

```
In[*]:= SetDirectory[
  "C:\\drorbn\\AcademicPensieve\\Talks\\LearningSeminarOnCategorification-2006"];
<< "Engine-Speedy.m";
<< "Objects.m";
$k = 0; ħ = 1;
```

» Warning: On Sep 4 2019 I swapped the operations ϵ and η . Some incompatibilities may arise in older notebooks.

```
In[*]:= s; h; t; Γ; dL; V; Xp; Xm
Once[Begin["MetaCalculi`"]; << "../..../Projects/MetaCalculi/MetaCalculi.m";
End[]];
ΓSimp = MetaCalculi`ΓSimp;
ΓR[i_, j_] := Γ[Xp[i, j]]; ΓR̄[i_, j_] := Γ[Xm[i, j]];
```

Out[*]= Xm

MetaCalculi` loading...

```
In[*]:= ΓR[1, 2]
```

$$Out[*]= \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Gamma & 1 & T_1 \end{pmatrix}$$

```
In[*]:= {Ci, Ri,j} /. U21
```

$$Out[*]= \{ \mathbb{E}_{\{\} \rightarrow \{i\}} [0, 0, e^{-\frac{b_i}{2}} + 0[\epsilon]^1], \mathbb{E}_{\{\} \rightarrow \{i,j\}} [a_j b_i, x_j y_i, 1] \}$$

```
In[*]:= cmi,j→k
```

$$Out[*]= \mathbb{E}_{\{i,j\} \rightarrow \{k\}} [a_k \alpha_i + a_k \alpha_j + b_k \beta_i + b_k \beta_j, y_k \eta_i + \frac{y_k \eta_j}{\mathcal{A}_i} + \frac{x_k \xi_i}{\mathcal{A}_j} + b_k \eta_j \xi_i + x_k \xi_j, 1 + 0[\epsilon]^1]$$

```
In[*]:= dmi,j→k
```

$$Out[*]= \mathbb{E}_{\{i,j\} \rightarrow \{k\}} [a_k \alpha_i + a_k \alpha_j + b_k \beta_i + b_k \beta_j, y_k \eta_i + \frac{y_k \eta_j}{\mathcal{A}_i} + \frac{x_k \xi_i}{\mathcal{A}_j} + (1 - B_k) \eta_j \xi_i + x_k \xi_j, 1 + 0[\epsilon]^1]$$

```
In[*]:= cRi_,j_ := E{i,j} [aj bi,  $\frac{1 - B_i}{b_i} x_j y_i, 1]$ 
```

```
In[*]:= (cR1,2 cR4,3 cR5,6 // cm1,4→1 // cm2,5→2 // cm3,6→3) ≡ (cR2,3 cR1,6 cR4,5 // cm1,4→1 // cm2,5→2 // cm3,6→3)
```

Out[*]= True

```
In[*]:= cR̄i_,j_ := E{i,j} [-aj bi,  $\frac{B_i - 1}{b_i B_i} x_j y_i, 1]$ 
```

```
In[*]:= cR1,3 cR̄2,4 // cm1,2→1 // cm3,4→3
```

$$Out[*]= \mathbb{E}_{\{\} \rightarrow \{1,3\}} [0, 0, 1 + 0[\epsilon]^1]$$

```
In[*]:= Gsp1[ω-, L-, Q-] // E := E{}→sp1[L, Q, ω + O[ε]];
Gsp1[ω-, L-, Q-] // Esp2[L2-, Q2-, P2-] := Gsp1[ω, L, Q] // E // Esp2[L2, Q2, P2]
```

```
In[*]:= G1 = Expand /@
```

