k / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon; unknotting number / amphicheiral	diagram	n^t_k / Today's / Rozansky's ρ_1^+	Alexander's ω^+	genus / ribbon; unknotting number / amphicheiral
	0_1^a / 0	1	0 / ✔; 0 / ✔		3_1^a / t	$t-1$	1 / ✘; 1 / ✘
	4_1^a / 0	$3-t$	1 / ✘; 1 / ✔		5_1^a / $2t^3+3t$	t^2-t+1	2 / ✘; 2 / ✘