Work in Progress!

The Brute and the Hidden Paradise

Abstract. There is expected to be a hidden paradise of poly-time computable knot polynomials lying just beyond the Alexander polynomial. I will describe my brute attempts to gain entry.

Why "expected"? Finite-type invariants include all coefficients of all quantum knot polynomials (appropriately parametrized), and each is computable in poly-time. Yet

d	2	3	4	5	6	7	8	⋯
known f.t. invts in $O(n^d)$	1	1	∞	3	4	8	11	⋯

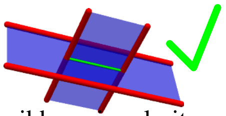
This is an unreasonable picture! So there ought to be further poly-time polynomial invariants.

Also. • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.

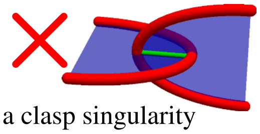
Rozansky

Why "paradise"? Foremost answer: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Secondary answer: may disprove {ribbon} = {slice}: (see [BN2])

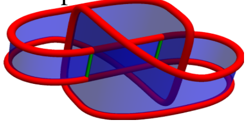


a ribbon singularity



a clasp singularity

example

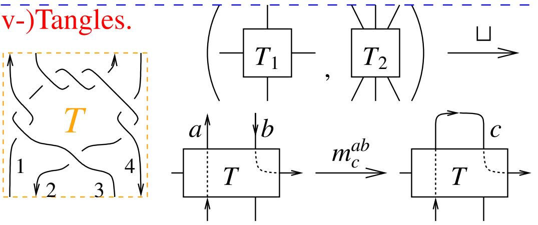


A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

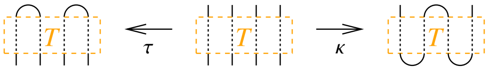
Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

(v-)Tangles.



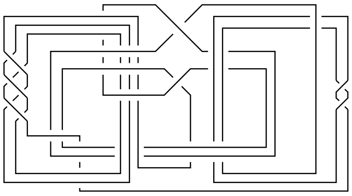
(meta-associativity: $m_x^{ab} /\!\!/ m_y^{xc} = m_x^{bc} /\!\!/ m_y^{ax}$)



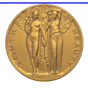
Gompf, Scharlemann, Thompson [GST]

$$\begin{tikzcd}
 & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n \\
\mathcal{T}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] \arrow[rr, "z"] & & \mathcal{A}_{2n} \arrow[ur, "\tau"] \arrow[dr, "\kappa"'] & \\
 & \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1
\end{tikzcd}$$

with $\mathcal{R} := \kappa(\tau^{-1}(1))$



Faster is better, leaner is meaner!



The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a : a\in S\})$:

$$\left({}_a\!\nwarrow\!\!\nearrow_b,\ {}_b\!\nwarrow\!\!\nearrow_a\right) \to \begin{array}{c|cc} 1 & a & b\\ \hline a & 1 & 1-t_a^{\pm1}\\ b & 0 & t_a^{\pm1}\end{array} \qquad T_1\sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

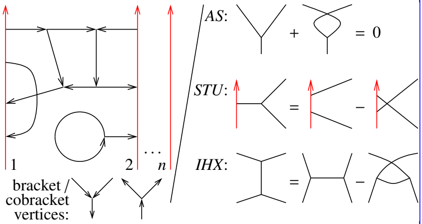
$$\begin{array}{c|ccc} \omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[t_a,t_b\to t_c]{m_c^{ab}} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

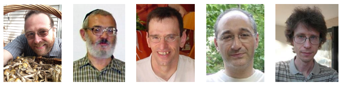
For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!

Dunfield: 1000-crossing fast.

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z\colon \left\{\begin{array}{c} S\text{-component}\\ (v/b\text{-})\text{tangles}\end{array}\right\} \to \mathcal{A}_S^v :=$$



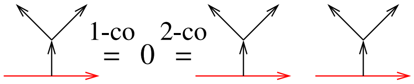


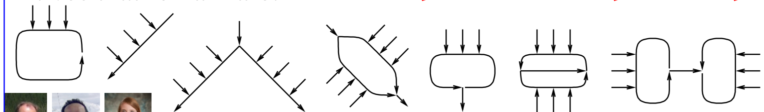
Etingof Kazhdan Haviv Enriquez Ševera

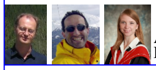
(it is enough to know Z on ⤱ and have disjoint union and stitching formulas) …exponential and too hard!

Idea. Look for "ideal" quotients of $\mathcal{A}_S^v$ that have poly-sized descriptions; …specifically, limit the co-brackets.

1-co and 2-co, aka TC and TC^2, on the right. The primitives that remain are:





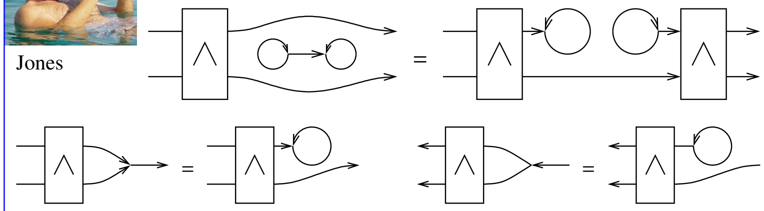


Alekseev, Torossian, Dancso

…manageable but still exponential!

Jones

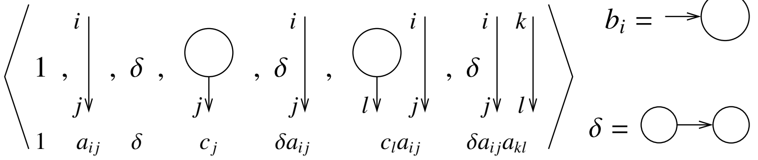
The 2D relations come from the relation with 2D Lie bialgebras:



We let $\mathcal{A}^{2,2}$ be $\mathcal{A}^v$ modulo 2-co and $2D$, and $z^{2,2}$ be the projection of $\log Z$ to $\mathcal{P}^{2,2} := \pi\mathcal{P}^v$, where $\mathcal{P}^v$ are the primitives of $\mathcal{A}^v$.

