

```
In[*]:= SetDirectory[
  "C:\\drorbn\\AcademicPensieve\\Talks\\LearningSeminarOnCategorification-2006"];
<< "Engine-Speedy.m";
<< "Objects.m";
$k = 0; ħ = 1;
```

» Warning: On Sep 4 2019 I swapped the operations  $\epsilon$  and  $\eta$ . Some incompatibilities may arise in older notebooks.

```
In[*]:= s; h; t; Γ; dL; V; Xp; Xm
Once[Begin["MetaCalculi`"]; << "../..../Projects/MetaCalculi/MetaCalculi.m";
End[]];
ΓSimp = MetaCalculi`ΓSimp;
ΓR[i_, j_] := Γ[Xp[i, j]]; ΓR̄[i_, j_] := Γ[Xm[i, j]];
```

Out[\*]= Xm

MetaCalculi` loading...

```
In[*]:= ΓR[1, 2]
```

$$Out[*]= \begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & 1 & 1 - T_1 \\ s_2 & 0 & T_1 \\ \Gamma & 1 & T_1 \end{pmatrix}$$

```
In[*]:= Ri,j
```

$$Out[*]= \mathbb{E}_{\{i,j\} \rightarrow \{i,j\}} [a_j b_i, x_j y_i, 1]$$

```
In[*]:= cmi,j→k
```

$$Out[*]= \mathbb{E}_{\{i,j\} \rightarrow \{k\}} [a_k \alpha_i + a_k \alpha_j + b_k \beta_i + b_k \beta_j, y_k \eta_i + \frac{y_k \eta_j}{\mathcal{A}_i} + \frac{x_k \xi_i}{\mathcal{A}_j} + b_k \eta_j \xi_i + x_k \xi_j, 1 + 0[\epsilon]^1]$$

```
In[*]:= dmi,j→k
```

$$Out[*]= \mathbb{E}_{\{i,j\} \rightarrow \{k\}} [a_k \alpha_i + a_k \alpha_j + b_k \beta_i + b_k \beta_j, y_k \eta_i + \frac{y_k \eta_j}{\mathcal{A}_i} + \frac{x_k \xi_i}{\mathcal{A}_j} + (1 - B_k) \eta_j \xi_i + x_k \xi_j, 1 + 0[\epsilon]^1]$$

```
In[*]:= cRi_,j_ := E{i,j}→{i,j} [aj bi,  $\frac{1 - B_i}{b_i} x_j y_i, 1]$ 
```

$$In[*]:= (cR_{1,2} cR_{4,3} cR_{5,6} // c_{m_{1,4} \rightarrow 1} // c_{m_{2,5} \rightarrow 2} // c_{m_{3,6} \rightarrow 3}) \equiv (cR_{2,3} cR_{1,6} cR_{4,5} // c_{m_{1,4} \rightarrow 1} // c_{m_{2,5} \rightarrow 2} // c_{m_{3,6} \rightarrow 3})$$

Out[\*]= True

```
In[*]:= cR̄i_,j_ := E{i,j}→{i,j} [-aj bi,  $\frac{B_i - 1}{b_i B_i} x_j y_i, 1]$ 
```

```
In[*]:= cR1,3 cR̄2,4 // cm1,2→1 // cm3,4→3
```

$$Out[*]= \mathbb{E}_{\{i\} \rightarrow \{1,3\}} [0, 0, 1 + 0[\epsilon]^1]$$

```
In[*]:= Gsp1[ $\omega$ _, L_, Q_] //  $\mathbb{E} := \mathbb{E}_{\{\} \rightarrow sp1}[L, Q, \omega + O[\epsilon]]$ ;
Gsp1[ $\omega$ _, L_, Q_] //  $\mathbb{E}_{sp2}[L2_, Q2_, P2_] := G_{sp1}[\omega, L, Q] // \mathbb{E} // \mathbb{E}_{sp2}[L2, Q2, P2]$ 
```

```
In[*]:= G1 = Expand /@
```

