Pensieve header: Alexander from Heisenberg.

In[∘]:=
```
SetDirectory[
 "C:\\drorbn\\AcademicPensieve\\Talks\\LearningSeminarOnCategorification-2006"]
```

Out[∘]=
```
C:\drorbn\AcademicPensieve\Talks\LearningSeminarOnCategorification-2006
```

In[∘]:=
```
𝔼A1_→B1_[ω1_, Q1_] ≡ 𝔼A2_→B2_[ω2_, Q2_] :=
 A1 === A2 ∧ B1 === B2 ∧ Simplify[ω1 == ω2] ∧ Simplify[Q1 == Q2]
```

In[∘]:=
```
𝔼A1_→B1_[ω1_, Q1_] 𝔼A2_→B2_[ω2_, Q2_] /; (A1 ⋂ A2 === {}) ∧ (B1 ⋂ B2 === {}) ^:=
 𝔼A1⋃A2→B1⋃B2[ω1 ω2, Q1 + Q2]
```

In[∘]:=
```
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I},
  I = IdentityMatrix@Length@B1;
  E1 = Table[∂i,j Q1, {i, A1}, {j, B1}]; E2 = Table[∂i,j Q2, {i, A2}, {j, B2}];
  F1 = Table[∂i,j Q1, {i, A1}, {j, A1}]; F2 = Table[∂i,j Q2, {i, A2}, {j, A2}];
  G1 = Table[∂i,j Q1, {i, B1}, {j, B1}]; G2 = Table[∂i,j Q2, {i, B2}, {j, B2}];
  Expand /@ 𝔼A1→B2[PowerExpand@Simplify[ω1 ω2 Det[I - F2.G1]^(-1/2)], Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).B2]
    ]]]
```

In[∘]:=
```
A_ \ B_ := Complement[A, B]
```

In[∘]:=
```
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* =!= A2) :=
 𝔼A1⋃(A2\B1*)→B1⋃A2*[ω1, Q1 + Sum[ξ* ξ, {ξ, A2 \ B1*}]] //
  𝔼B1*⋃A2→B2⋃(B1\A2*)[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