$$G_{\{1,2,3\}}[\omega, \{b_1, b_2, b_3\}] \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \{y_1, y_2, y_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \xi \end{pmatrix} \cdot \{x_1, x_2, x_3\}$$

```
Out[*]:= G{1,2,3}[ω, a1 b1 l11 + a2 b1 l12 + a3 b1 l13 + a1 b2 l21 + a2 b2 l22 + a3 b2 l23 + a1 b3 l31 + a2 b3 l32 + a3 b3 l33,
α x1 y1 + β x2 y1 + θ x3 y1 + γ x1 y2 + δ x2 y2 + ε x3 y2 + φ x1 y3 + ψ x2 y3 + ξ x3 y3]
```

```
In[*]:= G1 // E
```

```
Out[*]:= E{}→{1,2,3}[a1 b1 l11 + a2 b1 l12 + a3 b1 l13 + a1 b2 l21 + a2 b2 l22 + a3 b2 l23 + a1 b3 l31 + a2 b3 l32 + a3 b3 l33,
α x1 y1 + β x2 y1 + θ x3 y1 + γ x1 y2 + δ x2 y2 + ε x3 y2 + φ x1 y3 + ψ x2 y3 + ξ x3 y3, ω + O[ε]1]
```

```
In[*]:= G1 // cm1,2→0 // Simplify
```

```
Out[*]:= E{}→{0,3}[a0 (b0 (l11 + l12 + l21 + l22) + b3 (l31 + l32)) + a3 (b0 (l13 + l23) + b3 l33),
1 / (-1 + γ b0) (x3 ((-θ + (-α ε + γ θ) b0 - ε B011+121 B3131) y0 + (-ξ + (γ ξ - ε φ) b0) y3) +
x0 ((-β + (β γ - α δ) b0 - δ B011+121 B3131 - α B012+122 B3132 - γ B011+12+121+122 B3131+132) y0 +
(-ψ + (-δ φ + γ ψ) b0 - φ B012+122 B3132) y3), ω / (1 - γ b0) + O[ε]1]
```

```
In[*]:= E{}→r[L-, Q-, P-] // G := Gr[Normal@P, L, Q]
```

```
In[*]:= G1 // cm1,2→0 // G
```

```
Out[*]:= G{0,3}[ - ω / (-1 + γ b0), a0 b0 (l11 + l12 + l21 + l22) + a3 b0 (l13 + l23) + a0 b3 (l31 + l32) + a3 b3 l33,
1 / (-1 + γ b0) (-β x0 y0 + β γ b0 x0 y0 - α δ b0 x0 y0 - δ B011+121 B3131 x0 y0 - α B012+122 B3132 x0 y0 -
γ B011+12+121+122 B3131+132 x0 y0 - θ x3 y0 - α ε b0 x3 y0 + γ θ b0 x3 y0 - ε B011+121 B3131 x3 y0 -
ψ x0 y3 - δ φ b0 x0 y3 + γ ψ b0 x0 y3 - φ B012+122 B3132 x0 y3 - ξ x3 y3 + γ ξ b0 x3 y3 - ε φ b0 x3 y3)]
```

```
In[*]:= GS[ω-, L-, Q-] // GForm := Module[{M, i, j},
M = Table[Simplify[∂xi Q], {i, S}, {j, S}];
PrependTo[M, y# & /@ S];
M = Join[
{Prepend[x# & /@ S, ω]},
Transpose[M],
{Prepend[Table[∂ai L, {i, S}], "G"]}
];
MatrixForm[M]
];
```

```
In[*]:= GS1[ω1-, L1-, Q1-] ≡ GS2[ω2-, L2-, Q2-] :=
(Sort[S1] === Sort[S2]) ∧ Simplify[ω1 == ω2] ∧ Simplify[L1 == L2] ∧ Simplify[Q1 == Q2]
```

In[]:= **G1 // GForm**

Out[]//MatrixForm=

$$\begin{pmatrix} \omega & & & \\ \mathbf{y}_1 & \alpha & & \\ \mathbf{y}_2 & \gamma & & \\ \mathbf{y}_3 & \phi & & \\ \mathbb{G} & b_1 l_{11} + b_2 l_{21} + b_3 l_{31} & b_1 l_{12} + b_2 l_{22} + b_3 l_{32} & b_1 l_{13} + b_2 l_{23} + b_3 l_{33} \end{pmatrix}$$

In[]:= **G1 // cm_{1,2→0} // G // GForm**

Out[]//MatrixForm=

$$\begin{pmatrix} -\frac{\omega}{1+\gamma b_\theta} & & & \\ \mathbf{y}_\theta & \frac{\beta + (-\beta\gamma + \alpha\delta) b_\theta + \delta B_\theta^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_\theta^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_\theta^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1-\gamma b_\theta} & & \frac{\theta + (\alpha\epsilon - \gamma\theta) b_\theta + \epsilon B_\theta^{l_{11}+l_{21}} B_3^{l_{31}}}{1-\gamma b_\theta} \\ \mathbf{y}_3 & \frac{\psi + (\delta\phi - \gamma\psi) b_\theta + \phi B_\theta^{l_{12}+l_{22}} B_3^{l_{32}}}{1-\gamma b_\theta} & & \frac{\Xi + (-\gamma\Xi + \epsilon\phi) b_\theta}{1-\gamma b_\theta} \\ \mathbb{G} & b_\theta (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & & b_\theta (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

In[]:= **((G1 /. $\mathbf{y}_i \rightarrow \mathbf{y}_i / b_i$) // cm_{1,2→0}) /. $\mathbf{y}_i \rightarrow \mathbf{y}_i b_i$ // G // GForm**

Out[]//MatrixForm=

$$\begin{pmatrix} -\frac{\omega}{1+\gamma} & & & \\ \mathbf{y}_\theta & \frac{\beta - \beta\gamma + \alpha\delta + \delta B_\theta^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_\theta^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_\theta^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1-\gamma} & & \frac{\alpha\epsilon + \theta - \gamma\theta + \epsilon B_\theta^{l_{11}+l_{21}} B_3^{l_{31}}}{1-\gamma} \\ \mathbf{y}_3 & \frac{\delta\phi + \psi - \gamma\psi + \phi B_\theta^{l_{12}+l_{22}} B_3^{l_{32}}}{1-\gamma} & & \frac{\Xi - \frac{\epsilon\phi}{1+\gamma}}{1-\gamma} \\ \mathbb{G} & b_\theta (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & & b_\theta (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