$$G_{\{1,2,3\}}[\omega, \{b_1, b_2, b_3\}] \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \{y_1, y_2, y_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{x_1, x_2, x_3\}$$

```
Out[*]:= G{1,2,3}[ $\omega$ , a1 b1 l11 + a2 b1 l12 + a3 b1 l13 + a1 b2 l21 + a2 b2 l22 + a3 b2 l23 + a1 b3 l31 + a2 b3 l32 + a3 b3 l33,
 $\alpha x_1 y_1 + \beta x_2 y_1 + \theta x_3 y_1 + \gamma x_1 y_2 + \delta x_2 y_2 + \epsilon x_3 y_2 + \phi x_1 y_3 + \psi x_2 y_3 + \Xi x_3 y_3$ ]
```

```
In[*]:= G1 //  $\mathbb{E}$ 
```

```
Out[*]:=  $\mathbb{E}_{\{\} \rightarrow \{1,2,3\}}[a_1 b_1 l_{11} + a_2 b_1 l_{12} + a_3 b_1 l_{13} + a_1 b_2 l_{21} + a_2 b_2 l_{22} + a_3 b_2 l_{23} + a_1 b_3 l_{31} + a_2 b_3 l_{32} + a_3 b_3 l_{33},$ 
 $\alpha x_1 y_1 + \beta x_2 y_1 + \theta x_3 y_1 + \gamma x_1 y_2 + \delta x_2 y_2 + \epsilon x_3 y_2 + \phi x_1 y_3 + \psi x_2 y_3 + \Xi x_3 y_3, \omega + O[\epsilon]^1]$ 
```

```
In[*]:= G1 // cm1,2→0 // Simplify
```

```
Out[*]:=  $\mathbb{E}_{\{\} \rightarrow \{0,3\}}[a_0 (b_0 (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32})) + a_3 (b_0 (l_{13} + l_{23}) + b_3 l_{33}),$ 
 $\frac{1}{-1 + \gamma b_0} \left( x_3 \left( (-\theta + (-\alpha \epsilon + \gamma \theta) b_0 - \epsilon B_0^{11+121} B_3^{131}) y_0 + (-\Xi + (\gamma \Xi - \epsilon \phi) b_0) y_3 \right) + \right.$ 
 $x_0 \left( (-\beta + (\beta \gamma - \alpha \delta) b_0 - \delta B_0^{11+121} B_3^{131} - \alpha B_0^{12+122} B_3^{132} - \gamma B_0^{11+12+121+122} B_3^{131+132}) y_0 + \right.$ 
 $\left. \left. (-\psi + (-\delta \phi + \gamma \psi) b_0 - \phi B_0^{12+122} B_3^{132}) y_3 \right) \right), \frac{\omega}{1 - \gamma b_0} + O[\epsilon]^1]$ 
```

```
In[*]:=  $\mathbb{E}_{\{\} \rightarrow r}[L_, Q_, P_] // G := G_r[\text{Normal}@P, L, Q]$ 
```

```
In[*]:= G1 // cm1,2→0 // G
```

```
Out[*]:= G{0,3}[ $-\frac{\omega}{-1 + \gamma b_0}$ , a0 b0 (l11 + l12 + l21 + l22) + a3 b0 (l13 + l23) + a0 b3 (l31 + l32) + a3 b3 l33,
 $\frac{1}{-1 + \gamma b_0} \left( -\beta x_0 y_0 + \beta \gamma b_0 x_0 y_0 - \alpha \delta b_0 x_0 y_0 - \delta B_0^{11+121} B_3^{131} x_0 y_0 - \alpha B_0^{12+122} B_3^{132} x_0 y_0 - \right.$ 
 $\gamma B_0^{11+12+121+122} B_3^{131+132} x_0 y_0 - \theta x_3 y_0 - \alpha \epsilon b_0 x_3 y_0 + \gamma \theta b_0 x_3 y_0 - \epsilon B_0^{11+121} B_3^{131} x_3 y_0 -$ 
 $\psi x_0 y_3 - \delta \phi b_0 x_0 y_3 + \gamma \psi b_0 x_0 y_3 - \phi B_0^{12+122} B_3^{132} x_0 y_3 - \Xi x_3 y_3 + \gamma \Xi b_0 x_3 y_3 - \epsilon \phi b_0 x_3 y_3 \left. \right)$ ]
```

```
In[*]:= GS[ $\omega$ _, L_, Q_] // GForm := Module[{M, i, j},
M = Table[Simplify[ $\partial_{x_i, y_j} Q$ ], {i, S}, {j, S}];
PrependTo[M, y# & /@ S];
M = Join[
{Prepend[x# & /@ S,  $\omega$ ]},
Transpose[M],
{Prepend[Table[ $\partial_{a_i} L$ , {i, S}], "G"]}
];
MatrixForm[M]
];
```

```
In[*]:= GS1[ $\omega 1$ _, L1_, Q1_]  $\equiv$  GS2[ $\omega 2$ _, L2_, Q2_] :=
(Sort[S1] === Sort[S2])  $\wedge$  Simplify[ $\omega 1 = \omega 2$ ]  $\wedge$  Simplify[L1 == L2]  $\wedge$  Simplify[Q1 == Q2]
```

In[ ]:= **G1 // GForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} \omega & & & \\ \mathbf{y}_1 & \mathbf{x}_1 & & \mathbf{x}_2 & & \mathbf{x}_3 \\ \mathbf{y}_2 & \alpha & & \beta & & \theta \\ \mathbf{y}_3 & \gamma & & \delta & & \epsilon \\ \mathbf{y}_3 & \phi & & \psi & & \Xi \\ \mathbb{G} & b_1 l_{11} + b_2 l_{21} + b_3 l_{31} & & b_1 l_{12} + b_2 l_{22} + b_3 l_{32} & & b_1 l_{13} + b_2 l_{23} + b_3 l_{33} \end{pmatrix}$$