```
(*(𝔼A1_→B1_[ω1_,Q1_]//𝔼A2_→B2_[ω2_,Q2_])/; (B1*=!=A2):=Module[{L,R},
  L=Echo@𝔼A1⋃(A2\B1*)→B1⋃A2*[ω1,Q1+Sum[ξ*ξ,{ξ,A2\B1*}]];
  R=Echo@𝔼B1*⋃A2→B2⋃(B1\A2*)[ω2,Q2+Sum[z*z,{z,B1\A2*}]];
  L//R]*)
```

In[∘]:=
```
Hz@i_ := Sequence[pi, xi]; Hξ@i_ := Sequence[πi, ξi];
{p*, x*} = {π, ξ}; {π*, ξ*} = {p, x};
(u_i_)* := (u*)i; l_List* := #* & /@ l;
```

Our PBW ordering is $\{p, x\}$.

We are at $[p, x] = 1$ and $R = e^{t\otimes px - tp\otimes x}$. Let $a = p\,x$. Then $[a, x] = x$.

Claim. $e^{t\,a - tp\,x} = e^{t\,a}\, e^{t^{-1}\left(e^{-t}-1\right) tp\,x}$.

Proof. Use a 2d representation:

In[∘]:= $\rho a = \begin{pmatrix}1&0\\0&0\end{pmatrix}$; $\rho x = \begin{pmatrix}0&1\\0&0\end{pmatrix}$; ρa.ρx − ρx.ρa == ρx

Out[∘]= True

In[∘]:= MatrixExp[t ρa − tp ρx] == MatrixExp[t ρa].MatrixExp[t^{-1} (e^{-t} − 1) tp ρx]

Out[∘]= True

Claim. $e^{t\,p\,x} = \mathbb{O}\left[e^{(1-e^{-t})\,p\,x}\right]$.

Proof. True at $t = 0$, test ∂_t using $p\,x\,\mathbb{O}[f] = \mathbb{O}[p(x\,f - \partial_p f)]$:

In[∘]:= Simplify[$p\left(x\, e^{(1-e^{-t})\,p\,x} - \partial_p e^{(1-e^{-t})\,p\,x}\right) == \partial_t e^{(1-e^{-t})\,p\,x}$]

Out[∘]= True

Claim. $\mathbb{O}\left[e^{t\,a - \text{tp}\,x}\right] = e^{(1-e^{-t})\,p\,x + t^{-1}(e^{-t}-1)\,\text{tp}\,x}$

In[∘]:= Collect[$(1 - e^{-t})$ p x + t^{-1} (e^{-t} − 1) tp x, {tp}, Simplify]

Out[∘]= $\left(1 - e^{-t}\right) p\,x + \frac{\left(-1 + e^{-t}\right) \text{tp}\,x}{t}$

In[∘]:= $R_{i\_,j\_} := \mathbb{E}_{\{\}\to\{Hz@i,Hz@j\}}\left[1, \left(1 - T^{-1}\right) p_j\, x_j + \left(T^{-1} - 1\right) p_i\, x_j\right]$;
$\overline{R}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{Hz@i,Hz@j\}}\left[1, \left(1 - T\right) p_j\, x_j + \left(T - 1\right) p_i\, x_j\right]$;

In[∘]:= $R_{1,2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[1, \left(-1 + \frac{1}{T}\right) p_1\, x_2 + \left(1 - \frac{1}{T}\right) p_2\, x_2\right]$

In[∘]:= $hm_{i\_,j\_\to k\_} := \mathbb{E}_{\{H\xi@i,H\xi@j\}\to\{Hz@k\}}\left[1, -\xi_i\, \pi_j + (\pi_i + \pi_j)\, p_k + (\xi_i + \xi_j)\, x_k\right]$

In[∘]:= $hm_{1,2\to3}$

Out[∘]= $\mathbb{E}_{\{\pi_1,\xi_1,\pi_2,\xi_2\}\to\{p_3,x_3\}}\left[1, p_3\, (\pi_1 + \pi_2) - \pi_2\, \xi_1 + x_3\, (\xi_1 + \xi_2)\right]$

In[∘]:= $hm_{1,2\to1}$ // $hm_{1,3\to1}$

Out[∘]= $\mathbb{E}_{\{\pi_1,\pi_2,\pi_3,\xi_1,\xi_2,\xi_3\}\to\{p_1,x_1\}}\left[1, p_1\,\pi_1 + p_1\,\pi_2 + p_1\,\pi_3 - \pi_2\,\xi_1 - \pi_3\,\xi_1 + x_1\,\xi_1 - \pi_3\,\xi_2 + x_1\,\xi_2 + x_1\,\xi_3\right]$

In[∘]:= ($hm_{2,3\to2}$ // $hm_{1,2\to1}$) ≡ ($hm_{1,2\to1}$ // $hm_{1,3\to1}$)

Out[∘]= True

In[∘]:= $R_{1,2}\, R_{6,3}\, R_{4,5}$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,p_3,p_4,p_5,p_6,x_1,x_2,x_3,x_4,x_5,x_6\}}\left[1, \left(-1 + \frac{1}{T}\right) p_1\, x_2 + \left(1 - \frac{1}{T}\right) p_2\, x_2 + \left(1 - \frac{1}{T}\right) p_3\, x_3 + \left(-1 + \frac{1}{T}\right) p_6\, x_3 + \left(-1 + \frac{1}{T}\right) p_4\, x_5 + \left(1 - \frac{1}{T}\right) p_5\, x_5\right]$

In[∘]:= $hm_{1,6\to1}\, hm_{2,4\to2}\, hm_{3,5\to3}$