Main Claim. $z^{2,2}$ is poly-time computable.

Main Point. $\mathcal{P}^{2,2}$ is poly-size, so how hard can it be? Indeed, as a module over $\mathbb{Q}[\![b_i]\!]$, $\mathcal{P}^{2,2}$ is at most



Claim. $R_{jk} = e^{a_{jk}}e^{\rho_{jk}}$ is a solution of the Yang-Baxter / R3 equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in $\exp\mathcal{P}^{2,2}$, with $\rho_{jk} :=$

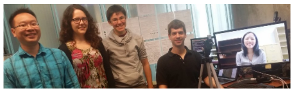
$$\psi(b_j)\left(-c_k + \frac{c_k a_{jk}}{b_j} - \frac{\delta a_{jk}a_{jk}}{b_j^2}\right) + \frac{\phi(b_j)\psi(b_k)}{b_k\phi(b_k)}\left(c_k a_{kk} - \frac{\delta a_{jk}a_{kk}}{b_j}\right),$$

and with $\phi(x) := e^{-x} - 1 = -x + x^2/2 - \ldots$, and $\psi(x) := ((x+2)e^{-x} - 2 + x)/(2x) = x^2/12 - x^3/24 + \ldots$.

Problem. How do we multiply in $\exp(\mathcal{P}^{2,2})$? How do we stitch?

Help Needed! Disorganized videos of talks in a private seminar are at ωεβ/PP.

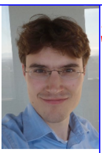
Many thanks: Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)



> "God created the knots, all else in topology is the work of mortals."
>
> Leopold Kronecker (modified)

`www.katlas.org`

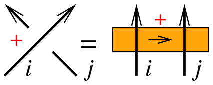
1-Smidgen sl_2 (with van der Veen). Let $\mathfrak{g}_1$ be the 4D Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over $\mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset 0$. In a certain sense, $\mathfrak{g}_1$ is more valuable than sl_2.


van der Veen

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon = 0$, or $\mathbb{Q}\langle b, c, u, w\rangle/([b, \cdot] = 0, [c, u] = u, [c, w] = -w, [u, w] = b$ with $r_{ij} = b_ic_j + u_iw_j$. It is $\mathfrak{a}^* \rtimes \mathfrak{a}$ where $\mathfrak{a}$ is the 2D Lie algebra $\mathbb{Q}\langle b, u\rangle$ and (c, w) is the dual basis of (b, u). It is even more valuable than $\mathfrak{g}_1$, but topology already got by other means almost everything $\mathfrak{g}_0$ has to give.

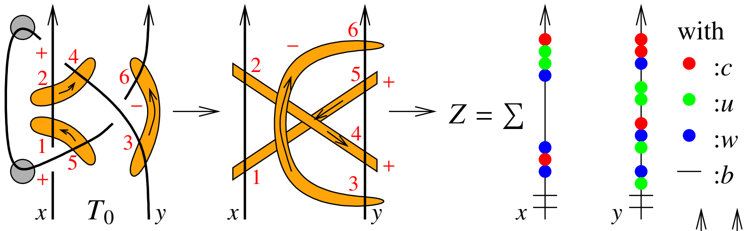
How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w\rangle/[w, c] = w$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (c, w) \mapsto (0, c \wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get $\mathfrak{g}_0$. At $k = 1$, get $[w, c] = w$, $[w, b'] = -\epsilon w$, $[c, u] = u$, $[b', u] = -\epsilon u$, $[b', c] = 0$, and $[u, w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is $\mathfrak{g}_1$.

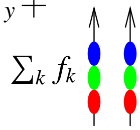
0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$$= R_{ij} = e^{r_{ij}} = e^{b_ic_j+u_iw_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$$

solves YB/R3, hence we get a tangle invariant:





Goal. Sort Z to be as on the right, with $f_k \in \mathbb{Q}[\![b_i]\!]$. Better, with $\zeta \in \mathbb{Q}[\![b_x, c_x, u_x, w_x, b_y, c_y, u_y, w_y]\!]$, write

$$Z = \mathbb{O}\left(\zeta | x\colon c_xu_xw_x,\ y\colon c_yu_yw_y\right) \qquad \text{(cuw form)}$$

Here $\mathbb{O}$ (*poly* | *specs*) plants the variables of *poly* in $\mathcal{S}(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(c_1^3u_1c_2e^{u_3}w_3^9 | x\colon w_3c_1,\ y\colon u_1u_3c_2\right) = w^9c^3 \otimes ue^uc \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$$

Lemma. $R_{ij} = e^{b_ic_j+u_iw_j} = \mathbb{O}\left(\exp\left(b_ic_j + \frac{e^{b_i}-1}{b_i}u_iw_j\right) | i\colon u_i,\ j\colon c_jw_j\right)$

$$= \sum_{m,n} \frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!}u^n \otimes c^mw^m.$$

Example. $Z(T_0) =$

$$\mathbb{O}\left(1\exp\left(b_5c_1 + \frac{e^{b_5}-1}{b_5}u_5w_1 + b_2c_4 + \frac{e^{b_2}-1}{b_2}u_2w_4 - b_3c_6 + \frac{e^{-b_3}-1}{b_3}u_3w_6\right) \Big|\ x\colon c_1w_1u_2,\ y\colon u_3c_4w_4u_5c_6w_6\right)$$

$$= \mathbb{O}\left(?|x\colon c_xu_xw_x,\ y\colon c_yu_yw_y\right)$$

$\mathbb{O}(\omega e^{L+Q})$: L bilinear in b_i and c_i, and Q a balanced quadratic in u_i and w_i with coefficients in $\mathbb{Q}(b_i, e^{b_i}) \ni \omega$.
"Admissible"

The Big $\mathfrak{g}_0$ Lemma. Under $[c, u] = u$, $[c, w] = -w$, and $[u, w] = b$:

1. $N^{c_1c_2} := \mathbb{O}(\zeta|c_1c_2) \overset{\rightarrow}{=} \mathbb{O}(\zeta/(c_2 \to c_1)|c_1)$ (trivial, also for b, u, w)