In[ ]:= **G1 // cm<sub>1,2→0</sub> // G // GForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{\omega}{1-\gamma b_\theta} & & \mathbf{x}_0 & & \mathbf{x}_3 \\ \mathbf{y}_0 & \frac{\beta + (-\beta\gamma + \alpha\delta) b_\theta + \delta B_\theta^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_\theta^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_\theta^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1-\gamma b_\theta} & & & \frac{\theta + (\alpha\epsilon - \gamma\theta) b_\theta + \epsilon B_\theta^{l_{11}+l_{21}} B_3^{l_{31}}}{1-\gamma b_\theta} \\ \mathbf{y}_3 & \frac{\psi + (\delta\phi - \gamma\psi) b_\theta + \phi B_\theta^{l_{12}+l_{22}} B_3^{l_{32}}}{1-\gamma b_\theta} & & & \frac{\Xi + (-\gamma\Xi + \epsilon\phi) b_\theta}{1-\gamma b_\theta} \\ \mathbb{G} & b_\theta (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & & & b_\theta (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

$$\text{In[ ]:= } \mathbf{G2} = \mathbb{G}_{\{\theta,3\}} \left[ \frac{\omega}{1-\gamma b_\theta}, \{b_\theta, b_3\} \cdot \begin{pmatrix} l_{11} + l_{12} + l_{21} + l_{22} & l_{13} + l_{23} \\ l_{31} + l_{32} & l_{33} \end{pmatrix} \cdot \{a_\theta, a_3\}, \{\mathbf{y}_0, \mathbf{y}_3\} \cdot \right.$$

$$\left. \begin{pmatrix} \frac{\beta + (\alpha\delta - \gamma\beta) b_\theta + \delta B_\theta^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_\theta^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_\theta^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1-\gamma b_\theta} & \frac{\theta + (\alpha\epsilon - \gamma\theta) b_\theta + \epsilon B_\theta^{l_{11}+l_{21}} B_3^{l_{31}}}{1-\gamma b_\theta} \\ \frac{\psi + (\delta\phi - \gamma\psi) b_\theta + \phi B_\theta^{l_{12}+l_{22}} B_3^{l_{32}}}{1-\gamma b_\theta} & \frac{\Xi + (\epsilon\phi - \gamma\Xi) b_\theta}{1-\gamma b_\theta} \end{pmatrix} \cdot \{\mathbf{x}_0, \mathbf{x}_3\} \right] // \mathbf{GForm}$$

$$(\mathbf{G1} // \mathbf{cm}_{1,2\rightarrow 0} // \mathbb{G}) \equiv \mathbf{G2}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{\omega}{1-\gamma b_\theta} & & \mathbf{x}_0 & & \mathbf{x}_3 \\ \mathbf{y}_0 & \frac{\beta + (-\beta\gamma + \alpha\delta) b_\theta + \delta B_\theta^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_\theta^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_\theta^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1-\gamma b_\theta} & & & \frac{\theta + (\alpha\epsilon - \gamma\theta) b_\theta + \epsilon B_\theta^{l_{11}+l_{21}} B_3^{l_{31}}}{1-\gamma b_\theta} \\ \mathbf{y}_3 & \frac{\psi + (\delta\phi - \gamma\psi) b_\theta + \phi B_\theta^{l_{12}+l_{22}} B_3^{l_{32}}}{1-\gamma b_\theta} & & & \frac{\Xi + (-\gamma\Xi + \epsilon\phi) b_\theta}{1-\gamma b_\theta} \\ \mathbb{G} & b_\theta (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & & & b_\theta (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

Out[ ]:= True

In[ ]:= **G1 // dm<sub>1,2→0</sub> // G // GForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{\omega}{1-\gamma + \gamma B_\theta} & & \mathbf{x}_0 & & \mathbf{x}_3 \\ \mathbf{y}_0 & \frac{\beta - \beta\gamma + \alpha\delta + (\beta\gamma - \alpha\delta) B_\theta + \delta B_\theta^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_\theta^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_\theta^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1-\gamma + \gamma B_\theta} & & & \frac{\alpha\epsilon + \theta - \gamma\theta + (-\alpha\epsilon + \gamma\theta) B_\theta + \epsilon B_\theta^{l_{11}+l_{21}} B_3^{l_{31}}}{1-\gamma + \gamma B_\theta} \\ \mathbf{y}_3 & \frac{\delta\phi + \psi - \gamma\psi + (-\delta\phi + \gamma\psi) B_\theta + \phi B_\theta^{l_{12}+l_{22}} B_3^{l_{32}}}{1-\gamma + \gamma B_\theta} & & & \frac{\Xi - \gamma\Xi + \epsilon\phi + (\gamma\Xi - \epsilon\phi) B_\theta}{1-\gamma + \gamma B_\theta} \\ \mathbb{G} & b_\theta (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & & & b_\theta (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

