Pensieve header: Heisenberg calculus; continues AlexanderFromHeisenberg.nb.

In[∘]:=
```
SetDirectory[
 "C:\\drorbn\\AcademicPensieve\\Talks\\LearningSeminarOnCategorification-2006"]
```

Out[∘]=
```
C:\drorbn\AcademicPensieve\Talks\LearningSeminarOnCategorification-2006
```

In[∘]:=
```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
```

In[∘]:=
```
𝔼_{s1_}[ωQ1__] ≡ 𝔼_{s2_}[ωQ2__] := s1 === s2 ∧ Simplify[{ωQ1} == {ωQ2}]
```

In[∘]:=
```
𝔼_{A1_→B1_}[ω1_, Q1_] 𝔼_{A2_→B2_}[ω2_, Q2_] ^:= 𝔼_{A1⋃A2→B1⋃B2}[ω1 ω2, Q1 + Q2]
```

In[∘]:=
```
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂_{i,j}Q1, {i, A1}, {j, B1}]; E2 = M[∂_{i,j}Q2, {i, A2}, {j, B2}];
  F1 = M[∂_{i,j}Q1, {i, A1}, {j, A1}]; F2 = M[∂_{i,j}Q2, {i, A2}, {j, A2}];
  G1 = M[∂_{i,j}Q1, {i, B1}, {j, B1}]; G2 = M[∂_{i,j}Q2, {i, B2}, {j, B2}];
  𝔼_{A1→B2}[CF[ω1 ω2 Det[I - F2.G1]^{1/2}], CF@Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1^T).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2^T.G1.Inverse[I - F2.G1].E2).B2]]]]
```