2a. $N^{uc} := \mathbb{O}(e^{\gamma c+\beta u}|uc) \overset{\rightarrow}{=} \mathbb{O}(e^{\gamma c+e^{-\gamma}\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c} = e^{\gamma c}e^{e^{-\gamma}\beta u}$

2b. $N^{wc} := \mathbb{O}(e^{\gamma c+\alpha w}|wc) \overset{\rightarrow}{=} \mathbb{O}(e^{\gamma c+e^{\gamma}\alpha w}|cw)$ …in the $\{ax + b\}$ group)

3. $\mathbb{O}(e^{\alpha w+\beta u}|wu) = \mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$ (the Weyl relations)

4. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u} = e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu = (1 + b\delta)^{-1}$

(a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)

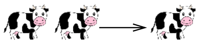
5. $\mathbb{O}(e^{\delta uw}|wu) = \mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)

6. $N^{wu} := \mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu) \overset{\rightarrow}{=} \mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$

Sneaky: α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition

$$c_iu_i\overline{w_ic_j}\,u_jw_j \xrightarrow{N_k^{w_ic_j}} c_i\,\overline{u_ic_k}\,\overline{w_ku_j}\,w_j \xrightarrow{N_k^{u_ic_k}/\!\!/N_k^{w_ku_j}} \overline{c_ic_k}\,\overline{u_ku_k}\,\overline{w_kw_j} \xrightarrow{N_k^{c_ic_k}/\!\!/-\!-/\!\!/N_k^{w_kw_j}} c_ku_kw_k$$



1-Smidgen Invariants. Much is the same:

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 2 are the same, yet

6. $\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:

$$\begin{aligned}\Lambda = &-\frac{1}{2}b\nu\left(\nu^2\alpha^2\beta^2 + 4\delta\nu\alpha\beta + 2\delta^2\right) - \frac{1}{2}\delta\nu^3(3b\delta + 2)\beta^2u^2\\ &- \frac{1}{2}b\delta^4\nu^3u^2w^2 - \delta^2\nu^3(2b\delta + 1)\beta u^2w\\ &- \nu^2(2b\delta + 1)(\nu\alpha\beta + 2\delta)\beta u - 2b\delta^2\nu^2(\nu\alpha\beta + \delta)uw\\ &+ \frac{1}{2}\delta\nu^3(b\delta + 2)\alpha^2w^2 + 2(\nu\alpha\beta + \delta)c + 2\delta\nu\beta cu + 2\delta^2\nu cuw\\ &+ 2\delta\nu\alpha cw + \delta^2\nu^3\alpha uw^2 + \nu^2(\nu\alpha\beta + 2\delta)\alpha w.\end{aligned}$$

Proof. A brutal hell.

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c, u)e^{\gamma c+\beta u}|uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta)e^{\gamma c+\beta u}|uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta)e^{\gamma c+e^{-\gamma}\beta u}|cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u, w)e^{\alpha w+\beta u+\delta uw}|wu\right) = \mathbb{O}\left(\epsilon P(\partial_\beta, \partial_\alpha)\nu e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$$

Note. Strand stitching requires a tiny extra step.

Finally, the values of the generators $\nwarrow\!\!\!\!\nearrow$, $\nearrow\!\!\!\!\nwarrow$, $\overrightarrow{n}$, $\overleftarrow{n}$, $\underrightarrow{u}$, and $\underleftarrow{u}$, are set by brutally solving many equations, non-uniquely.

Pragmatic Simplifications. Get rid of $\zeta = (e^b - 1)/b$ factors by rescaling $u \to \bar{u} = \zeta u$. Complement this with $\beta \to \bar{\beta} = \zeta^{-1}\beta$, $\delta \to \bar{\delta} = \zeta^{-1}\delta$, $\epsilon \to \bar{\epsilon} = \zeta^{-1}\epsilon$. Simplify further by naming $e^b \to t$; e.g., $\nu \to \bar{\nu} = (1 + (t - 1)\delta)^{-1}$. Get confused by renaming $(\bar{u}, \bar{\beta}, \bar{\delta}, \bar{\nu}) \to (u, \beta, \delta, \nu)$, and more confused by working with $\mu = \nu^{-1}$ and $\mathbb{E}(\omega, L, Q, P) := \omega^{-1}(1 + \epsilon\omega^{-4}P)e^{L+\omega^{-1}Q}$, where $\omega \in R := \mathbb{Q}(t_k)$, $L = \sum l_{ij}b_ic_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij}u_iw_j$ with $q_{ij} \in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

Rough complexity estimate, after $t_k \to t$. n: xing number; w: width, maybe $\sim \sqrt{n}$.

$$\underbrace{n}_{A}\overbrace{\sum_{d=0}^{4}\underbrace{w^{4-d}}_{E}\underbrace{w^d}_{F}}^{B}\underbrace{n^2}_{G} = n^3w^4 \in [n^5, n^7]$$

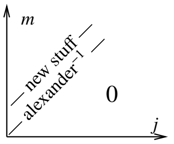
A: go over stitchings in order. B: multiplication ops per $N^{u_iw_j}$. d: deg of u_i, w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Expectation. Our invariant is the "1-higher diagonal" in the MMR expansion of the coloured Jones polynomial J_λ.


Melvin, Morton, Garoufalidis

Theorem ([BNG], conjectured [MM], elucidated [Ro]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of $sl(2)$. Writing

$$\left.\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$



"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right) \cdot A(K)(e^\hbar) = 1$.

Demo Programs for 0-Co.

