Pensieve header: Alexander from Heisenberg.

In[∘]:=
```
SetDirectory[
 "C:\\drorbn\\AcademicPensieve\\Talks\\LearningSeminarOnCategorification-2006"]
```

Out[∘]=
```
C:\drorbn\AcademicPensieve\Talks\LearningSeminarOnCategorification-2006
```

tex

\def\cellscale{0.78}
\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
\def\nbpdfEcho#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
\def\nbpdfSubsection#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}

pdf

In[∘]:=
```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
```

```
𝔼s1_[ωQ1__] ≡ 𝔼s2_[ωQ2__] := s1 === s2 ∧ Simplify[{ωQ1} == {ωQ2}]
```

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```
𝔼A1_→B1_[ω1_, Q1_] 𝔼A2_→B2_[ω2_, Q2_] ^:= 𝔼A1⋃A2→B1⋃B2[ω1 ω2, Q1 + Q2]
```

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In[∘]:=
```
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂i,j Q1, {i, A1}, {j, B1}]; E2 = M[∂i,j Q2, {i, A2}, {j, B2}];
  F1 = M[∂i,j Q1, {i, A1}, {j, A1}]; F2 = M[∂i,j Q2, {i, A2}, {j, A2}];
  G1 = M[∂i,j Q1, {i, B1}, {j, B1}]; G2 = M[∂i,j Q2, {i, B2}, {j, B2}];
  𝔼A1→B2[CF[ω1 ω2 Det[I - F2.G1]^(1/2)], CF@Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).B2]]]]
```

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In[∘]:=
```
A_ \ B_ := Complement[A, B];
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* =!= A2) :=
 𝔼A1⋃(A2\B1*)→B1⋃A2*[ω1, Q1 + Sum[ξ* ξ, {ξ, A2 \ B1*}]] //
  𝔼B1*⋃A2→B2⋃(B1\A2*)[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

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In[∘]:=
```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)i; l_List* := #* & /@ l;
```

A proof of the formula for R is at http://drorbn.net/cat20.

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In[ ]:=
```
R[i_,j_] := E[{}→{p_i,x_i,p_j,x_j}][T^(-1/2), (1 - T) p_j x_j + (T - 1) p_i x_j];
R̄[i_,j_] := E[{}→{p_i,x_i,p_j,x_j}][T^(1/2), (1 - T^-1) p_j x_j + (T^-1 - 1) p_i x_j];
C[i_] := E[{}→{p_i,x_i}][T^(-1/2), 0]; C̄[i_] := E[{}→{p_i,x_i}][T^(1/2), 0];
```

In[ ]:= C_1

Out[ ]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1}{\sqrt{T}}, 0\right]$

In[ ]:= $\{(R_{1,2})_h, (\bar{R}_{1,2})_h\}$

Out[ ]= $\left\{\mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[\frac{1}{\sqrt{T}}, (-1+T)\, p_1 x_2 + (1-T)\, p_2 x_2\right]_h,\right.$ $\left.\mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[\sqrt{T}, \left(-1+\frac{1}{T}\right) p_1 x_2 + \left(1-\frac{1}{T}\right) p_2 x_2\right]_h\right\}$

In[ ]:= $R_{1,2} \equiv \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[T^{-1/2}, \{x_1, x_2\}.\begin{pmatrix} 0 & 0 \\ T-1 & 1-T \end{pmatrix}.\{p_1, p_2\}\right]$

Out[ ]= True

A proof of the formula for hm is at http://drorbn.net/cat20.

pdf

In[ ]:=
```
hm[i_,j_→k_] := E[{π_i,ξ_i,π_j,ξ_j}→{p_k,x_k}][1, -ξ_i π_j + (π_i + π_j) p_k + (ξ_i + ξ_j) x_k]
```

In[ ]:= $hm_{1,2\to3}$

Out[ ]= $\mathbb{E}_{\{\pi_1,\xi_1,\pi_2,\xi_2\}\to\{p_3,x_3\}}[1, p_3 (\pi_1+\pi_2) - \pi_2 \xi_1 + x_3 (\xi_1+\xi_2)]$

pdf

In[ ]:=
```
E[{}→vs_][ωi_, Q_]_h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[ωi, 1 + Length@ps, 1 + Length@xs];
  M[[2 ;;, 2 ;;]] = Table[CF[∂_{i,j} Q], {i, ps}, {j, xs}];
  M[[2 ;;, 1]] = ps; M[[1, 2 ;;]] = xs;
  MatrixForm[M]_h]
```

tex

\parpic[r]{\scalebox{1}{\input{R3.pdftex_t}}}
{\red\bf Proof of Reidemeister 3.}
{\def\nbpdfOutput#1{\vskip
1mm\par\noindent\includegraphics[scale=\cellscale]{#1}\hfill\text{$\Box$\quad}}