Out[∘]= $\mathbb{E}_{\{\pi_1,\pi_2,\pi_3,\pi_4,\pi_5,\pi_6,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6\}\to\{p_1,p_2,p_3,x_1,x_2,x_3\}}\left[1, p_2\,(\pi_2 + \pi_4) + p_3\,(\pi_3 + \pi_5) + p_1\,(\pi_1 + \pi_6) - \pi_6\,\xi_1 - \pi_4\,\xi_2 - \pi_5\,\xi_3 + x_2\,(\xi_2 + \xi_4) + x_3\,(\xi_3 + \xi_5) + x_1\,(\xi_1 + \xi_6)\right]$

In[∘]:= $R_{1,2}\,R_{6,3}\,R_{4,5}$ // $hm_{1,6\to1}\,hm_{2,4\to2}\,hm_{3,5\to3}$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,p_3,x_1,x_2,x_3\}}\left[1,\ -p_1 x_2+\frac{p_1 x_2}{T}+p_2 x_2-\frac{p_2 x_2}{T}-p_1 x_3+\frac{p_1 x_3}{T}+\frac{p_2 x_3}{T^2}-\frac{p_2 x_3}{T}+p_3 x_3-\frac{p_3 x_3}{T^2}\right]$

In[∘]:= $(R_{1,2}\,R_{6,3}\,R_{4,5}$ // $hm_{1,6\to1}\,hm_{2,4\to2}\,hm_{3,5\to3}) \equiv (R_{2,3}\,R_{1,4}\,R_{5,6}$ // $hm_{1,5\to1}\,hm_{2,6\to2}\,hm_{3,4\to3})$

Out[∘]= True

In[∘]:= $\overline{R}_{1,2}\,R_{3,4}$ // $hm_{1,3\to1}\,hm_{2,4\to2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1,\ 0]$

In[∘]:= $\overline{R}_{1,4}\,R_{3,2}$ // $hm_{1,3\to1}\,hm_{2,4\to2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1,\ 0]$

In[∘]:= $\gamma1 = \mathbb{E}_{\{\}\to\{Hz[1],Hz[2],Hz[3]\}}\left[1,\ \{x_1,x_2,x_3\}.\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix}.\{p_1,p_2,p_3\}\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2,p_3,x_3\}}[1,\ p_3\ (\theta\, x_1+\epsilon\, x_2+\Xi\, x_3)+p_1\ (\alpha\, x_1+\gamma\, x_2+\phi\, x_3)+p_2\ (\beta\, x_1+\delta\, x_2+\psi\, x_3)]$

In[∘]:= Simplify /@ ($\gamma1$ // $hm_{1,2\to0}$)

Out[∘]= $\mathbb{E}_{\{\}\to\{p_0,p_3,x_0,x_3\}}\left[\frac{1}{1+\beta},\ \frac{p_0\ ((\alpha+\beta+\gamma+\beta\,\gamma+\delta-\alpha\,\delta)\ x_0+(\phi+\beta\,\phi+\psi-\alpha\,\psi)\ x_3)+p_3\ ((\epsilon+\beta\,\epsilon+\theta-\delta\,\theta)\ x_0+(\Xi+\beta\,\Xi-\theta\,\psi)\ x_3)}{1+\beta}\right]$

In[∘]:= MatrixForm@Simplify[
 Table[$\partial_{i,j}$ ($\gamma1$ // $hm_{1,2\to0}$)⟦2⟧, {j, {x_0, x_3}}, {i, {p_0, p_3}}]
]

Out[∘]//MatrixForm=

$$\begin{pmatrix}\frac{\alpha+\beta+\gamma+\beta\,\gamma+\delta-\alpha\,\delta}{1+\beta} & \frac{\epsilon+\beta\,\epsilon+\theta-\delta\,\theta}{1+\beta}\\ \frac{\phi+\beta\,\phi+\psi-\alpha\,\psi}{1+\beta} & \frac{\Xi+\beta\,\Xi-\theta\,\psi}{1+\beta}\end{pmatrix}$$

In[∘]:= Simplify[
 Table[$\partial_{i,j}$ ($\gamma1$ // $hm_{1,2\to0}$)⟦2⟧, {j, {x_0, x_3}}, {i, {p_0, p_3}}] == $\begin{pmatrix}1+\gamma-\frac{(1-\alpha)\ (1-\delta)}{1+\beta} & \epsilon+\frac{(1-\delta)\ \theta}{1+\beta}\\ \phi+\frac{(1-\alpha)\ \psi}{1+\beta} & \Xi-\frac{\theta\,\psi}{1+\beta}\end{pmatrix}$]

Out[∘]= True

In[∘]:= MatrixForm@Simplify[
 IdentityMatrix[2] - Table[$\partial_{i,j}$ ($\gamma1$ // $hm_{1,2\to0}$)⟦2⟧, {j, {x_0, x_3}}, {i, {p_0, p_3}}] /.
 Thread[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ} -> {1 - α, -β, -γ, 1 - δ, -θ, -ϵ, -ϕ, -ψ, 1 - Ξ}]
]

Out[∘]//MatrixForm=

$$\begin{pmatrix}\gamma-\frac{\alpha\,\delta}{-1+\beta} & \epsilon-\frac{\delta\,\theta}{-1+\beta}\\ \phi-\frac{\alpha\,\psi}{-1+\beta} & \Xi-\frac{\theta\,\psi}{-1+\beta}\end{pmatrix}$$

In[ ]:= Expand[1 + β - (1 - α) (1 - δ)]

Out[ ]= $\alpha + \beta + \delta - \alpha\,\delta$