Indiana-1611 handout on 161030

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Dror Bar-Natan: Talks: Indiana-1611: (thanks for accepting my invitation!) ωεβ:=http://drorbn.net/Indiana-1611/

A Poly-Time Knot Polynomial Via Solvable Approximation

Work in Progress!


Rozansky


Overbay

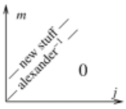
Abstract. Rozansky [Ro2] and Overbay [Ov] described a spectacular knot polynomial that failed to attract the attention it deserved as the first poly-time-computable knot polynomial since Alexander's [Al, 1928] and (in my opinion) as the second most likely knot polynomial (after Alexander's) to carry topological information. With Roland van der Veen, I will explain how to compute the Rozansky polynomial using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.


Melvin, Morton, Garoufalidis

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq 0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar)=1$.

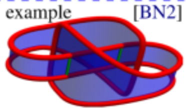


include specifics about the higher lines?

Why "spectacular"? Foremost reason: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C). Also, will bound genus and may disprove {ribbon} = {slice}.

Genus.


a ribbon singularity a clasp singularity

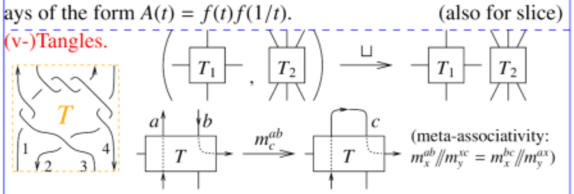


A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3=\partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

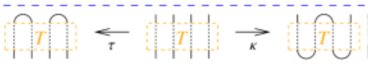
Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t)=f(t)f(1/t)$. (also for slice)

(v-)Tangles.



(meta-associativity: $m_x^{ab}/\!\!/m_y^{xc}=m_x^{bc}/\!\!/m_y^{ax}$)




Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccccc} & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n & \\ \mathcal{T}_{2n} \nearrow^{\tau} \searrow_{\kappa} & & \xrightarrow{z} & \mathcal{A}_{2n} \nearrow^{\tau} \searrow_{\kappa} & \text{with } \mathcal{R}:=\kappa(\tau^{-1}(1)) \\ & \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1 & \end{array}$$

Faster is better, leaner is meaner!

$a_+=-t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$

$\rho_+=5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T)\in R_S\times M_{S\times S}(R_S)=\left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S:=\mathbb{Z}(\{t_a : a\in S\})$:

$$\left({}_a\!\nearrow\!{}_b,\ {}_b\!\nwarrow\!{}_a\right)\to\begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-t_a^{\pm1}\\ b & 0 & t_a^{\pm1}\end{array}\qquad T_1\sqcup T_2\to\begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array}\ \xrightarrow[t_a,t_b\to t_c]{m_c^{ab}}\ \left(\begin{array}{c|cc}(1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

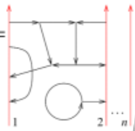
(Roland: "add to A the product of column b and row a, divide by $(1-A_{ab})$, delete column b and row a".)

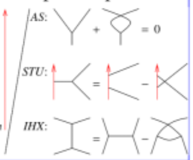
For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

(There are also formulas for strand doubling and strand reversal).

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z:\left\{\begin{array}{c}S\text{-component}\\ (v/b\text{-})\text{tangles}\end{array}\right\}\to\mathcal{A}_S^v:=$$

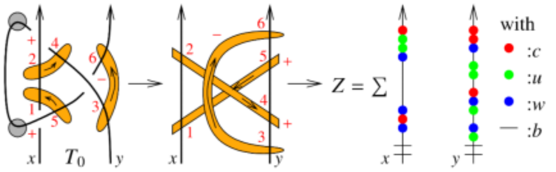





Etingof Kazhdan Haviv Enriquez Ševera

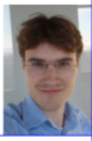
even better if A is Hopf.