In[]:= **G2 = G_{0,3} [$\frac{\omega}{1-\gamma b_\theta}$, {b₀, b₃} . ($\begin{matrix} l_{11} + l_{12} + l_{21} + l_{22} & l_{13} + l_{23} \\ l_{31} + l_{32} & l_{33} \end{matrix}$) . {a₀, a₃}, {y₀, y₃}.**

$$\left(\begin{pmatrix} \frac{\beta + (\alpha\delta - \gamma\beta) b_\theta + \delta B_\theta^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_\theta^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_\theta^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1-\gamma b_\theta} & \frac{\theta + (\alpha\epsilon - \gamma\theta) b_\theta + \epsilon B_\theta^{l_{11}+l_{21}} B_3^{l_{31}}}{1-\gamma b_\theta} \\ \frac{\psi + (\delta\phi - \gamma\psi) b_\theta + \phi B_\theta^{l_{12}+l_{22}} B_3^{l_{32}}}{1-\gamma b_\theta} & \frac{\Xi + (\epsilon\phi - \gamma\Xi) b_\theta}{1-\gamma b_\theta} \end{pmatrix} \cdot \{\mathbf{X}_\theta, \mathbf{X}_3\} \right) // \mathbb{GForm}$$

(G1 // cm_{1,2→0} // G) ≡ G2

Out[]//MatrixForm=

$$\begin{pmatrix} \frac{\omega}{1-\gamma b_\theta} & & & \\ \mathbf{y}_\theta & \frac{\beta + (-\beta\gamma + \alpha\delta) b_\theta + \delta B_\theta^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_\theta^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_\theta^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1-\gamma b_\theta} & & \frac{\theta + (\alpha\epsilon - \gamma\theta) b_\theta + \epsilon B_\theta^{l_{11}+l_{21}} B_3^{l_{31}}}{1-\gamma b_\theta} \\ \mathbf{y}_3 & \frac{\psi + (\delta\phi - \gamma\psi) b_\theta + \phi B_\theta^{l_{12}+l_{22}} B_3^{l_{32}}}{1-\gamma b_\theta} & & \frac{\Xi + (-\gamma\Xi + \epsilon\phi) b_\theta}{1-\gamma b_\theta} \\ \mathbb{G} & b_\theta (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & & b_\theta (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

Out[]:= True

In[]:= **G1 // dm_{1,2→0} // G // GForm**

Out[]//MatrixForm=

$$\begin{pmatrix} \frac{\omega}{1-\gamma + \gamma B_\theta} & & & \\ \mathbf{y}_\theta & \frac{\beta - \beta\gamma + \alpha\delta + (\beta\gamma - \alpha\delta) B_\theta + \delta B_\theta^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_\theta^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_\theta^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1-\gamma + \gamma B_\theta} & & \frac{\alpha\epsilon + \theta - \gamma\theta + (-\alpha\epsilon + \gamma\theta) B_\theta + \epsilon B_\theta^{l_{11}+l_{21}} B_3^{l_{31}}}{1-\gamma + \gamma B_\theta} \\ \mathbf{y}_3 & \frac{\delta\phi + \psi - \gamma\psi + (-\delta\phi + \gamma\psi) B_\theta + \phi B_\theta^{l_{12}+l_{22}} B_3^{l_{32}}}{1-\gamma + \gamma B_\theta} & & \frac{\Xi - \gamma\Xi + \epsilon\phi + (\gamma\Xi - \epsilon\phi) B_\theta}{1-\gamma + \gamma B_\theta} \\ \mathbb{G} & b_\theta (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & & b_\theta (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

In[*]:= (R1,4 R6,2 R3,5) // dm1,2->1 // dm1,3->1 // dm1,4->1 // dm5,6->2 // Simplify // C // CForm

Out[*]//MatrixForm=

$$\begin{pmatrix} \frac{B_1 B_2}{-1+B_1+B_2} & X_1 & X_2 \\ Y_1 & \frac{B_1+(-1+B_2) B_2}{B_2 (-1+B_1+B_2)} & -\frac{1}{-1+B_1+B_2} \\ Y_2 & -\frac{B_1}{-1+B_1+B_2} & \frac{1-B_1}{B_1 (-1+B_1+B_2)} \\ \mathbb{C} & b_1 - b_2 & -b_1 \end{pmatrix}$$

In[*]:= (cR1,4 cR6,2 cR3,5) // cm1,2->1 // cm1,3->1 // cm1,4->1 // cm5,6->2 // Simplify // C // CForm

Out[*]//MatrixForm=

$$\begin{pmatrix} \frac{B_1 B_2}{-1+B_1+B_2} & X_1 & X_2 \\ Y_1 & -\frac{(-1+B_1) (B_1+(-1+B_2) B_2)}{b_1 B_2 (-1+B_1+B_2)} & \frac{-1+B_1}{b_1 (-1+B_1+B_2)} \\ Y_2 & \frac{B_1 (-1+B_2)}{b_2 (-1+B_1+B_2)} & \frac{(-1+B_1) (-1+B_2)}{b_2 B_1 (-1+B_1+B_2)} \\ \mathbb{C} & b_1 - b_2 & -b_1 \end{pmatrix}$$

