Pensieve header: Alexander from Heisenberg.

In[∘]:=
```
SetDirectory[
 "C:\\drorbn\\AcademicPensieve\\Talks\\LearningSeminarOnCategorification-2006"]
```

Out[∘]=
```
C:\drorbn\AcademicPensieve\Talks\LearningSeminarOnCategorification-2006
```

tex

```
\def\cellscale{0.8}
\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
\def\nbpdfEcho#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
\def\nbpdfSubsection#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
```

pdf

```
𝔼_{s1_}[ω1_, Q1_] ≡ 𝔼_{s2_}[ω2_, Q2_] := s1 === s2 ∧ Simplify[ω1 == ω2 ∧ Q1 == Q2]
```

pdf

In[∘]:=
```
𝔼_{A1_→B1_}[ω1_, Q1_] 𝔼_{A2_→B2_}[ω2_, Q2_] /; (A1 ⋂ A2 === {}) ∧ (B1 ⋂ B2 === {}) ^:=
 𝔼_{A1⋃A2→B1⋃B2}[ω1 ω2, Q1 + Q2]
```

pdf

```
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I},
  I = IdentityMatrix@Length@B1;
  E1 = Table[∂_{i,j}Q1, {i, A1}, {j, B1}]; E2 = Table[∂_{i,j}Q2, {i, A2}, {j, B2}];
  F1 = Table[∂_{i,j}Q1, {i, A1}, {j, A1}]; F2 = Table[∂_{i,j}Q2, {i, A2}, {j, A2}];
  G1 = Table[∂_{i,j}Q1, {i, B1}, {j, B1}]; G2 = Table[∂_{i,j}Q2, {i, B2}, {j, B2}];
  Expand /@ 𝔼_{A1→B2}[PowerExpand@Simplify[ω1 ω2 Det[I - F2.G1]^(1/2)], Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).B2] ]] ]
```