Algebras and Invariants. Given any unital algebra A (typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



Good News. In theory, enough to know R, the cuaps, and stitching/multiplication $m^{ij}_k\colon A_i \otimes A_j \to A_k$.
Problem. Extract information out of Z.
Textbook Solution. Use representation theory … works, slowly.
Today's Solution. (with van der Veen) For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$$\left\{\begin{matrix}\text{ordered perturbed} \\ \text{Gaussian formulas}\end{matrix}\right\} \to \hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$


van der Veen

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and with $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w\rangle \supset 0$. (note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon = 0$, or $\mathbb{Q}\langle b, c, u, w\rangle/([b, \cdot] = 0, [c, u] = u, [c, w] = -w, [u, w] = b$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{b}^* \rtimes \mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c, w\rangle$ and (b, u) is the dual basis of (c, w). It is even more valuable than $\mathfrak{g}_1$, but topology already got by other means almost everything $\mathfrak{g}_0$ has to give.

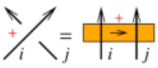
How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w\rangle/[w, c] = w$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (c, w) \mapsto (0, c \wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get $\mathfrak{g}_0$. At $k = 1$, get $[w, c] = w$, $[w, b'] = -\epsilon w$, $[c, u] = u$, $[b', u] = -\epsilon u$, $[b', c] = 0$, and $[u, w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of $poly$ in $S(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,

$$\mathbb{O}\left(c_1^3 u_1 c_2 e^{u_3} w_3^9 | x\colon w_3 c_1,\ y\colon u_1 u_3 c_2\right) = w^9 c^3 \otimes u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$$= R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$$
solves YB/R3.

Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \mathbb{O}\left(\exp\left(b_i c_j + \frac{e^{b_i}-1}{b_i} u_i w_j\right) | i\colon u_i,\ j\colon c_j w_j\right)$

$$= \sum_{m,n} \frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!} u^n \otimes c^m w^n.$$

Example. $Z(T_0) =$

$$\mathbb{O}\Big(\exp\left(b_5 c_1 + \tfrac{e^{b_5}-1}{b_5} u_5 w_1 + b_2 c_4 + \tfrac{e^{b_2}-1}{b_2} u_2 w_4 - b_3 c_6 + \tfrac{e^{-b_3}-1}{b_3} u_3 w_6\right) \Big| \\ x\colon c_1 w_1 u_2,\ y\colon u_3 c_4 w_4 u_5 c_6 w_6 \Big) = \mathbb{O}\left(? | x\colon c_x u_x w_x,\ y\colon c_y u_y w_y\right)$$

"cuw form"

Goal. Write ? as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S := \mathbb{Q}(b_i, e^{b_i})$, and $\omega \in R_S$.

The Big $\mathfrak{g}_0$ Lemma. Under $[c, u] = u$, $[c, w] = -w$, and $[u, w] = b$:

~~1. $N_k^{c_i c_j} := \mathbb{O}(\zeta | c_i c_j) \overset{\to}{=} \mathbb{O}(\zeta/(c_i, c_j \to c_k) | c_k)$~~
~~(Meaning, $N_k^{c_i c_j}\colon \zeta \mapsto (\zeta/(c_i, c_j \to c_k))$ and the diagram commutes. Trivial, also for b, u, w.)~~

$$S \xrightarrow{N_k^{c_i c_j}} S, \quad \mathbb{O}(-|c_i c_j)\colon S \to \mathcal{U}, \quad \mathbb{O}(-|c_k)\colon S \to \mathcal{U}$$

2a. $N^{uc} := \mathbb{O}(e^{\gamma c + \beta u} | uc) \overset{\to}{=} \mathbb{O}(e^{\gamma c + e^{-\gamma}\beta u} | cu)$ (means $e^{\beta u} e^{\gamma c} = e^{\gamma c} e^{e^{-\gamma}\beta u}$
2b. $N^{wc} := \mathbb{O}(e^{\gamma c + \alpha w} | wc) \overset{\to}{=} \mathbb{O}(e^{\gamma c + e^{\gamma}\alpha w} | cw)$ … in the $\{ax + b\}$ group)
3. $\mathbb{O}(e^{\alpha w + \beta u} | wu) = \mathbb{O}(e^{-b\alpha\beta + \alpha w + \beta u} | uw)$ (the Weyl relations)
4. $\mathbb{O}(e^{\delta uw} | wu) e^{\beta u} = e^{\nu\beta u} \mathbb{O}(e^{\delta uw} | wu)$, with $\nu = (1 + b\delta)^{-1}$
(a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)
5. $\mathbb{O}(e^{\delta uw} | wu) = \mathbb{O}(\nu e^{\nu\delta uw} | uw)$ (same techniques)
6. $N^{wu} := \mathbb{O}(e^{\beta u + \alpha w + \delta uw} | wu) \overset{\to}{=} \mathbb{O}(\nu e^{-b\nu\alpha\beta + \nu\alpha w + \nu\beta u + \nu\delta uw} | uw)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition

$$c_i u_i \overline{w_i c_j} u_j w_j \xrightarrow{N_k^{w_i c_j}} c_i \overline{u_i c_k} \overline{w_k u_j} w_j \xrightarrow{N_k^{u_i c_k} /\!/ N_k^{w_k u_j}} \overline{c_i c_k} \overline{u_k u_k} \overline{w_k w_j} \xrightarrow{N_k^{c_i c_k} /\!/ -- /\!/ N_k^{w_k w_j}} c_k u_k w_k$$

1-Smidgen Invariants. Much is the same:
The Big $\mathfrak{g}_1$ Lemma. Parts 2 and 6 are the same, yet
5. $\mathbb{O}\left(e^{\alpha w + \beta u + \delta uw} | wu\right) = \mathbb{O}\left(\nu(1 + \epsilon\nu\Lambda) e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} | cuw\right)$
Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:

$$\begin{aligned}\Lambda = & -b\nu\left(\nu^2\alpha^2\beta^2 + 4\delta\nu\alpha\beta + 2\delta^2\right)/2 - \delta\nu^3(3b\delta + 2)\beta^2 u^2/2 \\ & - b\delta^4\nu^3 u^2 w^2/2 - \delta^2\nu^3(2b\delta + 1)\beta u^2 w \\ & - \nu^2(2b\delta + 1)(\nu\alpha\beta + 2\delta)\beta u - 2b\delta^2\nu^2(\nu\alpha\beta + \delta)uw \\ & + \delta\nu^3(b\delta + 2)\alpha^2 w^2/2 + 2(\nu\alpha\beta + \delta)c + 2\delta\nu\beta cu + 2\delta^2\nu cuw \\ & + 2\delta\nu\alpha cw + \delta^2\nu^3\alpha uw^2 + \nu^2(\nu\alpha\beta + 2\delta)\alpha w.\end{aligned}$$

Proof. A brutal hell.
Problem. We now need to normal-order perturbed Gaussians!
Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c, u) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + \beta u} | uc) = \mathbb{O}(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + e^{-\gamma}\beta u} | cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u, w) e^{\alpha w + \beta u + \delta uw} | wu\right) = \mathbb{O}\left(\epsilon P(\partial_\beta, \partial_\alpha) \nu e^{\nu(-b\alpha\beta + \alpha w + \beta u + \delta uw)} | cuw\right)$$

Note. Strand stitching requires a tiny extra step.
Finally, the values of the generators ⤯, ⤰, $\overrightarrow{n}$, $\overleftarrow{n}$, $\underrightarrow{u}$, and $\underleftarrow{u}$, are set by brutally solving many equations, non-uniquely.

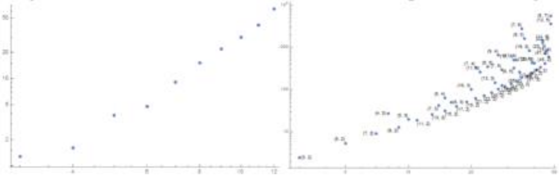
Pragmatic Simplifications. Get rid of $\zeta = (e^b - 1)/b$ factors by rescaling $u \to \bar{u} = \zeta u$. Complement this with $\beta \to \bar{\beta} = \zeta^{-1}\beta$, $\delta \to \bar{\delta} = \zeta^{-1}\delta$, $\epsilon \to \bar{\epsilon} = \zeta^{-1}\epsilon$. Simplify further by naming $e^b \to t$; e.g., $\nu \to \bar{\nu} = (1 + (t-1)\delta)^{-1}$. Get confused by renaming $(\bar{u}, \bar{\beta}, \bar{\delta}, \bar{\nu}) \to (u, \beta, \delta, \nu)$, and more confused by working with $\mu = \nu^{-1}$ and $\mathbb{E}(\omega, L, Q, P) := \omega^{-1}(1 + \epsilon\omega^{-4}P)e^{L + \omega^{-1}Q}$, where $\omega \in R := \mathbb{Q}(t_k)$, $L = \sum l_{ij} b_i c_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij} u_i w_j$ with $q_{ij} \in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