In[*]:= (rR[1, 4] rR[6, 2] rR[3, 5]) // dm[1, 2, 1] // dm[1, 3, 1] // dm[1, 4, 1] // dm[5, 6, 2]

Out[*]=

$$\begin{pmatrix} \frac{-1+T_1+T_2}{T_2} & S_1 & S_2 \\ S_1 & \frac{T_1}{-1+T_1+T_2} & \frac{(-1+T_1) T_2}{-1+T_1+T_2} \\ S_2 & \frac{-1+T_2}{-1+T_1+T_2} & -\frac{1-T_1-2 T_2+T_1 T_2}{-1+T_1+T_2} \\ \Gamma & \frac{T_1}{T_2} & \frac{1}{T_1} \end{pmatrix}$$

In[*]:= $\gamma_1 = \text{Expand} / @$

$$\Gamma_{1\{1,2,3\}}[\omega, \{b_1, b_2, b_3\}] \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \{y_1, y_2, y_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \xi \end{pmatrix} \cdot \{x_1, x_2, x_3\}$$

Out[*]= $\Gamma_{1\{1,2,3\}}[\omega, a_1 b_1 l_{11} + a_2 b_1 l_{12} + a_3 b_1 l_{13} + a_1 b_2 l_{21} + a_2 b_2 l_{22} + a_3 b_2 l_{23} + a_1 b_3 l_{31} + a_2 b_3 l_{32} + a_3 b_3 l_{33}, \alpha x_1 y_1 + \beta x_2 y_1 + \theta x_3 y_1 + \gamma x_1 y_2 + \delta x_2 y_2 + \epsilon x_3 y_2 + \phi x_1 y_3 + \psi x_2 y_3 + \xi x_3 y_3]$

In[*]:= $\Gamma_{1_{sp1}}[\omega, L, Q] // \mathbb{E} := \mathbb{E}_{\{\}} \rightarrow_{sp1}[L, Q, \omega + \mathbf{0}[\epsilon]];$

$\Gamma_{1_{sp1}}[\omega, L, Q] // \mathbb{E}_{sp2}[L2, Q2, P2] := \Gamma_{1_{sp1}}[\omega, L, Q] // \mathbb{E} // \mathbb{E}_{sp2}[L2, Q2, P2]$

In[*]:= $\mathbb{E}_{\{\}} \rightarrow_r[L, Q, P] // \Gamma_1 := \Gamma_{1_r}[\text{Normal}@P, L, Q]$

In[*]:= $\Gamma_{1_s}[\omega, L, Q] // \Gamma_1 \text{Form} := \text{Module}[\{M, i, j\},$
 $M = \text{Table}[\text{Simplify}[\partial_{x_i, y_j} Q], \{i, S\}, \{j, S\}];$
 $\text{PrependTo}[M, y_{\#} \& / @ S];$
 $M = \text{Join}[$
 $\{\text{Prepend}[x_{\#} \& / @ S, \omega]\},$
 $\text{Transpose}[M],$
 $\{\text{Prepend}[\text{Table}[\partial_{a_i} L, \{i, S\}], "r1"]\}$
 $];$
 $\text{MatrixForm}[M]$
 $];$

```
In[ ]:=  $\Gamma_{S1\_}[\omega1\_ , L1\_ , Q1\_ ] \equiv \Gamma_{S2\_}[\omega2\_ , L2\_ , Q2\_ ] :=$   
 $(\text{Sort}[S1] === \text{Sort}[S2]) \wedge \text{Simplify}[\omega1 == \omega2] \wedge \text{Simplify}[L1 == L2] \wedge \text{Simplify}[Q1 == Q2]$ 
```

```
In[ ]:=  $\Upsilon1 \equiv (\Upsilon1 // \mathbb{E} // \Gamma1)$ 
```

```
Out[ ]:= True
```

```
In[ ]:=  $\Upsilon1 // \text{cm}_{1,2 \rightarrow 0} // \Gamma1 // \Gamma1\text{Form}$ 
```

```
Out[ ]:= MatrixForm=
```

$$\begin{pmatrix} -\frac{\omega}{1+\gamma b_0} & X_0 & X_3 \\ y_0 & \frac{\beta + (-\beta\gamma + \alpha\delta) b_0 + \delta B_0^{111+121} B_3^{131} + \alpha B_0^{112+122} B_3^{132} + \gamma B_0^{111+112+121+122} B_3^{131+132}}{1-\gamma b_0} & \frac{\theta + (\alpha - \gamma\theta) b_0 + \epsilon B_0^{111+121} B_3^{131}}{1-\gamma b_0} \\ y_3 & \frac{\psi + (\delta\phi - \gamma\psi) b_0 + \phi B_0^{112+122} B_3^{132}}{1-\gamma b_0} & \frac{\Xi + (-\gamma\Xi + \epsilon\phi) b_0}{1-\gamma b_0} \\ \Gamma1 & b_0 (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & b_0 (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

```
In[ ]:=  $\Upsilon2 = \text{Expand} / @$ 
```

