Pensieve header: The Objects sans γ.

```
Echo@"Warning: On Sep 4 2019 I swapped the operations
   ϵ and η. Some incompatibilities may arise in older notebooks."
```

Program

The Objects

Program

Symmetric Algebra Objects

Program

```
sm_{i_,j_→k_} := 𝔼_{{i,j}→{k}}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) + y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_{i_→j_,k_} := 𝔼_{{i}→{j,k}}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) + η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_{i_} := 𝔼_{{i}→{i}}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sη_{i_} := 𝔼_{{}→{i}}[0];
sϵ_{i_} := 𝔼_{{i}→{}}[0];
```

Program

```
sσ_{i_→j_} := 𝔼_{{i}→{j}}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sϒ_{i_→j_,k_,l_,m_} := 𝔼_{{i}→{j,k,l,m}}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

Program

The CU Definitions

Program

$$c\Lambda = \left(\eta_i + \frac{e^{-\alpha_i - \epsilon\,\beta_i}\,\eta_j}{1 + \epsilon\,\eta_j\,\xi_i}\right) y_k + \left(\beta_i + \beta_j + \frac{\mathrm{Log}[1 + \epsilon\,\eta_j\,\xi_i]}{\epsilon}\right) b_k +$$
$$\left(\alpha_i + \alpha_j + \mathrm{Log}[1 + \epsilon\,\eta_j\,\xi_i]\right) a_k + \left(\frac{e^{-\alpha_j - \epsilon\,\beta_j}\,\xi_i}{1 + \epsilon\,\eta_j\,\xi_i} + \xi_j\right) x_k;$$

```
Define[cm_{i,j→k} = 𝔼_{{i,j}→{k}}[cΛ]]
```

Program

```
Define[cσ_{i→j} = sσ_{i,j} /. τ_i → 0, cϵ_i = sϵ_i, cη_i = sη_i, cΔ_{i→j,k} = sΔ_{i→j,k},
  cS_i = sS_i // sϒ_{i→1,2,3,4} // cm_{4,3→i} // cm_{i,2→i} // cm_{i,1→i}];
```

Program

Booting Up QU

Program

```
Define[aσ_{i→j} = 𝔼_{{i}→{j}}[a_j α_i + x_j ξ_i], bσ_{i→j} = 𝔼_{{i}→{j}}[b_j β_i + y_j η_i]]
```

Program

$$\text{Define}\Big[am_{i,j\to k} = \mathbb{E}_{\{i,j\}\to\{k\}}\Big[(\alpha_i + \alpha_j)\,a_k + \left(\mathcal{A}_j^{-1}\,\xi_i + \xi_j\right) x_k\Big],$$
$$bm_{i,j\to k} = \mathbb{E}_{\{i,j\}\to\{k\}}\Big[(\beta_i + \beta_j)\,b_k + \left(\eta_i + e^{-\epsilon\,\beta_i}\,\eta_j\right) y_k\Big]\Big]$$

Three types of inverses appear below!
$\overline{R}$ is the inverse of R in the algebra $\mathbb{B}\otimes\mathbb{A}$.
P is the inverse of R as a quadratic form, like how an element of $V^*\otimes V^*$ can be the inverse of an element of $V\otimes V$.
$\overline{\mathrm{aS}}$ is the inverse of aS as an operator form, like how an element of $V^*\otimes V$ can be the inverse of another element of $V^*\otimes V$.

Program

```
Define[R_{i,j} = 𝔼_{{}→{i,j}}[ℏ a_j b_i + Σ_{k=1}^{$k+1} (1 - e^{ϵ ℏ})^k (ℏ y_i x_j)^k / (k (1 - e^{k ϵ ℏ}))],
  R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i, 1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_{$k}[3] -
      (((R̄_{{i,j},0})_{$k} R_{1,2} (R̄_{{3,4},$k-1})_{$k}) // (bm_{i,1→i} am_{j,2→j}) // (bm_{i,3→i} am_{j,4→j}))[3]]],
  P_{i,j} = 𝔼_{{i,j}→{}}[β_i α_j / ℏ, η_i ξ_j / ℏ, 1 + If[$k == 0, 0, (P_{{i,j},$k-1})_{$k}[3] -
      (R_{1,2} // ((P_{{1,j},0})_{$k} (P_{{i,2},$k-1})_{$k}))[3]]]]
```

Program