Rough complexity estimate, after $t_k \to t$. n: xing number; w: width, maybe $\sim \sqrt{n}$.

$$\underbrace{n}_{A} \overbrace{\sum_{d=0}^{4} \underbrace{w^{4-d}}_{E} \underbrace{w^d}_{F} \underbrace{n^2}_{G}}^{B} = n^3 w^4 \in [n^5, n^7]$$

A: go over stitchings in order. B: multiplication ops per $N^{u_i w_j}$. d: deg of u_i, w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Experimental Analysis (ωεβ/Exp). Log-log plot of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial a, there is a polynomial e ~~(the "essence" of P) such that~~ s.t.

$$P = a^2\left((t-2+t^{-1})e + taa'\left(\frac{(4+t-t^2)(uw+(t-1)c)}{2(t-1)} - 1\right)\right).$$

Furthermore, a and e are symmetric under $t \to t^{-1}$, so let a_+ and e_+ be their "positive parts", so e.g., $e(t) = e_+(t) + e_+(t^{-1}) - e_+(0)$.
Power. On the 250 knots with at most 10 crossings, the pair (a, e) attains 250 distinct values, while (Khovanov, HOMFLYPT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).
Genus. Up to 12 crossings, always $\deg e_+ \leq 2g - 1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of e (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-crossing Alexander failures it does give the right answer.

Demo Programs for 0-Co.

ωεβ/Demo

The R-matrices

```
R^+_{0,i_,j_} := E[b_i c_j + b_i^-1 (e^(b_i) - 1) u_i w_j];
R^-_{0,i_,j_} := E[-b_i c_j + b_i^-1 (e^(-b_i) - 1) u_i w_j];
```

Utilities

```
CF[ω_. E[Q_]] := Simplify[ω] E[Simplify[Q]];
E /: E[Q1_] E[Q2_] := CF@E[Q1 + Q2];
ω1_. E[Q1_] ≡ ω2_. E[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

Normal Ordering Operators

```
N_{u_i_ c_j_ → k_}[ω_. E[Q_]] := CF[
   ω E[e^-γ β u_k + γ c_k + (Q /. c_j | u_i → 0)] /. {γ → ∂_{c_j} Q, β → ∂_{u_i} Q}];
N_{w_i_ c_j_ → k_}[ω_. E[Q_]] := CF[
   ω E[e^γ α w_k + γ c_k + (Q /. c_j | w_i → 0)] /. {γ → ∂_{c_j} Q, α → ∂_{w_i} Q}];
N_{w_i_ u_j_ → k_}[ω_. E[Q_]] := CF[
   ν ω E[-b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k + (Q /. w_i | u_j → 0)] /.
    ν → (1 + b_k δ)^-1 /.
   {α → ∂_{w_i} Q /. u_j → 0, β → ∂_{u_j} Q /. w_i → 0, δ → ∂_{w_i,u_j} Q}];
```

Stitching

```
m_{i_,j_→k_}[ω_. E[Q_]] := CF[Module[{x},
   (ω E[Q] /. b_{i|j} → b_k // N_{w_i c_j→x} // N_{u_i c_x→x} // N_{w_x u_j→x}) /.
    {c_i → c_k, w_j → w_k, y_x :→ y_k}]]
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Some calculations for T_0

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T0 = R^+_{0,5,1} R^+_{0,2,4} R^-_{0,3,6}
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$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5}) u_5 w_1}{b_5} + \frac{(-1+e^{b_2}) u_2 w_4}{b_2} + \frac{(-1+e^{-b_3}) u_3 w_6}{b_3}\right]$$

```
T0 // m_{1,2→1} // m_{3,4→3} // m_{3,5→3} // m_{3,6→3}
```

$$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{(-1+e^{b_1})(-1+e^{b_3}) u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_1} - \frac{e^{-b_3}(-1+e^{b_1})(b_3 u_1 - e^{b_3}(-1+e^{b_3}) b_1 u_3) w_3}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_1 b_3} + \frac{e^{-b_1}(-1+e^{b_3}) u_3 (-e^{b_1+b_3} w_1 + (e^{b_1}+e^{b_3}-e^{b_1+b_3}) w_3)}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3}) b_3}\Big]$$

Verifying meta-associativity

```
Q0 = E[Sum[f_i c_i, {i, 3}] + Sum[f_{i,j} u_i w_j, {i, 3}, {j, 3}]]
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$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