$$\text{In[ ]:=} \left( \mathbf{G3} = \mathbb{G}_{\{0,3\}} \left[ \frac{\omega}{1 + \gamma (B_0 - 1)}, \{b_0, b_3\} \cdot \begin{pmatrix} l_{11} + l_{12} + l_{21} + l_{22} & l_{13} + l_{23} \\ l_{31} + l_{32} & l_{33} \end{pmatrix} \cdot \{a_0, a_3\}, \right. \right. \\
 \left. \left. \{y_0, y_3\} \cdot \begin{pmatrix} \frac{\beta - \gamma \beta + \alpha \delta + (\gamma \beta - \alpha \delta) B_0 + \delta B_0^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_0^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_0^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1 + \gamma (B_0 - 1)} & \frac{\alpha \epsilon + \theta - \gamma \theta + (-\alpha \epsilon + \gamma \theta) B_0 + \epsilon B_0^{l_{11}+l_{21}} B_3^{l_{31}}}{1 + \gamma (B_0 - 1)} \\ \frac{\delta \phi + \psi - \gamma \psi + (-\delta \phi + \gamma \psi) B_0 + \phi B_0^{l_{12}+l_{22}} B_3^{l_{32}}}{1 + \gamma (B_0 - 1)} & \frac{\Xi - \gamma \Xi + \epsilon \phi + (\gamma \Xi - \epsilon \phi) B_0}{1 + \gamma (B_0 - 1)} \end{pmatrix} \right. \\
 \left. \left. \{x_0, x_3\} \right] // \mathbb{G}\text{Form} \right) \\
 (\mathbf{G1} // \mathbf{dm}_{1,2 \rightarrow 0} // \mathbb{G}) \equiv \mathbf{G3}$$

Out[ ]//MatrixForm=

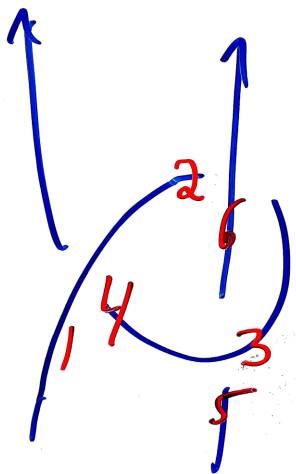
$$\begin{pmatrix} \omega & & \\ 1 + \gamma (-1 + B_0) & X_0 & X_3 \\ y_0 & \frac{\beta - \gamma \beta + \alpha \delta + (\gamma \beta - \alpha \delta) B_0 + \delta B_0^{l_{11}+l_{21}} B_3^{l_{31}} + \alpha B_0^{l_{12}+l_{22}} B_3^{l_{32}} + \gamma B_0^{l_{11}+l_{12}+l_{21}+l_{22}} B_3^{l_{31}+l_{32}}}{1 + \gamma (-1 + B_0)} & \frac{\alpha \epsilon + \theta - \gamma \theta + (-\alpha \epsilon + \gamma \theta) B_0 + \epsilon B_0^{l_{11}+l_{21}} B_3^{l_{31}}}{1 + \gamma (-1 + B_0)} \\ y_3 & \frac{\delta \phi + \psi - \gamma \psi + (-\delta \phi + \gamma \psi) B_0 + \phi B_0^{l_{12}+l_{22}} B_3^{l_{32}}}{1 + \gamma (-1 + B_0)} & \frac{\Xi - \gamma \Xi + \epsilon \phi + (\gamma \Xi - \epsilon \phi) B_0}{1 + \gamma (-1 + B_0)} \\ G & b_0 (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & b_0 (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

Out[ ]:= True

In[ ]:=  $\mathbf{R}_{1,2} // \mathbb{G} // \mathbb{G}\text{Form}$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & x_1 & x_2 \\ y_1 & 0 & 1 \\ y_2 & 0 & 0 \\ G & 0 & b_1 \end{pmatrix}$$



In[ ]:=  $(\mathbf{R}_{1,4} \bar{\mathbf{R}}_{6,2} \bar{\mathbf{R}}_{3,5}) // \mathbf{dm}_{1,2 \rightarrow 1} // \mathbf{dm}_{1,3 \rightarrow 1} // \mathbf{dm}_{1,4 \rightarrow 1} // \mathbf{dm}_{5,6 \rightarrow 2}$