$$\Gamma_{2\{1,2,3\}}[\omega, \{b_1, b_2, b_3\} \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \{y_1, y_2, y_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{x_1, x_2, x_3\}]$$

```
Out[ ]:=  $\Gamma_{2\{1,2,3\}}[\omega, a_1 b_1 l_{11} + a_2 b_1 l_{12} + a_3 b_1 l_{13} + a_1 b_2 l_{21} + a_2 b_2 l_{22} + a_3 b_2 l_{23} + a_1 b_3 l_{31} + a_2 b_3 l_{32} + a_3 b_3 l_{33},$   
 $\alpha x_1 y_1 + \beta x_2 y_1 + \theta x_3 y_1 + \gamma x_1 y_2 + \delta x_2 y_2 + \epsilon x_3 y_2 + \phi x_1 y_3 + \psi x_2 y_3 + \Xi x_3 y_3]$ 
```

```
In[ ]:=  $\Gamma_{2sp1\_}[\omega\_ , L\_ , Q\_ ] // \mathbb{E} := \mathbb{E}_{\{ \} \rightarrow sp1} [L, Q /. y_{i\_} \Rightarrow y_i / b_i, \omega + 0[\epsilon]];$ 
```

```
In[ ]:=  $\mathbb{E}_{\{ \} \rightarrow r\_} [L\_ , Q\_ , P\_ ] // \Gamma2 := \Gamma_{2r}[\text{Normal}@P, L, Q /. y_{i\_} \Rightarrow y_i b_i]$ 
```

```
In[ ]:=  $\Gamma_{S1\_}[\omega1\_ , L1\_ , Q1\_ ] \equiv \Gamma_{S2\_}[\omega2\_ , L2\_ , Q2\_ ] :=$   
 $(\text{Sort}[S1] === \text{Sort}[S2]) \wedge \text{Simplify}[\omega1 == \omega2] \wedge \text{Simplify}[L1 == L2] \wedge \text{Simplify}[Q1 == Q2]$ 
```

```
In[ ]:=  $\Upsilon2 \equiv (\Upsilon2 // \mathbb{E} // \Gamma2)$ 
```

```
Out[ ]:= True
```

```
In[ ]:=  $\Gamma_{2s\_}[\omega\_ , L\_ , Q\_ ] // \Gamma2\text{Form} := \text{Module}[\{M, i, j\},$   
 $M = \text{Table}[\text{Simplify}[\partial_{x_i, y_j} Q], \{i, S\}, \{j, S\}];$   
 $\text{PrependTo}[M, y_{\#} \& / @ S];$   
 $M = \text{Join}[$   
 $\{ \text{Prepend}[x_{\#} \& / @ S, \omega] \},$   
 $\text{Transpose}[M],$   
 $\{ \text{Prepend}[\text{Table}[\partial_{a_i} L, \{i, S\}], "r2"] \}$   
 $];$   
 $\text{MatrixForm}[M]$   
 $];$ 
```



```
In[ ]:= Γ3S_ [ω_, L_, Q_] // Γ3Form := Module[{M, i, j},
  M = Table[Simplify[∂xi, yj Q], {i, S}, {j, S}];
  PrependTo[M, y_# & /@ S];
  M = Join[
    {Prepend[x_# & /@ S, ω]},
    Transpose[M],
    {Prepend[Table[∂ai L, {i, S}], "Γ3"]}
  ];
  MatrixForm[M]
];
```

```
In[ ]:= CR1,2 // Γ3 // Γ3Form
```

Out[]//MatrixForm=

$$\begin{pmatrix} 1 & x_1 & x_2 \\ y_1 & 1 & 1 - B_1 \\ y_2 & 0 & B_1 \\ \Gamma 3 & 0 & b_1 \end{pmatrix}$$

```
In[ ]:= Γ3S1_ [ω1_, L1_, Q1_] ≡ Γ3S2_ [ω2_, L2_, Q2_] :=
  (Sort[S1] === Sort[S2]) ∧ Simplify[ω1 == ω2] ∧ Simplify[L1 == L2] ∧ Simplify[Q1 == Q2]
```

```
In[ ]:= γ3 ≡ (γ3 // E // Γ3)
```

Out[]:= True

```
In[ ]:= γ3 // E // CM1,2→0 // Γ3 // Γ3Form
```

Out[]//MatrixForm=

$$\begin{pmatrix} -\frac{\omega}{-1+\gamma} & x_0 & x_3 \\ y_0 & \beta - \frac{\alpha\delta}{-1+\gamma} & -\frac{\alpha\epsilon}{-1+\gamma} + \theta \\ y_3 & -\frac{\delta\phi}{-1+\gamma} + \psi & \Xi - \frac{\epsilon\phi}{-1+\gamma} \\ \Gamma 3 & b_0 (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & b_0 (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

```
In[ ]:= γ3 // E // CM2,1→0 // Γ3 // Γ3Form
```

Out[]//MatrixForm=

$$\begin{pmatrix} -\frac{\omega}{-1+\beta} & x_0 & x_3 \\ y_0 & \gamma - \frac{\alpha\delta}{-1+\beta} & \epsilon - \frac{\delta\theta}{-1+\beta} \\ y_3 & \phi - \frac{\alpha\psi}{-1+\beta} & \Xi - \frac{\theta\psi}{-1+\beta} \\ \Gamma 3 & b_0 (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & b_0 (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