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In[ ]:= $(R_{1,2}\, R_{4,3}\, R_{5,6}\ //\ hm_{1,4\to1}\, hm_{2,5\to2}\, hm_{3,6\to3}) == (R_{2,3}\, R_{1,6}\, R_{4,5}\ //\ hm_{1,4\to1}\, hm_{2,5\to2}\, hm_{3,6\to3})$

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Out[ ]= True

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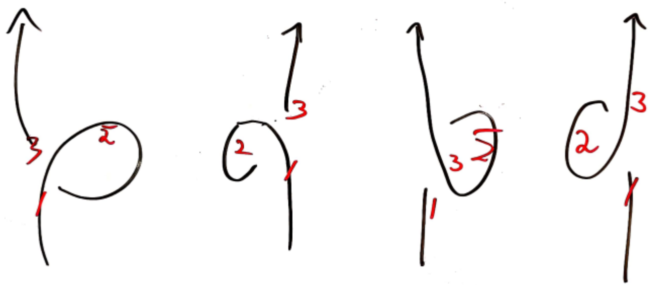
}

Reidemeister 2.

In[∘]:= $\{\overline{R}_{1,2}\, R_{3,4}$ // $hm_{1,3\to1}\, hm_{2,4\to2}$, $\overline{R}_{1,4}\, R_{3,2}$ // $hm_{1,3\to1}\, hm_{2,4\to2}\}$

Out[∘]= $\{\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1, 0], \mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1, 0]\}$

Reidemeister 1's.

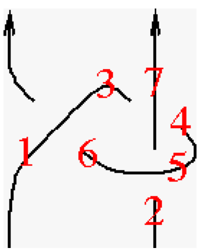


In[∘]:= $\{(R_{1,3}\,\overline{C}_2)$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}$, $(\overline{R}_{1,3}\, C_2)$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}$,
$(\overline{R}_{3,1}\,\overline{C}_2)$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}$, $(R_{3,1}\, C_2)$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}\}$

Out[∘]= $\{\mathbb{E}_{\{\}\to\{p_1,x_1\}}[1, 0], \mathbb{E}_{\{\}\to\{p_1,x_1\}}[1, 0], \mathbb{E}_{\{\}\to\{p_1,x_1\}}[1, 0], \mathbb{E}_{\{\}\to\{p_1,x_1\}}[1, 0]\}$

tex

{\red\bf The ``First Tangle''.}



pdf

In[∘]:= Factor /@ $(z = R_{1,6}\,\overline{C}_3\,\overline{R}_{7,4}\,\overline{R}_{5,2}$ // $hm_{1,3\to1}$ // $hm_{1,4\to1}$ // $hm_{1,5\to1}$ // $hm_{1,6\to1}$ // $hm_{2,7\to2})$

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Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{-1+2T}{T}, \frac{(-1+T)\,(p_1-p_2)\,(T x_1 - x_2)}{-1+2T}\right]$

tex

\vskip -3mm
\parpic[r]{\scalebox{1}{\input{FirstTangle.pdftex_t}}}

pdf

In[∘]:= z_h

pdf

Out[∘]= $\begin{pmatrix} \frac{-1+2T}{T} & x_1 & x_2 \\ p_1 & \frac{-T+T^2}{-1+2T} & \frac{1-T}{-1+2T} \\ p_2 & \frac{T-T^2}{-1+2T} & \frac{-1+T}{-1+2T} \end{pmatrix}_h$

tex

\parpic[r]{\scalebox{0.8}{\input{817.pdftex_t}}}
{\red\bf The knot 8_{17}.}

pdf

In[∘]:=
```
z = R̄12,1 R̄27 R̄83 R̄4,11 R16,5 R6,13 R14,9 R10,15;
Table[z = z // hm1k→1, {k, 2, 16}] // Last
```

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Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1-4T+8T^2-11T^3+8T^4-4T^5+T^6}{T^3}, 0\right]$

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{\red\bf Proof of Theorem 3, (3).}
{\def\nbpdfOutput#1{\vskip
1mm\par\noindent\includegraphics[scale=\cellscale]{#1}\hfill\text{$\Box$\quad}}

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In[∘]:= $\left\{\left(\gamma1 = \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2,p_3,x_3\}}\left[\omega, \{p_1, p_2, p_3\}.\begin{pmatrix}\alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi\end{pmatrix}.\{x_1, x_2, x_3\}\right]\right)_h, (\gamma1 \,//\, hm_{1,2\to0})_h\right\}$