pdf

In[∘]:=
```
A_ \ B_ := Complement[A, B]
```

pdf

In[∘]:=
```
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* =!= A2) :=
 𝔼_{A1⋃(A2\B1*)→B1⋃A2*}[ω1, Q1 + Sum[ξ* ξ, {ξ, A2 \ B1*}]] //
  𝔼_{B1*⋃A2→B2⋃(B1\A2*)}[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

pdf

In[∘]:=
```
Hz@i_ := Sequence[p_i, x_i]; Hξ@i_ := Sequence[π_i, ξ_i];
{p*, x*} = {π, ξ}; {π*, ξ*} = {p, x};
(u_{i_})* := (u*)_i; l_List* := #* & /@ l;
```

A proof of the formula for R is at http://drorbn.net/cat20.

pdf

In[∘]:=

$$R_{i\_,j\_} := \mathbb{E}_{\{\}\to\{\text{Hz@}i,\text{Hz@}j\}}\left[1,\ (1-T)\ p_j\ x_j + (T-1)\ p_i\ x_j\right];$$

$$\overline{R}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{\text{Hz@}i,\text{Hz@}j\}}\left[1,\ (1-T^{-1})\ p_j\ x_j + (T^{-1}-1)\ p_i\ x_j\right];$$

$$C_{i\_} := \mathbb{E}_{\{\}\to\{\text{Hz@}i\}}\left[T^{-1/2},\ 0\right];\ \ \overline{C}_{i\_} := \mathbb{E}_{\{\}\to\{\text{Hz@}i\}}\left[T^{1/2},\ 0\right];$$

In[∘]:= C_1

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1}{\sqrt{T}},\ 0\right]$

In[∘]:= $\left\{(R_{1,2})_h,\ (\overline{R}_{1,2})_h\right\}$

Out[∘]= $\left\{\mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[1,\ (-1+T)\ p_1\ x_2 + (1-T)\ p_2\ x_2\right]_h,\ \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[1,\ \left(-1+\frac{1}{T}\right)\ p_1\ x_2 + \left(1-\frac{1}{T}\right)\ p_2\ x_2\right]_h\right\}$

In[∘]:= $R_{1,2} \equiv \mathbb{E}_{\{\}\to\{\text{Hz}[1],\text{Hz}[2]\}}\left[1,\ \{x_1,\ x_2\}.\begin{pmatrix} 0 & 0 \\ T-1 & 1-T \end{pmatrix}.\{p_1,\ p_2\}\right]$

Out[∘]= True

A proof of the formula for hm is at http://drorbn.net/cat20.

pdf

In[∘]:=

$$hm_{i\_,j\_\to k\_} := \mathbb{E}_{\{\text{H}\xi\text{@}i,\text{H}\xi\text{@}j\}\to\{\text{Hz@}k\}}\left[1,\ -\xi_i\ \pi_j + (\pi_i + \pi_j)\ p_k + (\xi_i + \xi_j)\ x_k\right]$$

tex

Proof of Reidemeister 3:

{\def\nbpdfOutput#1{\vskip
1mm\par\noindent\includegraphics[scale=\cellscale]{#1}\hfill\text{$\Box$\quad}}

pdf

In[∘]:= $\left(R_{1,2}\ R_{6,3}\ R_{4,5}\ //\ hm_{1,6\to1}\ hm_{2,4\to2}\ hm_{3,5\to3}\right) \equiv \left(R_{2,3}\ R_{1,4}\ R_{5,6}\ //\ hm_{1,5\to1}\ hm_{2,6\to2}\ hm_{3,4\to3}\right)$

pdf

Out[∘]= True

tex

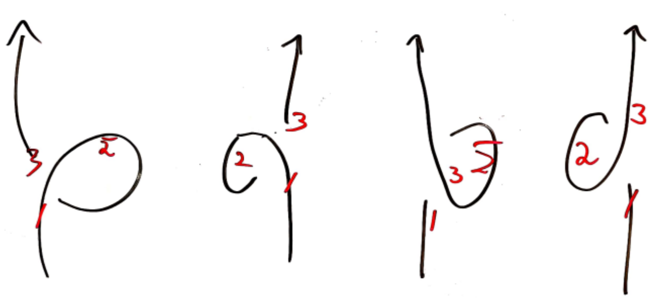
}

Reidemeister 2:

In[∘]:= $\left\{\overline{R}_{1,2}\ R_{3,4}\ //\ hm_{1,3\to1}\ hm_{2,4\to2},\ \overline{R}_{1,4}\ R_{3,2}\ //\ hm_{1,3\to1}\ hm_{2,4\to2}\right\}$

Out[∘]= $\left\{\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1,\ 0],\ \mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1,\ 0]\right\}$

Reidemeister 1's:



In[∘]:= $\{(R_{1,3}\,\overline{C}_2)$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}$, $(\overline{R}_{1,3}\,C_2)$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}$, $(\overline{R}_{3,1}\,\overline{C}_2)$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}$, $(R_{3,1}\,C_2)$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}\}$

Out[∘]= $\left\{\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\sqrt{T},0\right], \mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1}{\sqrt{T}},0\right], \mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1}{\sqrt{T}},0\right], \mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\sqrt{T},0\right]\right\}$

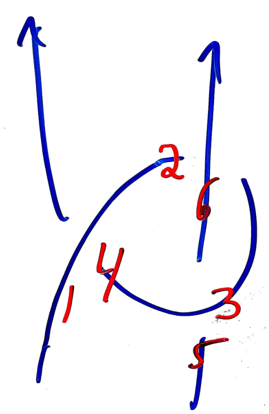
In[∘]:=
```
Kink_i_ := E_{}→{p_i,x_i}[T^(-1/2), 0]; Kinkbar_i_ := E_{}→{p_i,x_i}[T^(1/2), 0];
```

pdf

In[∘]:=
```
E_{}→vs_[ωi_, Q_]_h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[ωi, 1 + Length@ps, 1 + Length@xs];
  M[[2 ;;, 2 ;;]] = Table[Simplify[∂_{i,j}Q], {i, ps}, {j, xs}];
  M[[2 ;;, 1]] = ps; M[[1, 2 ;;]] = xs;
  MatrixForm[M]_h]
```

tex

The ``First Tangle''.



pdf

In[∘]:= Simplify /@ $((R_{1,4}\,\overline{R}_{6,2}\,\overline{R}_{3,5})$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}$ // $hm_{1,4\to1}$ // $hm_{5,6\to2})$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{1-2T}{T^2}, \frac{(-1+T)(p_1-p_2)(T x_1 - x_2)}{-1+2T}\right]$