In[]:= γ_3 // \mathbb{E} // \mathbf{CS}_1 // \mathbf{CS}_2 // \mathbf{CS}_3 // Γ_3 // $\Gamma_3\text{Form}$

Out[]//MatrixForm=

$$\left(\begin{array}{c} \beta \gamma B_1^{111+112} B_2^{122} - \alpha \delta B_1^{111+112} B_2^{122} - \delta B_1^{112} \left(\frac{1}{B_2}\right)^{121} B_2^{122} \left(\frac{1}{B_3}\right)^{131} + \delta \left(\frac{1}{B_1}\right)^{111} B_1^{111+112} \left(\frac{1}{B_2}\right)^{121} B_2^{122} \left(\frac{1}{B_3}\right)^{131} - \alpha B_1^{111} \left(\frac{1}{B_3}\right)^{132} + \alpha \left(\frac{1}{B_1}\right)^{112} B_1^{111+112} \left(\frac{1}{B_2}\right)^{122} B_2^{122} \left(\frac{1}{B_3}\right)^{132} - \end{array} \right)$$

```
In[ ]:=  $\Gamma_{4S}$ [ $\omega$ _,  $L$ _,  $Q$ _] //  $\Gamma_4\text{Form}$  := Module[{ $M$ ,  $i$ ,  $j$ ,  $SS = \text{Sort}[S]$ },
   $M = \text{Table}[\text{Simplify}[\partial_{y_i, x_j} Q], \{i, SS\}, \{j, SS\}];$ 
  PrependTo[ $M$ ,  $x_{\#} \& /@ SS$ ];
   $M = \text{Join}$ [
    {Prepend[ $y_{\#} \& /@ SS$ ,  $\omega$ ]},
    Transpose[ $M$ ],
    {Prepend[Table[ $\partial_{a_i} L$ , { $i, SS$ }], " $\Gamma_4$ "]}
  ];
  MatrixForm[ $M$ ]
];
```

In[]:= $\gamma_4 = \text{Expand} /@$

$$\Gamma_{4\{1,2,3\}}[\omega, \{b_1, b_2, b_3\} \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \{x_1, x_2, x_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \xi \end{pmatrix} \cdot \{y_1, y_2, y_3\}]$$

Out[]:= $\Gamma_{4\{1,2,3\}}[\omega, a_1 b_1 l_{11} + a_2 b_1 l_{12} + a_3 b_1 l_{13} + a_1 b_2 l_{21} + a_2 b_2 l_{22} + a_3 b_2 l_{23} + a_1 b_3 l_{31} + a_2 b_3 l_{32} + a_3 b_3 l_{33},$
 $\alpha x_1 y_1 + \gamma x_2 y_1 + \phi x_3 y_1 + \beta x_1 y_2 + \delta x_2 y_2 + \psi x_3 y_2 + \theta x_1 y_3 + \epsilon x_2 y_3 + \xi x_3 y_3]$

In[]:= γ_4 // $\Gamma_4\text{Form}$

Out[]//MatrixForm=

$$\left(\begin{array}{cccc} \omega & y_1 & y_2 & y_3 \\ x_1 & \alpha & \beta & \theta \\ x_2 & \gamma & \delta & \epsilon \\ x_3 & \phi & \psi & \xi \\ \Gamma_4 & b_1 l_{11} + b_2 l_{21} + b_3 l_{31} & b_1 l_{12} + b_2 l_{22} + b_3 l_{32} & b_1 l_{13} + b_2 l_{23} + b_3 l_{33} \end{array} \right)$$

```
In[ ]:=  $\Gamma_{4sp}$ [ $\omega$ _,  $L$ _,  $Q$ _] //  $\mathbb{E}$  :=
   $\mathbb{E}_{\{\} \rightarrow sp}[L, (Q /. y_{i\_} \Rightarrow y_i / b_i) - \text{Sum}[(e^{-\partial_{a_i} L} // . 12U) y_i x_i / b_i, \{i, sp\}], \omega^{-1} + 0[\epsilon]];$ 
```

```
In[ ]:=  $\mathbb{E}_{\{\} \rightarrow r}[L_-, Q_-, P_-] // \Gamma_4 :=$ 
   $\Gamma_{4r}[(\text{Normal}@P)^{-1}, L, (Q /. y_{i\_} \Rightarrow y_i b_i) + \text{Sum}[(e^{-\partial_{a_i} L} // . 12U) y_i x_i, \{i, r\}]]$ 
```

In[]:= {cR1,2 // Γ4 // Γ4Form, cR1,2 // Γ4 // Γ4Form}

$$\text{Out[]:=} \left\{ \begin{pmatrix} 1 & y_1 & y_2 \\ x_1 & 1 & \emptyset \\ x_2 & 1 - B_1 & B_1 \\ \Gamma 4 & \emptyset & b_1 \end{pmatrix}, \begin{pmatrix} 1 & y_1 & y_2 \\ x_1 & 1 & \emptyset \\ x_2 & 1 - \frac{1}{B_1} & \frac{1}{B_1} \\ \Gamma 4 & \emptyset & -b_1 \end{pmatrix} \right\}$$