Out[ ]:=  $\mathbb{E}_{\{1\} \rightarrow \{1,2\}} \left[ a_1 b_1 - a_2 b_1 - a_1 b_2, \right.$   
 $\frac{(B_1 - B_2 + B_2^2) x_1 y_1}{-B_2 + B_1 B_2 + B_2^2} + \frac{x_2 y_1}{1 - B_1 - B_2} - \frac{B_1 x_1 y_2}{-1 + B_1 + B_2} + \frac{(1 - B_1) x_2 y_2}{-B_1 + B_1^2 + B_1 B_2}, \frac{B_1 B_2}{-1 + B_1 + B_2} + O[\epsilon]^1 \left. \right]$

In[\*]:= (R1,4 R6,2 R3,5) // dm1,2->1 // dm1,3->1 // dm1,4->1 // dm5,6->2 // Simplify // C // CForm

Out[\*]//MatrixForm=

$$\begin{pmatrix} \frac{B_1 B_2}{-1+B_1+B_2} & X_1 & X_2 \\ Y_1 & \frac{B_1+(-1+B_2) B_2}{B_2 (-1+B_1+B_2)} & -\frac{1}{-1+B_1+B_2} \\ Y_2 & -\frac{B_1}{-1+B_1+B_2} & \frac{1-B_1}{B_1 (-1+B_1+B_2)} \\ \mathbb{C} & b_1 - b_2 & -b_1 \end{pmatrix}$$

In[\*]:= (cR1,4 cR6,2 cR3,5) // cm1,2->1 // cm1,3->1 // cm1,4->1 // cm5,6->2 // Simplify // C // CForm

Out[\*]//MatrixForm=

$$\begin{pmatrix} \frac{B_1 B_2}{-1+B_1+B_2} & X_1 & X_2 \\ Y_1 & -\frac{(-1+B_1) (B_1+(-1+B_2) B_2)}{b_1 B_2 (-1+B_1+B_2)} & \frac{-1+B_1}{b_1 (-1+B_1+B_2)} \\ Y_2 & \frac{B_1 (-1+B_2)}{b_2 (-1+B_1+B_2)} & \frac{(-1+B_1) (-1+B_2)}{b_2 B_1 (-1+B_1+B_2)} \\ \mathbb{C} & b_1 - b_2 & -b_1 \end{pmatrix}$$

In[\*]:= (rR[1, 4] rR[6, 2] rR[3, 5]) // dm[1, 2, 1] // dm[1, 3, 1] // dm[1, 4, 1] // dm[5, 6, 2]

Out[\*]=

$$\begin{pmatrix} \frac{-1+T_1+T_2}{T_2} & S_1 & S_2 \\ S_1 & \frac{T_1}{-1+T_1+T_2} & \frac{(-1+T_1) T_2}{-1+T_1+T_2} \\ S_2 & \frac{-1+T_2}{-1+T_1+T_2} & -\frac{1-T_1-2 T_2+T_1 T_2}{-1+T_1+T_2} \\ \Gamma & \frac{T_1}{T_2} & \frac{1}{T_1} \end{pmatrix}$$

In[\*]:=  $\gamma_1 = \text{Expand} / @$

$$\Gamma_{1\{1,2,3\}}[\omega, \{b_1, b_2, b_3\}] \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \{y_1, y_2, y_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \xi \end{pmatrix} \cdot \{x_1, x_2, x_3\}$$

Out[\*]=  $\Gamma_{1\{1,2,3\}}[\omega, a_1 b_1 l_{11} + a_2 b_1 l_{12} + a_3 b_1 l_{13} + a_1 b_2 l_{21} + a_2 b_2 l_{22} + a_3 b_2 l_{23} + a_1 b_3 l_{31} + a_2 b_3 l_{32} + a_3 b_3 l_{33}, \alpha x_1 y_1 + \beta x_2 y_1 + \theta x_3 y_1 + \gamma x_1 y_2 + \delta x_2 y_2 + \epsilon x_3 y_2 + \phi x_1 y_3 + \psi x_2 y_3 + \xi x_3 y_3]$

In[\*]:=  $\Gamma_{1_{sp1}}[\omega, L, Q] // \mathbb{E} := \mathbb{E}_{\{\}} \rightarrow_{sp1}[L, Q, \omega + \mathbf{0}[\epsilon]];$

$\Gamma_{1_{sp1}}[\omega, L, Q] // \mathbb{E}_{sp2}[L2, Q2, P2] := \Gamma_{1_{sp1}}[\omega, L, Q] // \mathbb{E} // \mathbb{E}_{sp2}[L2, Q2, P2]$

In[\*]:=  $\mathbb{E}_{\{\}} \rightarrow_r[L, Q, P] // \Gamma_1 := \Gamma_{1_r}[\text{Normal}@P, L, Q]$