ωεβ/Demo

The *R*-matrices

$$R^{+}_{0,i\_,j\_} := \mathbb{E}\left[b_i\, c_j + b_i^{-1}\left(\mathrm{e}^{b_i} - 1\right) u_i\, w_j\right];$$

$$R^{-}_{0,i\_,j\_} := \mathbb{E}\left[-b_i\, c_j + b_i^{-1}\left(\mathrm{e}^{-b_i} - 1\right) u_i\, w_j\right];$$

Utilities

$$\text{CF}[\omega\_.\ \mathbb{E}[Q\_]] := \text{Simplify}[\omega]\ \mathbb{E}[\text{Simplify}[Q]];$$

$$\mathbb{E}\ /\!:\ \mathbb{E}[Q1\_]\ \mathbb{E}[Q2\_] := \text{CF}@\mathbb{E}[Q1 + Q2];$$

$$\omega 1\_.\ \mathbb{E}[Q1\_] \equiv \omega 2\_.\ \mathbb{E}[Q2\_] := \text{Simplify}[\omega 1 == \omega 2 \wedge Q1 == Q2];$$

Normal Ordering Operators

$$\begin{aligned}&N_{u_{i\_}\ c_{j\_}\to k\_}[\omega\_.\ \mathbb{E}[Q\_]] := \text{CF}\Big[\\&\quad \omega\,\mathbb{E}[\mathrm{e}^{-\gamma}\beta\, u_k + \gamma\, c_k + (Q\ /.\ c_j\ |\ u_i \to 0)]\ /.\ \left\{\gamma \to \partial_{c_j}Q,\ \beta \to \partial_{u_i}Q\right\}\Big];\end{aligned}$$

$$\begin{aligned}&N_{w_{i\_}\ c_{j\_}\to k\_}[\omega\_.\ \mathbb{E}[Q\_]] := \text{CF}\Big[\\&\quad \omega\,\mathbb{E}[\mathrm{e}^{\gamma}\alpha\, w_k + \gamma\, c_k + (Q\ /.\ c_j\ |\ w_i \to 0)]\ /.\ \left\{\gamma \to \partial_{c_j}Q,\ \alpha \to \partial_{w_i}Q\right\}\Big];\end{aligned}$$

$$\begin{aligned}&N_{w_{i\_}\ u_{j\_}\to k\_}[\omega\_.\ \mathbb{E}[Q\_]] := \text{CF}\Big[\\&\quad \nu\,\omega\,\mathbb{E}[-b_k\,\nu\,\alpha\,\beta + \nu\,\beta\, u_k + \nu\,\delta\, u_k\, w_k + \nu\,\alpha\, w_k + (Q\ /.\ w_i\ |\ u_j \to 0)]\ /.\\&\qquad \nu \to (1 + b_k\,\delta)^{-1}\ /.\\&\qquad \left\{\alpha \to \partial_{w_i}Q\ /.\ u_j \to 0,\ \beta \to \partial_{u_j}Q\ /.\ w_i \to 0,\ \delta \to \partial_{w_i,u_j}Q\right\}\Big];\end{aligned}$$

Stitching

$$\begin{aligned}&m_{i\_,j\_\to k\_}[\omega\_.\ \mathbb{E}[Q\_]] := \text{CF}\Big[\text{Module}\Big[\{x\},\\&\quad \left(\omega\,\mathbb{E}[Q]\ /.\ b_{i|j} \to b_k\ //\ N_{w_i\ c_j\to x}\ //\ N_{u_i\ c_x\to x}\ //\ N_{w_x\ u_j\to x}\right)\ /.\\&\quad \{c_i \to c_k,\ w_j \to w_k,\ y\_{}_x :\to y_k\}\Big]\Big]\end{aligned}$$

Some calculations for T_0

$$T_{0,0} = R^{+}_{0,5,1}\,R^{+}_{0,2,4}\,R^{-}_{0,3,6}$$

$$\mathbb{E}\left[b_5\, c_1 + b_2\, c_4 - b_3\, c_6 + \frac{\left(-1+\mathrm{e}^{b_5}\right) u_5\, w_1}{b_5} + \frac{\left(-1+\mathrm{e}^{b_2}\right) u_2\, w_4}{b_2} + \frac{\left(-1+\mathrm{e}^{-b_3}\right) u_3\, w_6}{b_3}\right]$$

$$T_{0,1} = T_{0,0}\ //\ N_{u_3\ c_4\to 4}$$

$$\mathbb{E}\left[b_5\, c_1 + b_2\, c_4 - b_3\, c_6 + \frac{\left(-1+\mathrm{e}^{b_5}\right) u_5\, w_1}{b_5} + \frac{\left(-1+\mathrm{e}^{b_2}\right) u_2\, w_4}{b_2} + \frac{\mathrm{e}^{-b_2}\left(-1+\mathrm{e}^{-b_3}\right) u_4\, w_6}{b_3}\right]$$

$$T_{0,2} = T_{0,1}\ //\ N_{w_4\ u_5\to 4}$$

$$\mathbb{E}\left[b_5\, c_1 + b_2\, c_4 + \frac{\left(-1+\mathrm{e}^{b_5}\right)\left(-\left(-1+\mathrm{e}^{b_2}\right) b_4\, u_2 + b_2\, u_4\right) w_1}{b_2\, b_5} + \frac{\left(-1+\mathrm{e}^{b_2}\right) u_2\, w_4}{b_2} - \frac{b_3^2\, c_6 + \mathrm{e}^{-b_2-b_3}\left(-1+\mathrm{e}^{b_3}\right) u_4\, w_6}{b_3}\right]$$