In[]:= {cR2,1 // Γ4 // Γ4Form, cR2,1 // Γ4 // Γ4Form}

$$\text{Out[]:=} \left\{ \begin{pmatrix} 1 & y_1 & y_2 \\ x_1 & B_2 & 1 - B_2 \\ x_2 & \emptyset & 1 \\ \Gamma 4 & b_2 & \emptyset \end{pmatrix}, \begin{pmatrix} 1 & y_1 & y_2 \\ x_1 & \frac{1}{B_2} & 1 - \frac{1}{B_2} \\ x_2 & \emptyset & 1 \\ \Gamma 4 & -b_2 & \emptyset \end{pmatrix} \right\}$$

In[]:= $\Gamma 4_{S1}[\omega 1_, L1_, Q1_] \equiv \Gamma 4_{S2}[\omega 2_, L2_, Q2_] :=$
 $(\text{Sort}[S1] == \text{Sort}[S2]) \wedge \text{Simplify}[\omega 1 == \omega 2] \wedge \text{Simplify}[L1 == L2] \wedge \text{Simplify}[Q1 == Q2]$

In[]:= $\Upsilon 4 \equiv (\Upsilon 4 // \mathbb{E} // \Gamma 4)$

Out[]:= True

In[]:= $\Upsilon 4 // \mathbb{E} // \text{cm}_{1,2 \rightarrow \emptyset} // \Gamma 4 // \Gamma 4\text{Form}$

Out[]//MatrixForm=

$$\begin{pmatrix} \omega - \beta \omega & y_0 & y_3 \\ x_0 & \gamma - \frac{\alpha \delta}{-1 + \beta} & \varepsilon - \frac{\delta \theta}{-1 + \beta} \\ x_3 & \phi - \frac{\alpha \psi}{-1 + \beta} & \Xi - \frac{\theta \psi}{-1 + \beta} \\ \Gamma 4 & b_0 (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & b_0 (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

In[]:= Simplify /@ ($\Upsilon 4 // \mathbb{E} // \text{cm}_{1,2 \rightarrow \emptyset} // \Gamma 4$)

$$\text{Out[]:=} \frac{\Gamma 4_{\{0,3\}}[\omega - \beta \omega, a_0 (b_0 (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32})) + a_3 (b_0 (l_{13} + l_{23}) + b_3 l_{33}), x_0 ((-1 + \beta) \gamma - \alpha \delta) y_0 + ((-1 + \beta) \varepsilon - \delta \theta) y_3 + x_3 ((-1 + \beta) \phi - \alpha \psi) y_0 + ((-1 + \beta) \Xi - \theta \psi) y_3]}{-1 + \beta}$$

In[]:= $\text{tr}[i_][\Gamma 4_{is}[\omega_, \sigma_, \lambda_]] := \text{Module}[\{\alpha, \theta, \psi, \Xi\},$
 $\begin{pmatrix} \alpha & \theta \\ \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{x_i, y_i} \lambda & \partial_{x_i} \lambda \\ \partial_{y_i} \lambda & \lambda \end{pmatrix} /. (x | y)_i \rightarrow \emptyset;$
 $\Gamma 4_{\text{Complement}[is, \{i\}]}[\omega (1 - \alpha), \sigma /. a_i \rightarrow \emptyset, \Xi + \psi * \theta / (1 - \alpha)]]];$

In[]:= Simplify /@ ($\Upsilon 4 // \mathbb{E} // \text{cm}_{1,2 \rightarrow \emptyset} // \Gamma 4 // \text{tr}[\emptyset]$) // Γ4Form

Out[]//MatrixForm=

$$\begin{pmatrix} (1 + \beta (-1 + \gamma) - \gamma - \alpha \delta) \omega & y_3 \\ x_3 & \frac{(1 + \beta (-1 + \gamma) - \gamma - \alpha \delta) \Xi + \varepsilon (\phi - \beta \phi + \alpha \psi) + \theta (\delta \phi + \psi - \gamma \psi)}{1 + \beta (-1 + \gamma) - \gamma - \alpha \delta} \\ \Gamma 4 & b_0 (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

In[]:= Scatter[v_i_, ε_] := Module[{z}, (E_{i} → {z})[0, 0, v_z] ε // cm_{z, i → i}][[3]] // Normal]

In[]:= Table[Scatter[v, cR1,2], {v, {y1, y2, x1, x2}}

$$\text{Out[]:=} \{y_1, y_2, x_1 + (1 - B_1) x_2, B_1 x_2\}$$

$$\text{In[*]:= } \left(\gamma 4p = \text{Expand} /@ \Gamma 4_{\{1,2,3\}} \left[\mathbf{1}, \{b_1, b_2, b_3\} \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \right. \right. \\ \left. \left. \{x_1, x_2, x_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \varepsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{y_1, y_2, y_3\} \right] // \Gamma 4\text{Form} \right)$$