pdf

In[∘]:= $((R_{1,4}\,\overline{R}_{6,2}\,\overline{R}_{3,5})$ // $hm_{1,2\to1}$ // $hm_{1,3\to1}$ // $hm_{1,4\to1}$ // $hm_{5,6\to2})_h$

pdf

Out[∘]= $$\begin{pmatrix} \frac{1}{T^2}-\frac{2}{T} & x_1 & x_2 \\ p_1 & \frac{(-1+T)\,T}{-1+2T} & \frac{1-T}{-1+2T} \\ p_2 & -\frac{(-1+T)\,T}{-1+2T} & \frac{-1+T}{-1+2T} \end{pmatrix}_h$$

tex

The knot 8_{17}.

pdf

In[∘]:=
```
z = R̄12,1 R̄27 R̄83 R̄4,11 R16,5 R6,13 R14,9 R10,15;
Do[z = z // hm1k→1, {k, 2, 16}];
Simplify /@ z
```

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[-11+\frac{1}{T^3}-\frac{4}{T^2}+\frac{8}{T}+8\,T-4\,T^2+T^3,\ 0\right]$

tex

{\red\bf Proof of Theorem 3, (3).}
{\def\nbpdfOutput#1{\vskip
1mm\par\noindent\includegraphics[scale=\cellscale]{#1}\hfill\text{$\Box$\quad}}

pdf

In[∘]:= $\left\{\left(\gamma 1=\mathbb{E}_{\{\}\to\{Hz[1],Hz[2],Hz[3]\}}\left[\omega,\ \{p_1,p_2,p_3\}.\begin{pmatrix}\alpha&\beta&\theta\\\gamma&\delta&\epsilon\\\phi&\psi&\Xi\end{pmatrix}.\{x_1,x_2,x_3\}\right]\right)_h,\ \left(\gamma 1\ //\ hm_{1,2\to 0}\right)_h\right\}$

pdf

Out[∘]= $\left\{\begin{pmatrix}\omega&x_1&x_2&x_3\\p_1&\alpha&\beta&\theta\\p_2&\gamma&\delta&\epsilon\\p_3&\phi&\psi&\Xi\end{pmatrix}_h,\ \begin{pmatrix}\omega+\gamma\,\omega&x_0&x_3\\p_0&\frac{\alpha+\beta+\gamma+\beta\,\gamma+\delta-\alpha\,\delta}{1+\gamma}&\frac{\epsilon-\alpha\,\epsilon+\theta+\gamma\,\theta}{1+\gamma}\\p_3&\frac{\phi-\delta\,\phi+\psi+\gamma\,\psi}{1+\gamma}&\frac{\Xi+\gamma\,\Xi-\epsilon\,\phi}{1+\gamma}\end{pmatrix}_h\right\}$

tex

}

In[∘]:= $\text{Simplify}\left[\text{Table}\left[\partial_{i,j}\left(\gamma 1\ //\ hm_{1,2\to 0}\right)[[2]],\ \{i,\{p_0,p_3\}\},\ \{j,\{x_0,x_3\}\}\right]==\begin{pmatrix}1+\beta-\frac{(1-\alpha)\,(1-\delta)}{1+\gamma}&\theta+\frac{(1-\alpha)\,\epsilon}{1+\gamma}\\\psi+\frac{(1-\delta)\,\phi}{1+\gamma}&\Xi-\frac{\epsilon\,\phi}{1+\gamma}\end{pmatrix}\right]$

Out[∘]= True

In[∘]:=
```
MatrixForm@Simplify[
  IdentityMatrix[2] - Table[∂i,j (γ1 // hm1,2→0)[[2]], {i, {p0, p3}}, {j, {x0, x3}}] /.
   Thread[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ} -> {1 - α, -β, -γ, 1 - δ, -θ, -ϵ, -ϕ, -ψ, 1 - Ξ}]
 ]
```

Out[∘]//MatrixForm=
$$\begin{pmatrix}\beta-\frac{\alpha\,\delta}{-1+\gamma}&-\frac{\alpha\,\epsilon}{-1+\gamma}+\theta\\-\frac{\delta\,\phi}{-1+\gamma}+\psi&\Xi-\frac{\epsilon\,\phi}{-1+\gamma}\end{pmatrix}$$

Recycling

Our PBW ordering is $\{p, x\}$.

We are at $[p, x] = 1$ and $R = e^{-t\otimes px+tp\otimes x}$. Let $a = p\,x$. Then $[a, x] = x$.

Claim. $e^{-t\,a+tp\,x} = e^{-t\,a}\,e^{t^{-1}(e^t-1)\,tp\,x}$.

Proof. Use a 2d representation:

In[∘]:= $\rho a=\begin{pmatrix}1&0\\0&0\end{pmatrix};\ \rho x=\begin{pmatrix}0&1\\0&0\end{pmatrix};\ \rho a.\rho x-\rho x.\rho a==\rho x$

Out[∘]= True

In[ ]:=
```
Simplify[MatrixExp[-t ρa + tp ρx] == MatrixExp[-t ρa].MatrixExp[t^-1 (e^t - 1) tp ρx]]
```
Out[ ]= True

Claim. $e^{t p x} = \mathbb{O}\left[e^{(1-e^{-t}) p x}\right]$.

Proof. True at $t=0$, test ∂_t using $p\,x\,\mathbb{O}[f] = \mathbb{O}[p(x f - \partial_p f)]$:

In[ ]:=
```
Simplify[p (x e^((1-e^-t) p x) - ∂_p e^((1-e^-t) p x)) == ∂_t e^((1-e^-t) p x)]
```
Out[ ]= True

Claim. $\mathbb{O}\left[e^{-t a + \mathrm{tp}\, x}\right] = e^{(e^t - 1) p x + t^{-1}(e^t - 1)\,\mathrm{tp}\, x}$

In[ ]:=
```
Collect[(1 - e^-t) p x + t^-1 (e^-t - 1) tp x, {tp}, Simplify]
```
Out[ ]= $\left(1 - e^{-t}\right) \mathrm{p\,x} + \frac{\left(-1 + e^{-t}\right) \mathrm{tp\,x}}{\mathrm{t}}$

The Trefoil

In[ ]:=
```
Z31 = R_{1,5} R_{6,2} R_{3,7} C̄_4 Kink̄_8 Kink̄_9 Kink̄_10;
Do[Z31 = Z31 // hm_{1,r→1}, {r, 2, 10}];
Simplify /@ Z31
```
Out[ ]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[T^2\left(1 - T + T^2\right), 0\right]$