In[∘]:=
```
A_ \ B_ := Complement[A, B];
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* =!= A2) :=
 𝔼_{A1⋃(A2\B1*)→B1⋃A2*}[ω1, Q1 + Sum[ξ* ξ, {ξ, A2 \ B1*}]] //
  𝔼_{B1*⋃A2→B2⋃(B1\A2*)}[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

In[∘]:=
```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)_i; l_List* := #* & /@ l;
```

A proof of the formula for R is at http://drorbn.net/cat20.

In[∘]:=
```
R_{i_,j_} := 𝔼_{{}→{p_i,x_i,p_j,x_j}}[T^{-1/2}, (1 - T) p_j x_j + (T - 1) p_i x_j];
R̄_{i_,j_} := 𝔼_{{}→{p_i,x_i,p_j,x_j}}[T^{1/2}, (1 - T^{-1}) p_j x_j + (T^{-1} - 1) p_i x_j];
C_{i_} := 𝔼_{{}→{p_i,x_i}}[T^{-1/2}, 0]; C̄_{i_} := 𝔼_{{}→{p_i,x_i}}[T^{1/2}, 0];
```

In[∘]:= C₁

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1}{\sqrt{T}}, 0\right]$

In[∘]:= $\{(R_{1,2})_h,\ (\overline{R}_{1,2})_h\}$

Out[∘]= $\left\{\mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[\frac{1}{\sqrt{T}},\ (-1+T)\ p_1\, x_2 + (1-T)\ p_2\, x_2\right]_h,\ \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[\sqrt{T},\ \left(-1+\frac{1}{T}\right) p_1\, x_2 + \left(1-\frac{1}{T}\right) p_2\, x_2\right]_h\right\}$

In[∘]:= $R_{1,2} \equiv \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[T^{-1/2},\ \{x_1,x_2\}.\begin{pmatrix}0 & 0\\ T-1 & 1-T\end{pmatrix}.\{p_1,p_2\}\right]$

Out[∘]= True

A proof of the formula for hm is at http://drorbn.net/cat20.

In[∘]:= $hm_{i\_,j\_\to k\_} := \mathbb{E}_{\{\pi_i,\xi_i,\pi_j,\xi_j\}\to\{p_k,x_k\}}[1,\ -\xi_i\,\pi_j + (\pi_i+\pi_j)\ p_k + (\xi_i+\xi_j)\ x_k]$

In[∘]:= $hm_{1,2\to 3}$

Out[∘]= $\mathbb{E}_{\{\pi_1,\xi_1,\pi_2,\xi_2\}\to\{p_3,x_3\}}[1,\ p_3\ (\pi_1+\pi_2) - \pi_2\,\xi_1 + x_3\ (\xi_1+\xi_2)]$

In[∘]:=
```
𝔼_{}→vs_[ωi_, Q_]_h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[ωi, 1 + Length@ps, 1 + Length@xs];
  M[[2 ;;, 2 ;;]] = Table[CF[∂_{i,j} Q], {i, ps}, {j, xs}];
  M[[2 ;;, 1]] = ps; M[[1, 2 ;;]] = xs;
  MatrixForm[M]_h]
```

Reidemeister 3.

In[∘]:= $(R_{1,2}\, R_{4,3}\, R_{5,6}\ //\ hm_{1,4\to 1}\, hm_{2,5\to 2}\, hm_{3,6\to 3}) == (R_{2,3}\, R_{1,6}\, R_{4,5}\ //\ hm_{1,4\to 1}\, hm_{2,5\to 2}\, hm_{3,6\to 3})$

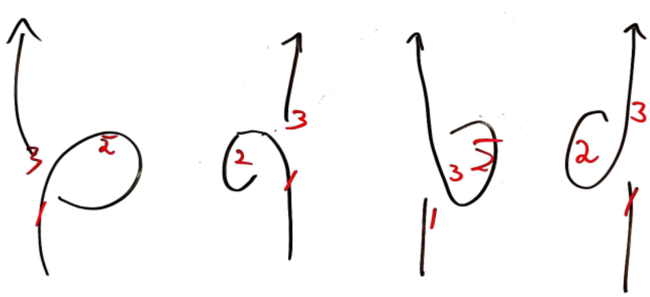
Out[∘]= True

Reidemeister 2.

In[∘]:= $\{\overline{R}_{1,2}\, R_{3,4}\ //\ hm_{1,3\to 1}\, hm_{2,4\to 2},\ \overline{R}_{1,4}\, R_{3,2}\ //\ hm_{1,3\to 1}\, hm_{2,4\to 2}\}$

Out[∘]= $\{\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1,0],\ \mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1,0]\}$

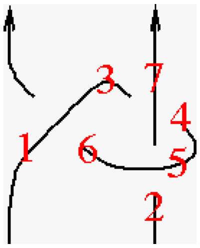
Reidemeister 1's.



In[∘]:= $\{(R_{1,3}\,\overline{C}_2)\ //\ hm_{1,2\to 1}\ //\ hm_{1,3\to 1},\ (\overline{R}_{1,3}\, C_2)\ //\ hm_{1,2\to 1}\ //\ hm_{1,3\to 1},\ (\overline{R}_{3,1}\,\overline{C}_2)\ //\ hm_{1,2\to 1}\ //\ hm_{1,3\to 1},\ (R_{3,1}\, C_2)\ //\ hm_{1,2\to 1}\ //\ hm_{1,3\to 1}\}$