$$T_{0,2}\ //\ N_{w_1\ u_2\to 1}$$

$$\begin{aligned}&\frac{1}{1-\frac{\left(-1+\mathrm{e}^{b_2}\right)\left(-1+\mathrm{e}^{b_5}\right) b_1\, b_4}{b_2\, b_5}}\ \mathbb{E}\Bigg[\frac{1}{b_3\left(\left(-1+\mathrm{e}^{b_2}\right)\left(-1+\mathrm{e}^{b_5}\right) b_1\, b_4 - b_2\, b_5\right)}\\&\quad \Big(b_3\, b_5\left(\left(-1+\mathrm{e}^{b_2}\right)\left(-1+\mathrm{e}^{b_5}\right) b_1\, b_4 - b_2\, b_5\right) c_1 +\\&\qquad b_2\, b_3\left(\left(-1+\mathrm{e}^{b_2}\right)\left(-1+\mathrm{e}^{b_5}\right) b_1\, b_4 - b_2\, b_5\right) c_4 +\\&\qquad \left(-1+\mathrm{e}^{b_2}\right)\left(-1+\mathrm{e}^{b_5}\right) b_3\, b_4\, u_1\, w_1 - \left(-1+\mathrm{e}^{b_5}\right) b_2\, b_3\, u_4\, w_1 -\\&\qquad \left(-1+\mathrm{e}^{b_2}\right) b_3\, b_5\, u_1\, w_4 + \left(-1+\mathrm{e}^{b_2}\right)\left(-1+\mathrm{e}^{b_5}\right) b_1\, b_3\, u_4\, w_4 -\\&\qquad \left(\left(-1+\mathrm{e}^{b_2}\right)\left(-1+\mathrm{e}^{b_5}\right) b_1\, b_4 - b_2\, b_5\right)\\&\qquad\ \left(b_3^2\, c_6 + \mathrm{e}^{-b_2-b_3}\left(-1+\mathrm{e}^{b_3}\right) u_4\, w_6\right)\Big)\Bigg]\end{aligned}$$

$$T_{0,0}\ //\ m_{1,2\to 1}\ //\ m_{3,4\to 3}\ //\ m_{3,5\to 3}\ //\ m_{3,6\to 3}$$

$$\begin{aligned}&\frac{1}{1-\left(-1+\mathrm{e}^{b_1}\right)\left(-1+\mathrm{e}^{b_3}\right)}\ \mathbb{E}\Bigg[b_3\, c_1 + b_1\, c_3 - b_3\, c_3 +\\&\quad \frac{\left(-1+\mathrm{e}^{b_1}\right)\left(-1+\mathrm{e}^{b_3}\right) u_1\, w_1}{\left(-\mathrm{e}^{b_1}-\mathrm{e}^{b_3}+\mathrm{e}^{b_1+b_3}\right) b_1} - \frac{\mathrm{e}^{-b_3}\left(-1+\mathrm{e}^{b_1}\right)\left(b_3\, u_1 - \mathrm{e}^{b_3}\left(-1+\mathrm{e}^{b_3}\right) b_1\, u_3\right) w_3}{\left(-\mathrm{e}^{b_1}-\mathrm{e}^{b_3}+\mathrm{e}^{b_1+b_3}\right) b_1\, b_3} +\\&\quad \frac{\mathrm{e}^{-b_1}\left(-1+\mathrm{e}^{b_3}\right) u_3\left(-\mathrm{e}^{b_1+b_3}\, w_1 + \left(\mathrm{e}^{b_1}+\mathrm{e}^{b_3}-\mathrm{e}^{b_1+b_3}\right) w_3\right)}{\left(-\mathrm{e}^{b_1}-\mathrm{e}^{b_3}+\mathrm{e}^{b_1+b_3}\right) b_3}\Bigg]\end{aligned}$$

Verifying meta-associativity

$$Q0 = \mathbb{E}[\text{Sum}[f_i\, c_i,\ \{i, 3\}] + \text{Sum}[f_{i,j}\, u_i\, w_j,\ \{i, 3\},\ \{j, 3\}]]$$

$$\begin{aligned}&\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} +\\&\quad u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]\end{aligned}$$

$$(Q0\ //\ m_{1,2\to 1}\ //\ m_{1,3\to 1}) \equiv (Q0\ //\ m_{2,3\to 2}\ //\ m_{1,2\to 1})$$

```
True
```

Testing R3

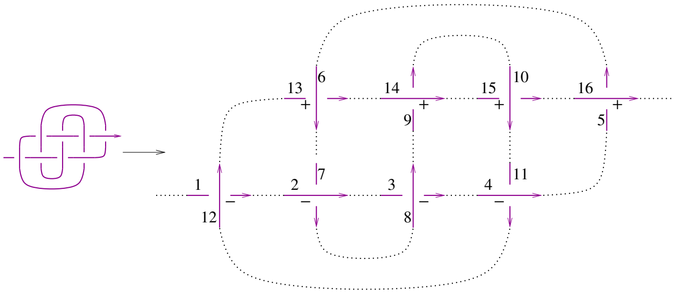
$$t1 = R^{+}_{0,1,2}\,R^{+}_{0,3,4}\,R^{+}_{0,5,6}\ //\ m_{3,5\to x}\ //\ m_{1,6\to y}\ //\ m_{2,4\to z}$$

$$\mathbb{E}\left[b_x\,(c_y + c_z) + \frac{\left(-1+\mathrm{e}^{b_x}\right) u_x\,(w_y + w_z)}{b_x} + \frac{b_y^2\, c_z + \left(-1+\mathrm{e}^{b_y}\right) u_y\, w_z}{b_y}\right]$$

$$t1 \equiv (R^{+}_{0,1,2}\,R^{+}_{0,3,4}\,R^{+}_{0,5,6}\ //\ m_{1,3\to x}\ //\ m_{2,5\to y}\ //\ m_{4,6\to z})$$

```
True
```

8_{17}



$$z1 = R^{-}_{0,12,1}\,R^{-}_{0,2,7}\,R^{-}_{0,8,3}\,R^{-}_{0,4,11}\,R^{+}_{0,16,5}\,R^{+}_{0,6,13}\,R^{+}_{0,14,9}\,R^{+}_{0,10,15};$$

$$\text{Do}[z1 = (z1\ //\ m_{1,n\to 1})\ /.\ b\_ \to b,\ \{n, 2, 16\}];$$

$$\{\text{CF}@z1,\ \text{KnotData}[\{8, 17\},\ \text{"AlexanderPolynomial"}][t]\}$$

$$\left\{-\frac{\mathrm{e}^{3b}\,\mathbb{E}[0]}{1-4\,\mathrm{e}^{b}+8\,\mathrm{e}^{2b}-11\,\mathrm{e}^{3b}+8\,\mathrm{e}^{4b}-4\,\mathrm{e}^{5b}+\mathrm{e}^{6b}},\ 11 - \frac{1}{t^3} + \frac{4}{t^2} - \frac{8}{t} - 8t + 4t^2 - t^3\right\}$$

Demo Programs for 1-Co.