```
Define[aS_i = (aσ_{i→2} R̄_{1,i}) // P_{1,2},
  aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i, 1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_{$k}[3] -
      ((aS̄_{{i},0})_{$k} // aS_i // (aS̄_{{i},$k-1})_{$k})[3]]]]
```

(was $\mathrm{aS}_j = \overline{R}_{i,j} \sim B_i \sim P_{i,j}$).

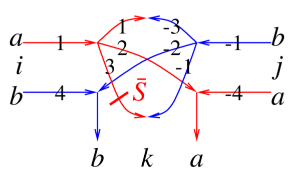
Program

```
Define[bS_i = bσ_{i→1} R_{i,2} // aS_2 // P_{1,2},
  bS̄_i = bσ_{i→1} R_{i,2} // aS̄_2 // P_{1,2},
  aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{3,i},
  bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{i,3}]
```

(was $\mathrm{bS}_i = R_{i,1} \sim B_1 \sim \mathrm{aS}_1 \sim B_1 \sim P_{i,1}$, $\overline{\mathrm{bS}}_i = R_{i,1} \sim B_1 \sim \overline{\mathrm{aS}}_1 \sim B_1 \sim P_{i,1}$).

Program

The Drinfel'd double:



Program

```
Define[
  dm_{i,j→k} = ((sΥ_{i→4,4,1,1} // aΔ_{1→1,2} // aΔ_{2→2,3} // aS̄_3) (sΥ_{j→-1,-1,-4,-4} // bΔ_{-1→-1,-2} // bΔ_{-2→-2,-3})) //
    (P_{-1,3} P_{-3,1} am_{2,-4→k} bm_{4,-2→k})]
```

Program

```
Define[dσ_{i→j} = aσ_{i→j} bσ_{i→j},
  dϵ_i = sϵ_i, dη_i = sη_i,
  dS_i = sΥ_{i→1,1,2,2} // (bS̄_1 aS_2) // dm_{2,1→i},
  dS̄_i = sΥ_{i→1,1,2,2} // (bS_1 aS̄_2) // dm_{2,1→i},
  dΔ_{i→j,k} = (bΔ_{i→3,1} aΔ_{i→2,4}) // (dm_{3,4→k} dm_{1,2→j})]
```

Program

In[∘]:=

```
Define[C_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/2} e^{-ℏ ϵ a_i/2}]_{$k},
  C̄_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/2} e^{ℏ ϵ a_i/2}]_{$k},
  Kink_i = (R_{1,3} C̄_2) // dm_{1,2→1} // dm_{1,3→i},
  Kink̄_i = (R̄_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i}]
```

Program

Note. $t == \epsilon a - b$ and $b == -t + \epsilon a$.

Program

```
Define[b2t_i = 𝔼_{{i}→{i}}[α_i a_i + β_i (ϵ a_i - t_i) + ξ_i x_i + η_i y_i],
  t2b_i = 𝔼_{{i}→{i}}[α_i a_i + τ_i (ϵ a_i - b_i) + ξ_i x_i + η_i y_i]]
```

Program

The Knot Tensors

Program

```
Define[kR_{i,j} = (R_{i,j} // (b2t_i b2t_j)) /. t_{i|j} → t,
  kR̄_{i,j} = (R̄_{i,j} // (b2t_i b2t_j)) /. {t_{i|j} → t, T_{i|j} → T},
  km_{i,j→k} = ((t2b_i t2b_j) // dm_{i,j→k} // b2t_k) /. {t_k → t, T_k → T, τ_{i|j} → 0},
  kC_i = (C_i // b2t_i) /. T_i → T,
  kC̄_i = (C̄_i // b2t_i) /. T_i → T,
  kKink_i = (Kink_i // b2t_i) /. {t_i → t, T_i → T},
  kKink̄_i = (Kink̄_i // b2t_i) /. {t_i → t, T_i → T}]
```