Out[∘]= $\{\mathbb{E}_{\{\}\to\{p_1,x_1\}}[1,0],\ \mathbb{E}_{\{\}\to\{p_1,x_1\}}[1,0],\ \mathbb{E}_{\{\}\to\{p_1,x_1\}}[1,0],\ \mathbb{E}_{\{\}\to\{p_1,x_1\}}[1,0]\}$

The “First Tangle”



In[∘]:= $\text{Factor} \,/@\, \left(z = R_{1,6}\, \overline{C}_3\, \overline{R}_{7,4}\, \overline{R}_{5,2} \;//\; hm_{1,3\to1} \;//\; hm_{1,4\to1} \;//\; hm_{1,5\to1} \;//\; hm_{1,6\to1} \;//\; hm_{2,7\to2}\right)$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{-1+2\,T}{T},\ \frac{(-1+T)\ (p_1-p_2)\ (T\,x_1-x_2)}{-1+2\,T}\right]$

In[∘]:= z_h

Out[∘]= $$\begin{pmatrix} \frac{-1+2\,T}{T} & x_1 & x_2 \\ p_1 & \frac{-T+T^2}{-1+2\,T} & \frac{1-T}{-1+2\,T} \\ p_2 & \frac{T-T^2}{-1+2\,T} & \frac{-1+T}{-1+2\,T} \end{pmatrix}_h$$

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In[∘]:= $z = \overline{R}_{12,1}\,\overline{R}_{27}\,\overline{R}_{83}\,\overline{R}_{4,11}\,R_{16,5}\,R_{6,13}\,R_{14,9}\,R_{10,15};$

$\text{Table}[z = z \;//\; hm_{1k\to1},\ \{k,\ 2,\ 16\}] \;//\; \text{Last}$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6}{T^3},\ 0\right]$

Proof of the hm formula.

In[∘]:= $\left\{\left(\gamma 1 = \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2,p_3,x_3\}}\left[\omega,\ \{p_1,\ p_2,\ p_3\}.\begin{pmatrix}\alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi\end{pmatrix}.\{x_1,\ x_2,\ x_3\}\right]\right)_h,\ \left(\gamma 1 \;//\; hm_{1,2\to0}\right)_h\right\}$

Out[∘]= $$\left\{\begin{pmatrix} \omega & x_1 & x_2 & x_3 \\ p_1 & \alpha & \beta & \theta \\ p_2 & \gamma & \delta & \epsilon \\ p_3 & \phi & \psi & \Xi \end{pmatrix}_h,\ \begin{pmatrix} \omega+\gamma\,\omega & x_0 & x_3 \\ p_0 & \frac{\alpha+\beta+\gamma+\beta\,\gamma+\delta-\alpha\,\delta}{1+\gamma} & \frac{\epsilon-\alpha\,\epsilon+\theta+\gamma\,\theta}{1+\gamma} \\ p_3 & \frac{\phi-\delta\,\phi+\psi+\gamma\,\psi}{1+\gamma} & \frac{\Xi+\gamma\,\Xi-\epsilon\,\phi}{1+\gamma} \end{pmatrix}_h \right.$$

$\}$

In[∘]:= $\text{Simplify}\Big[$

$\text{Table}\left[\partial_{i,j}\left(\gamma 1 \;//\; hm_{1,2\to0}\right)[\![2]\!],\ \{i,\ \{p_0,\ p_3\}\},\ \{j,\ \{x_0,\ x_3\}\}\right] == \begin{pmatrix} 1+\beta-\frac{(1-\alpha)\ (1-\delta)}{1+\gamma} & \theta+\frac{(1-\alpha)\ \epsilon}{1+\gamma} \\ \psi+\frac{(1-\delta)\ \phi}{1+\gamma} & \Xi-\frac{\epsilon\,\phi}{1+\gamma} \end{pmatrix}\Big]$

Out[∘]= True

In[∘]:=
```
MatrixForm@Simplify[
  IdentityMatrix[2] - Table[∂i,j (γ1 // hm1,2→0) [[2]], {i, {p0, p3}}, {j, {x0, x3}}] /.
   Thread[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ} -> {1 - α, -β, -γ, 1 - δ, -θ, -ϵ, -ϕ, -ψ, 1 - Ξ}]
 ]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \beta - \frac{\alpha\delta}{-1+\gamma} & -\frac{\alpha\epsilon}{-1+\gamma} + \Theta \\ -\frac{\delta\phi}{-1+\gamma} + \psi & \Xi - \frac{\epsilon\phi}{-1+\gamma} \end{pmatrix}$$