ωεβ/Demo

The Λόγος

$$\begin{aligned}&\Lambda[k\_] := (1 - t_k)\left(\alpha^2\beta^2 + 4\,\alpha\,\beta\,\delta\,\mu + 2\,\delta^2\mu^2\right)/2 + 2\,\mu^2\,(\alpha\,\beta + \delta\,\mu)\, c_k -\\&\quad \beta\,(2\mu - 1)\,(\alpha\,\beta + 2\,\delta\,\mu)\, u_k + 2\,\beta\,\delta\,\mu^2\, c_k\, u_k - \beta^2\,\delta\,(3\mu - 1)\, u_k^2/2 +\\&\quad \alpha\,(\alpha\,\beta + 2\,\delta\,\mu)\, w_k + 2\,\alpha\,\delta\,\mu^2\, c_k\, w_k - 2\,(t_k - 1)\,\delta^2\,(\alpha\,\beta + \delta\,\mu)\, u_k\, w_k +\\&\quad 2\,\delta^2\mu^2\, c_k\, u_k\, w_k - \beta\,\delta^2\,(2\mu - 1)\, u_k^2\, w_k + \alpha^2\,\delta\,(1 + \mu)\, w_k^2/2 +\\&\quad \alpha\,\delta^2\, u_k\, w_k^2 - (t_k - 1)\,\delta^4\, u_k^2\, w_k^2/2;\end{aligned}$$

Differential Polynomials

$$\begin{aligned}&DP_{x\_\to D_{\alpha\_},\,y\_\to D_{\beta\_}}[P\_][f\_] := (\ast\ \text{means } P[\partial_\alpha,\partial_\beta][f]\ \ast)\\&\quad \text{Total}[\text{CoefficientRules}[P,\ \{x, y\}]\ /.\\&\qquad (\{m\_, n\_\} \to c\_) :\to c\, D[f,\ \{\alpha, m\},\ \{\beta, n\}]]\end{aligned}$$

Utilities

$$\begin{aligned}&\text{CF}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \text{Expand}\ /@\ \text{Together}\ /@\\&\quad \mathbb{E}[\omega\ /.\ b_{l\_} :\to \text{Log}[t_l],\ L,\ Q\ /.\ b_{l\_} :\to \text{Log}[t_l],\\&\quad\ P\ /.\ b_{l\_} :\to \text{Log}[t_l]];\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\ /\!:\ \mathbb{E}[\omega 1\_, L1\_, Q1\_, P1\_]\ \mathbb{E}[\omega 2\_, L2\_, Q2\_, P2\_] :=\\&\quad \text{CF}@\mathbb{E}\left[\omega 1\,\omega 2,\ L1 + L2,\ \omega 2\, Q1 + \omega 1\, Q2,\ \omega 2^4\, P1 + \omega 1^4\, P2\right];\end{aligned}$$

Normal Ordering Operators

$$\begin{aligned}&N_{u_{i\_}\ c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \text{With}\Big[\{q = \mathrm{e}^{-\gamma}\beta\, u_k + \gamma\, c_k\},\ \text{CF}\Big[\\&\quad \mathbb{E}\Big[\omega,\ \gamma\, c_k + (L\ /.\ c_j \to 0),\ \omega\,\mathrm{e}^{-\gamma}\beta\, u_k + (Q\ /.\ u_i \to 0),\\&\qquad \mathrm{e}^{-q}\, DP_{c_j\to D_\gamma,\,u_i\to D_\beta}[P][\mathrm{e}^{q}]\Big]\ /.\ \left\{\gamma \to \partial_{c_j}L,\ \beta \to \omega^{-1}\partial_{u_i}Q\right\}\Big]\Big];\end{aligned}$$

$$\begin{aligned}&N_{w_{i\_}\ c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \text{With}\Big[\{q = \mathrm{e}^{\gamma}\alpha\, w_k + \gamma\, c_k\},\ \text{CF}\Big[\\&\quad \mathbb{E}\Big[\omega,\ \gamma\, c_k + (L\ /.\ c_j \to 0),\ \omega\,\mathrm{e}^{\gamma}\alpha\, w_k + (Q\ /.\ w_i \to 0),\\&\qquad \mathrm{e}^{-q}\, DP_{c_j\to D_\gamma,\,w_i\to D_\alpha}[P][\mathrm{e}^{q}]\Big]\ /.\ \left\{\gamma \to \partial_{c_j}L,\ \alpha \to \omega^{-1}\partial_{w_i}Q\right\}\Big]\Big];\end{aligned}$$

$$\begin{aligned}&N_{w_{i\_}\ u_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] :=\\&\quad \text{With}\Big[\left\{q = (1 - t_k)\,\mu^{-1}\alpha\,\beta + \mu^{-1}\beta\, u_k + \mu^{-1}\delta\, u_k\, w_k + \mu^{-1}\alpha\, w_k\right\},\ \text{CF}\Big[\\&\quad \mathbb{E}\Big[\mu\,\omega,\ L,\ \mu\,\omega\, q + \mu\,(Q\ /.\ w_i\ |\ u_j \to 0),\\&\qquad \mu^4\,\mathrm{e}^{-q}\, DP_{w_i\to D_\alpha,\,u_j\to D_\beta}[P][\mathrm{e}^{q}] + \omega^4\,\Lambda[k]\Big]\ /.\\&\qquad \mu \to 1 + (t_k - 1)\,\delta\ /.\\&\qquad \left\{\alpha \to \omega^{-1}\left(\partial_{w_i}Q\ /.\ u_j \to 0\right),\ \beta \to \omega^{-1}\left(\partial_{u_j}Q\ /.\ w_i \to 0\right),\right.\\&\qquad \left.\delta \to \omega^{-1}\partial_{w_i,u_j}Q\right\}\Big]\Big];\end{aligned}$$