In[\*]:=  $\Gamma_{1_s}[\omega, L, Q] // \Gamma_1 \text{Form} := \text{Module}[\{M, i, j\},$   
 $M = \text{Table}[\text{Simplify}[\partial_{x_i, y_j} Q], \{i, S\}, \{j, S\}];$   
 $\text{PrependTo}[M, y_{\#} \& / @ S];$   
 $M = \text{Join}[$   
 $\{\text{Prepend}[x_{\#} \& / @ S, \omega]\},$   
 $\text{Transpose}[M],$   
 $\{\text{Prepend}[\text{Table}[\partial_{a_i} L, \{i, S\}], "r1"]\}$   
 $];$   
 $\text{MatrixForm}[M]$   
 $];$

```
In[ ]:=  $\Gamma_{S1\_}[\omega1\_ , L1\_ , Q1\_ ] \equiv \Gamma_{S2\_}[\omega2\_ , L2\_ , Q2\_ ] :=$   

 $(\text{Sort}[S1] === \text{Sort}[S2]) \wedge \text{Simplify}[\omega1 == \omega2] \wedge \text{Simplify}[L1 == L2] \wedge \text{Simplify}[Q1 == Q2]$ 
```

```
In[ ]:=  $\Upsilon1 \equiv (\Upsilon1 // \mathbb{E} // \Gamma1)$ 
```

```
Out[ ]:= True
```

```
In[ ]:=  $\Upsilon1 // \text{cm}_{1,2 \rightarrow 0} // \Gamma1 // \Gamma1\text{Form}$ 
```

```
Out[ ]:= MatrixForm=
```

$$\begin{pmatrix} -\frac{\omega}{1+\gamma b_\theta} & X_\theta & X_3 \\ y_\theta & \frac{\beta + (-\beta\gamma + \alpha\delta) b_\theta + \delta B_\theta^{111+121} B_3^{311} + \alpha B_\theta^{112+122} B_3^{322} + \gamma B_\theta^{111+112+121+122} B_3^{321+132}}{1-\gamma b_\theta} & \frac{\theta + (\alpha - \gamma\theta) b_\theta + \epsilon B_\theta^{111+121} B_3^{311}}{1-\gamma b_\theta} \\ y_3 & \frac{\psi + (\delta\phi - \gamma\psi) b_\theta + \phi B_\theta^{112+122} B_3^{322}}{1-\gamma b_\theta} & \frac{\Xi + (-\gamma\Xi + \epsilon\phi) b_\theta}{1-\gamma b_\theta} \\ \Gamma1 & b_\theta (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & b_\theta (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

```
In[ ]:=  $\Upsilon2 = \text{Expand} / @$ 
```

$$\Gamma_{2\{1,2,3\}}[\omega, \{b_1, b_2, b_3\} \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \{y_1, y_2, y_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} \cdot \{x_1, x_2, x_3\}]$$

```
Out[ ]:=  $\Gamma_{2\{1,2,3\}}[\omega, a_1 b_1 l_{11} + a_2 b_1 l_{12} + a_3 b_1 l_{13} + a_1 b_2 l_{21} + a_2 b_2 l_{22} + a_3 b_2 l_{23} + a_1 b_3 l_{31} + a_2 b_3 l_{32} + a_3 b_3 l_{33},$   

 $\alpha x_1 y_1 + \beta x_2 y_1 + \theta x_3 y_1 + \gamma x_1 y_2 + \delta x_2 y_2 + \epsilon x_3 y_2 + \phi x_1 y_3 + \psi x_2 y_3 + \Xi x_3 y_3]$ 
```

```
In[ ]:=  $\Gamma_{2sp1\_}[\omega\_ , L\_ , Q\_ ] // \mathbb{E} := \mathbb{E}_{\{ \} \rightarrow sp1} [L, Q / . y_{i\_} \Rightarrow y_i / b_i, \omega + 0[\epsilon]];$ 
```

```
In[ ]:=  $\mathbb{E}_{\{ \} \rightarrow r\_} [L\_ , Q\_ , P\_ ] // \Gamma2 := \Gamma_{2r}[\text{Normal}@P, L, Q / . y_{i\_} \Rightarrow y_i b_i]$ 
```

```
In[ ]:=  $\Gamma_{2S1\_}[\omega1\_ , L1\_ , Q1\_ ] \equiv \Gamma_{2S2\_}[\omega2\_ , L2\_ , Q2\_ ] :=$   

 $(\text{Sort}[S1] === \text{Sort}[S2]) \wedge \text{Simplify}[\omega1 == \omega2] \wedge \text{Simplify}[L1 == L2] \wedge \text{Simplify}[Q1 == Q2]$ 
```

```
In[ ]:=  $\Upsilon2 \equiv (\Upsilon2 // \mathbb{E} // \Gamma2)$ 
```

```
Out[ ]:= True
```

```
In[ ]:=  $\Gamma_{2S\_}[\omega\_ , L\_ , Q\_ ] // \Gamma2\text{Form} := \text{Module}[\{M, i, j\},$   

 $M = \text{Table}[\text{Simplify}[\partial_{x_i, y_j} Q], \{i, S\}, \{j, S\}];$   

 $\text{PrependTo}[M, y_{\#} \& / @ S];$   

 $M = \text{Join}[$   

 $\{ \text{Prepend}[x_{\#} \& / @ S, \omega] \},$   

 $\text{Transpose}[M],$   

 $\{ \text{Prepend}[\text{Table}[\partial_{a_i} L, \{i, S\}], "r2"] \}$   

 $];$   

 $\text{MatrixForm}[M]$   

 $];$ 
```



```
In[ ]:=  $\Gamma3_S[\omega_, L_, Q_] // \Gamma3Form := Module[{M, i, j},
  M = Table[Simplify[\partial_{x_i, y_j} Q], {i, S}, {j, S}];
  PrependTo[M, y_# & /@ S];
  M = Join[
    {Prepend[x_# & /@ S, \omega]},
    Transpose[M],
    {Prepend[Table[\partial_{a_i} L, {i, S}], "\Gamma3"]}
  ];
  MatrixForm[M]
];$ 
```

```
In[ ]:=  $CR_{1,2} // \Gamma3 // \Gamma3Form$ 
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & x_1 & x_2 \\ y_1 & 1 & 1 - B_1 \\ y_2 & 0 & B_1 \\ \Gamma3 & 0 & b_1 \end{pmatrix}$$