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Out[∘]= $\left\{\begin{pmatrix}\omega & x_1 & x_2 & x_3 \\ p_1 & \alpha & \beta & \theta \\ p_2 & \gamma & \delta & \epsilon \\ p_3 & \phi & \psi & \Xi\end{pmatrix}_h, \begin{pmatrix}\omega+\gamma\,\omega & x_0 & x_3 \\ p_0 & \frac{\alpha+\beta+\gamma+\beta\,\gamma+\delta-\alpha\,\delta}{1+\gamma} & \frac{\epsilon-\alpha\,\epsilon+\theta+\gamma\,\theta}{1+\gamma} \\ p_3 & \frac{\phi-\delta\,\phi+\psi+\gamma\,\psi}{1+\gamma} & \frac{\Xi+\gamma\,\Xi-\epsilon\,\phi}{1+\gamma}\end{pmatrix}_h\right\}$

tex

}

In[∘]:= $\text{Simplify}\left[\text{Table}\left[\partial_{i,j}\left(\gamma1 \,//\, hm_{1,2\to0}\right)[[2]], \{i, \{p_0, p_3\}\}, \{j, \{x_0, x_3\}\}\right] == \begin{pmatrix}1+\beta-\frac{(1-\alpha)(1-\delta)}{1+\gamma} & \theta+\frac{(1-\alpha)\,\epsilon}{1+\gamma} \\ \psi+\frac{(1-\delta)\,\phi}{1+\gamma} & \Xi-\frac{\epsilon\,\phi}{1+\gamma}\end{pmatrix}\right]$

Out[∘]= True

In[∘]:=
```
MatrixForm@Simplify[
  IdentityMatrix[2] - Table[∂i,j (γ1 // hm1,2→0)[[2]], {i, {p0, p3}}, {j, {x0, x3}}] /.
   Thread[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ} -> {1 - α, -β, -γ, 1 - δ, -θ, -ϵ, -ϕ, -ψ, 1 - Ξ}]
 ]
```

Out[∘]//MatrixForm= $\begin{pmatrix}\beta-\frac{\alpha\,\delta}{-1+\gamma} & -\frac{\alpha\,\epsilon}{-1+\gamma}+\theta \\ -\frac{\delta\,\phi}{-1+\gamma}+\psi & \Xi-\frac{\epsilon\,\phi}{-1+\gamma}\end{pmatrix}$

Recycling

Our PBW ordering is $\{p, x\}$.

We are at $[p, x] = 1$ and $R = e^{-t\otimes px+tp\otimes x}$. Let $a = p\,x$. Then $[a, x] = x$.

Claim. $e^{-t\,a+tp\,x} = e^{-t\,a}\,e^{t^{-1}(e^t-1)\,tp\,x}$.

Proof. Use a 2d representation:

In[∘]:= $\rho a = \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}$; $\rho x = \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$; $\rho a.\rho x - \rho x.\rho a == \rho x$

Out[∘]= True

In[∘]:=
```
Simplify[MatrixExp[-t ρa + tp ρx] == MatrixExp[-t ρa].MatrixExp[t^-1 (e^t - 1) tp ρx]]
```

Out[∘]= True

Claim. $e^{t p x} = \mathbb{O}\left[e^{(1-e^{-t}) p x}\right]$.

Proof. True at $t = 0$, test ∂_t using $p\,x\,\mathbb{O}[f] = \mathbb{O}[p(x f - \partial_p f)]$:

In[∘]:=
```
Simplify[p (x e^((1-e^-t) p x) - ∂_p e^((1-e^-t) p x)) == ∂_t e^((1-e^-t) p x)]
```

Out[∘]= True

Claim. $\mathbb{O}[e^{-t a + \mathrm{tp}\, x}] = e^{(e^t - 1) p x + t^{-1}(e^t - 1)\mathrm{tp}\, x}$

In[∘]:=
```
Collect[(1 - e^-t) p x + t^-1 (e^-t - 1) tp x, {tp}, Simplify]
```

Out[∘]= $\left(1 - e^{-t}\right) \mathrm{p\,x} + \frac{\left(-1 + e^{-t}\right) \mathrm{tp\,x}}{\mathrm{t}}$

The Trefoil

In[∘]:=
```
Z31 = R_{1,5} R_{6,2} R_{3,7} C̄_4;
Do[Z31 = Z31 // hm_{1,r→1}, {r, 2, 7}];
Simplify /@ Z31
```

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1, x_1\}}\left[-1 + \frac{1}{T} + T,\ 0\right]$