Stitching

$$\begin{aligned}&m_{i\_,j\_\to k\_}[Z\_] := \text{Module}\Big[\{x, y, z\},\\&\quad Z\ //\ N_{w_i\ c_j\to x}\ //\ N_{w_x\ u_j\to y}\ //\ \text{ReplaceAll}[\{c_{x|y} \to c_x,\ w_j \to w_y\}]\ //\\&\qquad N_{u_i\ c_x\to x}\ //\ \text{ReplaceAll}\left[z\_{}_{i|j|x|y} \to z_k\right]\ //\ \text{CF}\Big]\end{aligned}$$

The Generators

$$\mathrm{R}^{+}_{i\_,j\_} := \mathbb{E}\Big[1,\ b_i\, c_j,\ u_i\, w_j,\ -c_i\,(t_i-1)^2/2 - c_i^2\,(t_i-1)^2/2 + c_i\, c_j\,\left(t_j^2 - t_i - 2\right)/2 - c_j\, u_i\, w_i/2 + c_i\,(1-t_i)\, u_i\, w_i - u_i^2\, w_i^2/2 + u_i\, w_j + c_j\, t_i\, u_i\, w_j/2 + c_i\,(t_i-2)\, t_i\, u_i\, w_j + c_i\,(1+t_j)\, u_j\, w_j/2 + (t_i-1)\, u_i^2\, w_i\, w_j - (t_i-2)\, t_i\, u_i^2\, w_j^2/2\Big];$$

$$\mathrm{R}^{-}_{i\_,j\_} := \mathbb{E}\Big[1,\ -b_i\, c_j,\ -t_i^{-1}\, u_i\, w_j,\ c_i\,(t_i-1)^2/2 + c_i^2\,(t_i-1)^2/2 + c_i\, c_j\,\left(2 + t_i - t_j^2\right)/2 + c_j\, u_i\, w_i/2 + c_i\,(t_i-1)\, u_i\, w_i + u_i^2\, w_i^2/2 + \left(1 - t_i^{-1}\right) u_i\, w_j/2 + c_i\,\left(2\, t_i - 5 + 3\, t_i^{-1}\right) u_i\, w_j/2 + c_j\,\left(t_i^{-1} + 1 - t_i^{-1}\, t_j^2\right) u_i\, w_j/2 - c_i\,(t_j+1)\, u_j\, w_j/2 + \left(2 - 3\, t_i^{-1}\right) u_i^2\, w_i\, w_j/2 + \left(1 + 2\, t_i^{-2} - 3\, t_i^{-1}\right) u_i^2\, w_j^2/2 - t_i^{-1}\,(1+t_j)\, u_i\, u_j\, w_j^2/2\Big];$$

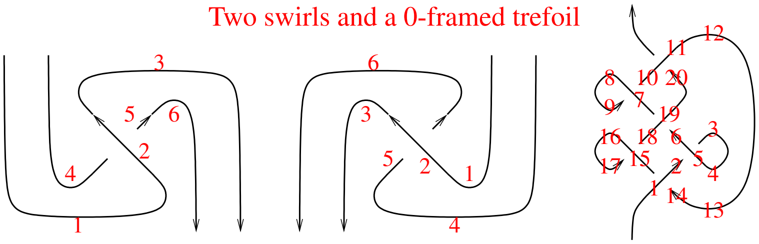
$$\mathrm{ur}_{i\_} := \mathbb{E}\left[t_i^{-1/4},\ 0,\ 0,\ c_i\, t_i/4 + u_i\, w_i/8\right];$$

$$\mathrm{nr}_{i\_} := \mathbb{E}\left[t_i^{1/4},\ 0,\ 0,\ -c_i\, t_i^3/4 - t_i^2\, u_i\, w_i/8\right];$$

$$\mathrm{ul}_{i\_} := \mathbb{E}\left[t_i^{1/4},\ 0,\ 0,\ c_i\, t_i\,(4+t_i)/4 - t_i^2\, u_i\, w_i/8\right];$$

$$\mathrm{nl}_{i\_} := \mathbb{E}\left[t_i^{-1/4},\ 0,\ 0,\ -c_i\,\left(1 + 4\, t_i^{-1}\right)/4 + u_i\, w_i/8\right];$$

Two swirls and a 0-framed trefoil



$$\mathrm{t2} = \mathrm{ur}_1\, \mathrm{R}^{-}_{2,5}\, \mathrm{nr}_3\, \mathrm{ur}_4\, \mathrm{nr}_6\ //\ \mathrm{m}_{1,2\to1}\ //\ \mathrm{m}_{1,3\to1}\ //\ \mathrm{m}_{4,5\to4}\ //\ \mathrm{m}_{4,6\to4}$$

$$\mathbb{E}\Big[1,\ -b_1\, c_4,\ -\frac{u_1\, w_4}{t_1},\ \frac{c_1}{2} + \frac{c_1^2}{2} + c_1\, c_4 - c_1\, t_1 - c_1^2\, t_1 + \frac{1}{2}\, c_1\, c_4\, t_1 + \frac{1}{2}\, c_1\, t_1^2 + \frac{1}{2}\, c_1^2\, t_1^2 - \frac{1}{2}\, c_1\, c_4\, t_4^2 - c_1\, u_1\, w_1 + \frac{1}{2}\, c_4\, u_1\, w_1 + c_1\, t_1\, u_1\, w_1 + \frac{1}{2}\, u_1^2\, w_1^2 + \frac{3\, u_1\, w_4}{8} - \frac{5}{2}\, c_1\, u_1\, w_4 + \frac{1}{2}\, c_4\, u_1\, w_4 - \frac{u_1\, w_4}{2\, t_1} + \frac{3\, c_1\, u_1\, w_4}{2\, t_1} + \frac{c_4\, u_1\, w_4}{2\, t_1} - \frac{1}{8}\, t_1\, u_1\, w_4 + c_1\, t_1\, u_1\, w_4 + \frac{t_4\, u_1\, w_4}{8\, t_1} + \frac{t_4^2\, u_1\, w_4}{8\, t_1} - \frac{c_4\, t_4^2\, u_1\, w_4}{2\, t_1} - \frac{1}{2}\, c_1\, u_4\, w_4 - \frac{1}{2}\, c_1\, t_4\, u_4\, w_4 + u_1^2\, w_1\, w_4 - \frac{3\, u_1^2\, w_1\, w_4}{2\, t_1} + \frac{1}{2}\, u_1^2\, w_4^2 + \frac{u_1^2\, w_4^2}{t_1^2} - \frac{3\, u_1^2\, w_4^2}{2\, t_1} - \frac{u_1\, u_4\, w_4^2}{2\, t_1} - \frac{t_4\, u_1\, u_4\, w_4^2}{2\, t_1}\Big]$$