Out[*]/MatrixForm=

$$\begin{pmatrix} 1 & y_1 & y_2 & y_3 \\ x_1 & \alpha & \beta & \theta \\ x_2 & \gamma & \delta & \varepsilon \\ x_3 & \phi & \psi & \Xi \\ \Gamma 4 & b_1 l_{11} + b_2 l_{21} + b_3 l_{31} & b_1 l_{12} + b_2 l_{22} + b_3 l_{32} & b_1 l_{13} + b_2 l_{23} + b_3 l_{33} \end{pmatrix}$$

In[*]:= $\gamma 4p // \mathbb{E}$

$$\text{Out[*]= } \mathbb{E}_{\{\} \rightarrow \{1,2,3\}} \left[a_1 b_1 l_{11} + a_2 b_1 l_{12} + a_3 b_1 l_{13} + a_1 b_2 l_{21} + a_2 b_2 l_{22} + a_3 b_2 l_{23} + a_1 b_3 l_{31} + a_2 b_3 l_{32} + a_3 b_3 l_{33}, \right. \\ \frac{\alpha x_1 y_1}{b_1} - \frac{B_1^{11} B_2^{21} B_3^{31} x_1 y_1}{b_1} + \frac{\gamma x_2 y_1}{b_1} + \frac{\phi x_3 y_1}{b_1} + \frac{\beta x_1 y_2}{b_2} + \frac{\delta x_2 y_2}{b_2} - \\ \frac{B_1^{12} B_2^{22} B_3^{32} x_2 y_2}{b_2} + \frac{\psi x_3 y_2}{b_2} + \frac{\theta x_1 y_3}{b_3} + \frac{\varepsilon x_2 y_3}{b_3} + \frac{\Xi x_3 y_3}{b_3} - \frac{B_1^{13} B_2^{23} B_3^{33} x_3 y_3}{b_3}, \mathbf{1} + \mathbf{O}[\epsilon]^1 \left. \right]$$

In[*]:= `Table[Scatter[v, $\gamma 4p // \mathbb{E}$], {v, {x1, x2, x3}}]`

$$\text{Out[*]= } \{ \alpha x_1 + \gamma x_2 + \phi x_3, \beta x_1 + \delta x_2 + \psi x_3, \theta x_1 + \varepsilon x_2 + \Xi x_3 \}$$

In[*]:= `Table[Scatter[v, ReplacePart[3 -> 1][$\gamma 4p // \mathbb{E} // \text{cm}_{1,2 \rightarrow 1}$]], {v, {x1, x3}}] // Simplify`

$$\text{Out[*]= } \left\{ \frac{\left((-1 + \beta) \gamma - \alpha \delta \right) x_1 + \left((-1 + \beta) \phi - \alpha \psi \right) x_3}{-1 + \beta}, \frac{\left((-1 + \beta) \varepsilon - \delta \theta \right) x_1 + \left((-1 + \beta) \Xi - \theta \psi \right) x_3}{-1 + \beta} \right\}$$

$$\text{In[*]:= } \left(\gamma 4pp = \text{Expand} /@ \Gamma 4_{\{1,2,3\}} \left[\omega^{-1}, \{b_1, b_2, b_3\} \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \right. \right. \\ \left. \left. \{x_1, x_2, x_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \varepsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{y_1, y_2, y_3\} \right] // \Gamma 4\text{Form} \right)$$

Out[*]/MatrixForm=

$$\begin{pmatrix} \frac{1}{\omega} & y_1 & y_2 & y_3 \\ x_1 & \alpha & \beta & \theta \\ x_2 & \gamma & \delta & \varepsilon \\ x_3 & \phi & \psi & \Xi \\ \Gamma 4 & b_1 l_{11} + b_2 l_{21} + b_3 l_{31} & b_1 l_{12} + b_2 l_{22} + b_3 l_{32} & b_1 l_{13} + b_2 l_{23} + b_3 l_{33} \end{pmatrix}$$

In[*]:= `Table[Scatter[v, $\gamma 4pp // \mathbb{E}$], {v, {x1, x2, x3}}]`

$$\text{Out[*]= } \{ \alpha \omega x_1 + \gamma \omega x_2 + \phi \omega x_3, \beta \omega x_1 + \delta \omega x_2 + \psi \omega x_3, \theta \omega x_1 + \varepsilon \omega x_2 + \Xi \omega x_3 \}$$

In[*]:= `($\gamma 4pp // \mathbb{E} // \text{cm}_{1,2 \rightarrow 1}$)[[3]]`

$$\text{Out[*]= } -\frac{\omega}{-1 + \beta} + \mathbf{O}[\epsilon]^1$$

In[]:= **Table**[**Scatter**[**v**, **γ4pp** // **E** // **cm_{1,2→1}**], {**v**, {**x₁**, **x₃**}}] // **Simplify**

$$\text{Out[]} = \left\{ \frac{\omega \left((\gamma - \beta \gamma + \alpha \delta) \mathbf{x}_1 + (\phi - \beta \phi + \alpha \psi) \mathbf{x}_3 \right)}{(-1 + \beta)^2}, \frac{\omega \left((\varepsilon - \beta \varepsilon + \delta \theta) \mathbf{x}_1 + (\Xi - \beta \Xi + \theta \psi) \mathbf{x}_3 \right)}{(-1 + \beta)^2} \right\}$$