Claim. $e^{\pi p + \xi x} = e^{-t\pi\xi/2} e^{\pi p} e^{\xi x}$

Proof. Use a 3D representation:

In[∘]:= $\rho p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $\rho x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; $\rho t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; ρp.ρx - ρx.ρp == ρt

In[∘]:=
```
MatrixForm /@
 Simplify /@ {MatrixExp[π ρp + ξ ρx], MatrixExp[-π ξ ρt / 2].MatrixExp[π ρp].MatrixExp[ξ ρx]}
```

Out[∘]= $\left\{ \begin{pmatrix} 1 & \pi & \frac{\pi\xi}{2} \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pi & \frac{\pi\xi}{2} \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix} \right\}$

In[∘]:= $S2O_{i\_} := \mathbb{E}_{\{\pi_i,\xi_i\}\to\{p_i,x_i\}}[1, \pi_i p_i + \xi_i x_i - t \pi_i \xi_i / 2]$;
$O2S_{i\_} := \mathbb{E}_{\{\pi_i,\xi_i\}\to\{p_i,x_i\}}[1, \pi_i p_i + \xi_i x_i + t \pi_i \xi_i / 2]$;

In[∘]:= $S2O_1$ // $O2S_1$

Out[∘]= $\mathbb{E}_{\{\pi_1,\xi_1\}\to\{p_1,x_1\}}[1, p_1 \pi_1 + x_1 \xi_1]$

In[∘]:= $\left\{ \left(\gamma 1 = \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2,p_3,x_3\}}\left[\omega, \{p_1, p_2, p_3\}.\begin{pmatrix} \alpha & \beta & \Theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix}.\{x_1, x_2, x_3\}\right]\right)_h, \right.$
$\left. (\gamma 1 \text{ // } S2O_1 \text{ // } S2O_2 \text{ // } hm_{1,2\to 0} \text{ // } O2S_0)_h \right\}$

Out[∘]= $\left\{ \begin{pmatrix} \omega & x_1 & x_2 & x_3 \\ p_1 & \alpha & \beta & \Theta \\ p_2 & \gamma & \delta & \epsilon \\ p_3 & \phi & \psi & \Xi \end{pmatrix}_h, \begin{pmatrix} \frac{1}{4}\left(-4\omega + 2t\beta\omega - 4\gamma\omega + 2t\gamma\omega + 2t\beta\gamma\omega - t^2\beta\gamma\omega - 2t\alpha\delta\omega + t^2\alpha\delta\omega\right) & \cdots & \frac{-4\alpha-4\beta}{-4+2t\beta-4} \\ p_0 & & \\ p_3 & & \frac{4\phi-2t\beta\phi-}{4-2t\beta+4} \end{pmatrix} \right.$

Recycling

Our PBW ordering is $\{p, x\}$.

We are at $[p, x] = 1$ and $R = e^{-t\otimes px + tp\otimes x}$. Let $a = p\,x$. Then $[a, x] = x$.

Claim. $e^{-ta + tp\,x} = e^{-ta} e^{t^{-1}(e^t - 1)\,tp\,x}$.

Proof. Use a 2d representation:

In[∘]:= $\rho a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $\rho x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; ρa.ρx - ρx.ρa == ρx

Out[∘]= True

```
In[ ]:= Simplify[MatrixExp[-t ρa + tp ρx] == MatrixExp[-t ρa].MatrixExp[t^-1 (e^t - 1) tp ρx]]
Out[ ]= True
```

Claim. $e^{t\,p\,x} = \mathbb{O}\left[e^{(1-e^{-t})\,p\,x}\right]$.

Proof. True at $t = 0$, test ∂_t using $p\,x\,\mathbb{O}[f] = \mathbb{O}[p(x\,f - \partial_p f)]$:

```
In[ ]:= Simplify[p (x e^((1-e^-t) p x) - ∂_p e^((1-e^-t) p x)) == ∂_t e^((1-e^-t) p x)]
Out[ ]= True
```

Claim. $\mathbb{O}[e^{-t\,a+\mathrm{tp}\,x}] = e^{(e^t-1)\,p\,x+t^{-1}(e^t-1)\,\mathrm{tp}\,x}$

```
In[ ]:= Collect[(1 - e^-t) p x + t^-1 (e^-t - 1) tp x, {tp}, Simplify]
Out[ ]= (1 - e^-t) p x + ((-1 + e^-t) tp x)/t
```

The Trefoil

```
In[ ]:= Z31 = R_{1,5} R_{6,2} R_{3,7} C̄_4;
        Do[Z31 = Z31 // hm_{1,r→1}, {r, 2, 7}];
        Simplify /@ Z31
Out[ ]= 𝔼_{{}→{p1,x1}}[-1 + 1/T + T, 0]
```