$$\mathrm{t2} == \left(\mathrm{ul}_1\, \mathrm{R}^{-}_{2,5}\, \mathrm{nl}_3\, \mathrm{ul}_4\, \mathrm{nl}_6\ //\ \mathrm{m}_{1,2\to1}\ //\ \mathrm{m}_{1,3\to1}\ //\ \mathrm{m}_{4,5\to4}\ //\ \mathrm{m}_{4,6\to4}\right)$$

True

$$\mathrm{z2} = \mathrm{R}^{+}_{1,14}\, \mathrm{R}^{-}_{5,2}\, \mathrm{nr}_3\, \mathrm{ul}_4\, \mathrm{R}^{+}_{19,6}\, \mathrm{R}^{-}_{7,10}\, \mathrm{nl}_8\, \mathrm{ur}_9\, \mathrm{R}^{+}_{11,20}\, \mathrm{nr}_{12}\, \mathrm{ul}_{13}\, \mathrm{R}^{-}_{15,18}\, \mathrm{nl}_{16}\, \mathrm{ur}_{17};$$

$$(\mathrm{Do}[\mathrm{z2} = \mathrm{z2}\ //\ \mathrm{m}_{1,k\to1},\ \{k, 2, 20\}];\ \mathrm{z2} = \mathrm{z2}\ /.\ \mathrm{a\_1} :\to \mathrm{a})$$

$$\mathbb{E}\Big[-1 + \frac{1}{t} + t,\ 0,\ 0,\ -16 + \frac{9\, c}{2} - \frac{2\, c}{t^4} + \frac{1}{t^3} + \frac{11\, c}{2\, t^3} - \frac{4}{t^2} - \frac{8\, c}{t^2} + \frac{10}{t} + \frac{4\, c}{t} + 18\, t - 10\, c\, t - 14\, t^2 + 8\, c\, t^2 + 7\, t^3 - \frac{3\, c\, t^3}{2} - 2\, t^4 - 2\, c\, t^4 + 2\, c\, t^5 - \frac{c\, t^6}{2} - 4\, u\, w + \frac{2\, u\, w}{t^4} - \frac{7\, u\, w}{2\, t^3} + \frac{9\, u\, w}{2\, t^2} + \frac{u\, w}{2\, t} + 6\, t\, u\, w - 2\, t^2\, u\, w - \frac{1}{2}\, t^3\, u\, w + \frac{3}{2}\, t^4\, u\, w - \frac{1}{2}\, t^5\, u\, w\Big]$$

```
FromCoefficientRules[
 CoefficientRules[z2⟦4⟧, {c, u, w}] /.
  {(e_ → a_) :→ (e → Simplify[a])}, {c, u, w}]
```

$$-\frac{\left(1-t+t^2\right)^2 \left(-1+2\, t-3\, t^2+2\, t^3\right)}{t^3} - \frac{c\, \left(1-t+t^2\right)^3 \left(4+t-5\, t^2-t^3+t^4\right)}{2\, t^4} - \frac{\left(1-t+t^2\right)^3 \left(-4-5\, t+t^3\right) u\, w}{2\, t^4}$$

Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • What about links / closed components? • Fully digest the "expansion" theorem. • Explore the (non-)dependence on *R*. • Is there a canonical *R*? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" *q*-algebra. • *k*-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

References.

[BN1] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BN2] D. Bar-Natan, *Polynomial Time Knot Polynomial,* research proposal for the 2017 Killam Fellowship, ωεβ/K17.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I, II, IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO4, arXiv:1405.1956, arXiv:1405.1955, arXiv:1511.05624.

[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture,* Invent. Math. **125** (1996) 103–133.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[En] B. Enriquez, *A Cohomological Construction of Quantization Functors of Lie Bialgebras,* Adv. in Math. **197-2** (2005) 430-479, arXiv:math/0212325.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica **2** (1996) 1–41, arXiv:q-alg/9506005.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[GPV] M. Goussarov, M. Polyak, and O. Viro, *Finite type invariants of classical and virtual knots,* Topology **39** (2000) 1045–1068, arXiv:math.GT/9810073.

[Ha] A. Haviv, *Towards a diagrammatic analogue of the Reshetikhin-Turaev link invariants,* Hebrew University PhD thesis, Sep. 2002, arXiv:math.QA/0211031.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function,* Commun. Math. Phys. **169** (1995) 501–520.

[PV] M. Polyak and O. Viro, *Gauss Diagram Formulas for Vassiliev Invariants,* Inter. Math. Res. Notices **11** (1994) 445–453.

[Ro] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I,* Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Se] P. Ševera, *Quantization of Lie Bialgebras Revisited,* Sel. Math., NS, to appear, arXiv:1401.6164.

Disclaimer. This is all quite new. The overall picture is correct, yet some details might be somewhat off. Many pieces are certainly not in their final form yet.

Dire Warning. On Tuesday night agents of the Evil Galactic Empire will lock all participants of this workshop in separate sound proof, electro-magnetically sealed, neutrino hardened, and gravitational wave resistant secret rooms in Hotel Les Sources. In the rooms they will place identical countable sequences of numbered boxes, each one containing a real number (the same sequence of real numbers in each room). By morning, each participant must open all but one of their boxes in the order of their liking, and guess the number in the remaining one. If more than one participant guesses wrong, breakfast will be poisoned. Do Something! We must devise a strategy during Tuesday's hike or else we will miss Thomas' talks!

ωεβ/Joker

— "Saw Omega" from Alfonso Gracia-Saz from Mira Bernstein from ωεβ/SO (spoilers). Deadly serious.