```
In[ ]:=  $\Gamma3_{S1}[\omega1_, L1_, Q1_] \equiv \Gamma3_{S2}[\omega2_, L2_, Q2_] :=$ 
  (Sort[S1] === Sort[S2]) \wedge Simplify[\omega1 == \omega2] \wedge Simplify[L1 == L2] \wedge Simplify[Q1 == Q2]
```

```
In[ ]:=  $\Upsilon3 \equiv (\Upsilon3 // \mathbb{E} // \Gamma3)$ 
```

Out[ ]:= True

```
In[ ]:=  $\Upsilon3 // \mathbb{E} // cm_{1,2 \rightarrow 0} // \Gamma3 // \Gamma3Form$ 
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{\omega}{-1+\gamma} & x_0 & x_3 \\ y_0 & \beta - \frac{\alpha \delta}{-1+\gamma} & -\frac{\alpha \epsilon}{-1+\gamma} + \theta \\ y_3 & -\frac{\delta \phi}{-1+\gamma} + \psi & \Xi - \frac{\epsilon \phi}{-1+\gamma} \\ \Gamma3 & b_0 (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & b_0 (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$

```
In[ ]:=  $\Upsilon3 // \mathbb{E} // cm_{2,1 \rightarrow 0} // \Gamma3 // \Gamma3Form$ 
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{\omega}{-1+\beta} & x_0 & x_3 \\ y_0 & \gamma - \frac{\alpha \delta}{-1+\beta} & \epsilon - \frac{\delta \theta}{-1+\beta} \\ y_3 & \phi - \frac{\alpha \psi}{-1+\beta} & \Xi - \frac{\theta \psi}{-1+\beta} \\ \Gamma3 & b_0 (l_{11} + l_{12} + l_{21} + l_{22}) + b_3 (l_{31} + l_{32}) & b_0 (l_{13} + l_{23}) + b_3 l_{33} \end{pmatrix}$$



```
In[ ]:=  $\Gamma4_{S_}$  [ $\omega_$ ,  $L_$ ,  $Q_$ ] //  $\Gamma4Form := Module[{M, i, j, SS = Sort[S]},
  M = Table[Simplify[ $\partial_{y_i, x_j} Q$ ], {i, SS}, {j, SS}];
  PrependTo[M,  $x_{\#}$  & /@ SS];
  M = Join[
    {Prepend[ $y_{\#}$  & /@ SS,  $\omega$ ]},
    Transpose[M],
    {Prepend[Table[ $\partial_{a_i} L$ , {i, SS}], " $\Gamma4$ "]}
  ];
  MatrixForm[M]
];$ 
```

```
In[ ]:=  $\gamma4 = Expand /@$ 
```

$$\Gamma4_{\{1,2,3\}} [\omega, \{b_1, b_2, b_3\} \cdot \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \{a_1, a_2, a_3\}, \{x_1, x_2, x_3\} \cdot \begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \xi \end{pmatrix} \cdot \{y_1, y_2, y_3\}]$$

```
Out[ ]:=  $\Gamma4_{\{1,2,3\}} [\omega, a_1 b_1 l_{11} + a_2 b_1 l_{12} + a_3 b_1 l_{13} + a_1 b_2 l_{21} + a_2 b_2 l_{22} + a_3 b_2 l_{23} + a_1 b_3 l_{31} + a_2 b_3 l_{32} + a_3 b_3 l_{33},$   