```
(Q0 // m_{1,2→1} // m_{1,3→1}) ≡ (Q0 // m_{2,3→2} // m_{1,2→1})
True
```

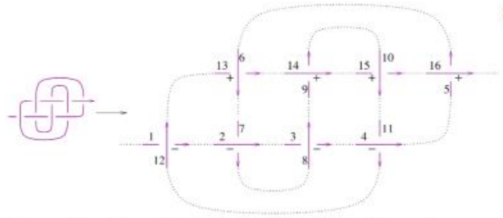
Testing R3

```
t1 = R^+_{0,1,2} R^+_{0,3,4} R^+_{0,5,6} // m_{3,5→x} // m_{1,6→y} // m_{2,4→z}
```

$$\mathbb{E}\left[b_x (c_y + c_z) + \frac{(-1+e^{b_x}) u_x (w_y + w_z)}{b_x} + \frac{b_y^2 c_z + (-1+e^{b_y}) u_y w_z}{b_y}\right]$$

```
t1 ≡ (R^+_{0,1,2} R^+_{0,3,4} R^+_{0,5,6} // m_{1,3→x} // m_{2,5→y} // m_{4,6→z})
True
```

8_{17}




```
z1 = R^-_{0,12,1} R^-_{0,2,7} R^-_{0,8,3} R^-_{0,4,11} R^+_{0,16,5} R^+_{0,6,13} R^+_{0,14,9} R^+_{0,10,15};
Do[z1 = (z1 // m_{1,n→1}) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$$

Demo Programs for 1-Co.

ωεβ/Demo

The Λόγος

```
Λ[k_] := (1 - t_k) (α^2 β^2 + 4 α β δ μ + 2 δ^2 μ^2) / 2 + 2 μ^2 (α β + δ μ) c_k -
   β (2 μ - 1) (α β + 2 δ μ) u_k + 2 β δ μ^2 c_k u_k - β^2 δ (3 μ - 1) u_k^2 / 2 +
   α (α β + 2 δ μ) w_k + 2 α δ μ^2 c_k w_k - 2 (t_k - 1) δ^2 (α β + δ μ) u_k w_k +
   2 δ^2 μ^2 c_k u_k w_k - β δ^2 (2 μ - 1) u_k^2 w_k + α^2 δ (1 + μ) w_k^2 / 2 +
   α δ^2 u_k w_k^2 - (t_k - 1) δ^4 u_k^2 w_k^2 / 2;
```

The Generators

```
R^+_{i_,j_} := E[1, b_i c_j, u_i w_j,
   -c_i (t_i - 1)^2 / 2 - c_i^2 (t_i - 1)^2 / 2 + c_i c_j (t_j^2 - t_i - 2) / 2 -
    c_j u_i w_i / 2 + c_i (1 - t_i) u_i w_i - u_i^2 w_i^2 / 2 + u_i w_j + c_j t_i u_i w_j / 2 +
    c_i (t_i - 2) t_i u_i w_j + c_i (1 + t_j) u_j w_j / 2 + (t_i - 1) u_i^2 w_i w_j -
    (t_i - 2) t_i u_i^2 w_j^2 / 2];
R^-_{i_,j_} := E[1, -b_i c_j, -t_i^-1 u_i w_j,
   c_i (t_i - 1)^2 / 2 + c_i^2 (t_i - 1)^2 / 2 + c_i c_j (2 + t_i - t_j^2) / 2 +
    c_j u_i w_i / 2 + c_i (t_i - 1) u_i w_i + u_i^2 w_i^2 / 2 + (1 - t_i^-1) u_i w_j / 2 +
    c_i (2 t_i - 5 + 3 t_i^-1) u_i w_j / 2 + c_j (t_i^-1 + 1 - t_i^-1 t_j^2) u_i w_j / 2 -
    c_i (t_j + 1) u_j w_j / 2 + (2 - 3 t_i^-1) u_i^2 w_i w_j / 2 +
    (1 + 2 t_i^-2 - 3 t_i^-1) u_i^2 w_j^2 / 2 - t_i^-1 (1 + t_j) u_i u_j w_j^2 / 2];
ur_i_ := E[t_i^(-1/4), 0, 0, c_i t_i / 4 + u_i w_i / 8];
nr_i_ := E[t_i^(1/4), 0, 0, -c_i t_i^3 / 4 - t_i^2 u_i w_i / 8];
ul_i_ := E[t_i^(1/4), 0, 0, c_i t_i (4 + t_i) / 4 - t_i^2 u_i w_i / 8];
nl_i_ := E[t_i^(-1/4), 0, 0, -c_i (1 + 4 t_i^-1) / 4 + u_i w_i / 8];
```