 $\alpha x_1 y_1 + \gamma x_2 y_1 + \phi x_3 y_1 + \beta x_1 y_2 + \delta x_2 y_2 + \psi x_3 y_2 + \theta x_1 y_3 + \epsilon x_2 y_3 + \xi x_3 y_3]$ 
```

```
In[ ]:=  $\gamma4 // \Gamma4Form$ 
```

```
Out[ ]//MatrixForm=
```

$$\begin{pmatrix} \omega & y_1 & y_2 & y_3 \\ x_1 & \alpha & \beta & \theta \\ x_2 & \gamma & \delta & \epsilon \\ x_3 & \phi & \psi & \xi \\ \Gamma4 & b_1 l_{11} + b_2 l_{21} + b_3 l_{31} & b_1 l_{12} + b_2 l_{22} + b_3 l_{32} & b_1 l_{13} + b_2 l_{23} + b_3 l_{33} \end{pmatrix}$$

```
In[ ]:=  $\Gamma4_{sp_}$  [ $\omega_$ ,  $L_$ ,  $Q_$ ] //  $\mathbb{E} :=$   

 $\mathbb{E}_{\{\} \rightarrow sp} [L, (Q /. y_{i_} \Rightarrow y_i / b_i) - Sum[(e^{-\partial_{a_i} L} /. 12U) y_i x_i / b_i, \{i, sp\}], \omega^{-1} + 0[\epsilon]];$ 
```

```
In[ ]:=  $\mathbb{E}_{\{\} \rightarrow r_}$  [ $L_$ ,  $Q_$ ,  $P_$ ] //  $\Gamma4 :=$   

 $\Gamma4_r [(Normal@P)^{-1}, L, (Q /. y_{i_} \Rightarrow y_i b_i) + Sum[(e^{-\partial_{a_i} L} /. 12U) y_i x_i, \{i, r\}]]$ 
```

```
In[ ]:= { $cR_{1,2} // \Gamma4 // \Gamma4Form$ ,  $\overline{cR}_{1,2} // \Gamma4 // \Gamma4Form$ }
```

```
Out[ ]:= {  $\begin{pmatrix} 1 & y_1 & y_2 \\ x_1 & 1 & \theta \\ x_2 & 1 - B_1 & B_1 \\ \Gamma4 & \theta & b_1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & y_1 & y_2 \\ x_1 & 1 & \theta \\ x_2 & 1 - \frac{1}{B_1} & \frac{1}{B_1} \\ \Gamma4 & \theta & -b_1 \end{pmatrix}$  }
```

```
In[ ]:= { $cR_{2,1} // \Gamma4 // \Gamma4Form$ ,  $\overline{cR}_{2,1} // \Gamma4 // \Gamma4Form$ }
```

```
Out[ ]:= {  $\begin{pmatrix} 1 & y_1 & y_2 \\ x_1 & B_2 & 1 - B_2 \\ x_2 & \theta & 1 \\ \Gamma4 & b_2 & \theta \end{pmatrix}$ ,  $\begin{pmatrix} 1 & y_1 & y_2 \\ x_1 & \frac{1}{B_2} & 1 - \frac{1}{B_2} \\ x_2 & \theta & 1 \\ \Gamma4 & -b_2 & \theta \end{pmatrix}$  }
```

```
In[ ]:=  $\Gamma4_{S1_} [\omega1_$ ,  $L1_$ ,  $Q1_]$   $\equiv \Gamma4_{S2_} [\omega2_$ ,  $L2_$ ,  $Q2_]$  :=  

 $(Sort[S1] === Sort[S2]) \wedge Simplify[\omega1 == \omega2] \wedge Simplify[L1 == L2] \wedge Simplify[Q1 == Q2]$ 
```

In[ ]:=  $\gamma_4 \equiv (\gamma_4 // \mathbb{E} // \Gamma_4)$

Out[ ]:= True

In[ ]:=  $\gamma_4 // \mathbb{E} // \mathbf{cm}_{1,2 \rightarrow 0} // \Gamma_4 // \Gamma_4 \text{Form}$

Out[ ]//MatrixForm=

$$\left( \begin{array}{ccc} \omega - \beta \omega & \mathbf{y}_0 & \mathbf{y}_3 \\ \mathbf{x}_0 & \gamma - \frac{\alpha \delta}{-1+\beta} & \epsilon - \frac{\delta \theta}{-1+\beta} \\ \mathbf{x}_3 & \phi - \frac{\alpha \psi}{-1+\beta} & \xi - \frac{\theta \psi}{-1+\beta} \\ \Gamma_4 & \mathbf{b}_0 (l_{11} + l_{12} + l_{21} + l_{22}) + \mathbf{b}_3 (l_{31} + l_{32}) & \mathbf{b}_0 (l_{13} + l_{23}) + \mathbf{b}_3 l_{33} \end{array} \right)$$