Differential Polynomials

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := (* means P[∂_α,∂_β][f] *)
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

Utilities

```
CF[𝔼[ω_, L_, Q_, P_]] := Expand /@ Together /@
    𝔼[ω /. b_l_ ⧴ Log[t_l], L, Q /. b_l_ ⧴ Log[t_l],
     P /. b_l_ ⧴ Log[t_l]];
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

Normal Ordering Operators

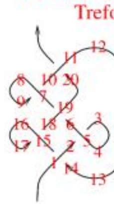
```
N_{u_i_ c_j_→k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^-γ β u_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω e^-γ β u_k + (Q /. u_i → 0),
      e^-q DP_{c_j→D_γ, u_i→D_β}[P][e^q]] /. {γ → ∂_{c_j} L, β → ω^-1 ∂_{u_i} Q}]];
N_{w_i_ c_j_→k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^γ α w_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω e^γ α w_k + (Q /. w_i → 0),
      e^-q DP_{c_j→D_γ, w_i→D_α}[P][e^q]] /. {γ → ∂_{c_j} L, α → ω^-1 ∂_{w_i} Q}]];
N_{w_i_ u_j_→k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{q = (1 - t_k) μ^-1 α β + μ^-1 β u_k + μ^-1 δ u_k w_k + μ^-1 α w_k}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. w_i | u_j → 0),
       μ^4 e^-q DP_{w_i→D_α, u_j→D_β}[P][e^q] + ω^4 Λ[k]] /.
      μ → 1 + (t_k - 1) δ /.
     {α → ω^-1 (∂_{w_i} Q /. u_j → 0), β → ω^-1 (∂_{u_j} Q /. w_i → 0),
      δ → ω^-1 ∂_{w_i,u_j} Q}]];
```

Stitching

```
m_{i_,j_→k_}[Z_] := Module[{x, y, z},
  Z // N_{w_i c_j→x} // N_{w_x u_j→y} // ReplaceAll[{c_{x|y} → c_x, w_j → w_y}] //
    N_{u_i c_x→x} // ReplaceAll[z_{_i|j|x|y} → z_k] // CF]
```

The 0-Framed Trefoil

```
z2 = R⁺_{1,14} R⁻_{5,2} nr_3 ul_4 R⁺_{19,6} R⁻_{7,10} nl_8 ur_9 R⁺_{11,20}
   nr_12 ul_13 R⁻_{15,18} nl_16 ur_17;
(Do[z2 = z2 // m_{1,k→1}, {k, 2, 20}];
 z2 = z2 /. a_1 ⧴ a)
```



$$\mathrm{E}\Big[-1+\tfrac{1}{t}+t,\ 0,\ 0,\ -16+\tfrac{9c}{2}-\tfrac{2c}{t^4}+\tfrac{1}{t^3}+\tfrac{11c}{2t^3}-\tfrac{4}{t^2}-\tfrac{8c}{t^2}+\tfrac{10}{t}+\tfrac{4c}{t}+18t-10ct-14t^2+8ct^2+7t^3-\tfrac{3ct^3}{2}-2t^4-2ct^4+2ct^5-\tfrac{ct^6}{2}-4uw+\tfrac{2uw}{t^4}-\tfrac{7uw}{2t^3}+\tfrac{9uw}{2t^2}+\tfrac{uw}{2t}+6tuw-2t^2uw-\tfrac{1}{2}t^3uw+\tfrac{3}{2}t^4uw-\tfrac{1}{2}t^5uw\Big]$$

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Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

Disclaimer. This is all quite new. The overall picture is correct, yet some details might be somewhat off. Many pieces are certainly not in their final form yet. **Help Needed!**

~~Propaganda~~

diagram	n^t_k Alexander a_+ / *P-essence* e_+	genus / ribbon / unknotting number / amphicheiral	diagram	n^t_k Alexander a_+ / *P-essence* e_+	genus / ribbon / unknotting number / amphicheiral

at least two data lines.