Indiana-1611 handout on 161025

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A Poly-Time Knot Polynomial Via Solvable Approximation

Abstract. Rozansky [Ro2] and Overbay [Ov] described a spectacular knot polynomial that failed to attract the attention it deserved as the first poly-time-computable knot polynomial since Alexander's [Al, 1928] and (in my opinion) as the second most likely knot polynomial (after Alexander's) to carry topological information. With Roland van der Veen, I will explain how to compute the Rozansky polynomial using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.

Roz

over

MMR box

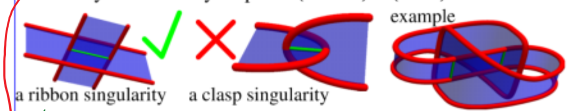
Rozansky

genus box

why "spectacular"?

~~Paradise!~~ Foremost reason: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Secondary reason: may disprove {ribbon} = {slice}:



a ribbon singularity a clasp singularity

re-expand

A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

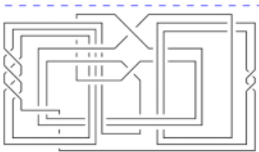
Conjecture. Some slice knots are not ribbon.

In [BN2] I list 5 criteria an invariant needs to meet to have a fair chance of detecting non-ribbons. Ours meets all 5.

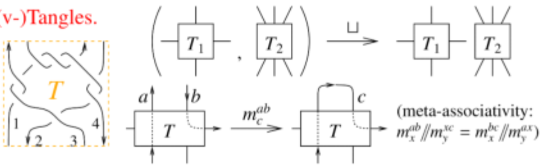
[GST]: Gompf, Scharlemann, Thompson:

$\rho_+ = -t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$

$\rho_+ = 5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$



(v-)Tangles.

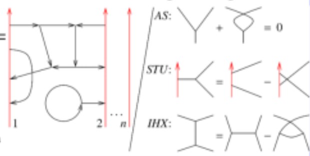


(meta-associativity: $m_x^{ab}/\!\!/m_y^{xc} = m_x^{bc}/\!\!/m_y^{ax}$)

bring back the box.

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z:\begin{Bmatrix}S\text{-component}\\(v/b\text{-})\text{tangles}\end{Bmatrix}\to\mathcal{A}_S^v:=$$



Etingof Kazhdan Haviv Enriquez Ševera

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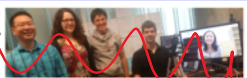
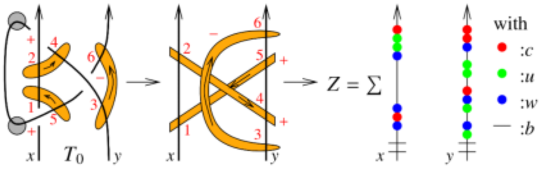
Videos of a 4-hour version of this talk are at ωεβ/LD. Videos of private seminar meetings are at ωεβ/PP.



Many thanks: Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

Algebras and Invariants. Given any unital algebra A (typically, $A\sim\hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R\in A\otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:

A: Aside: even better if A is Hopf.



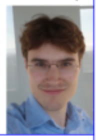
Good News. In theory, enough to know R, the cuaps, and stitching/multiplication $m_k^{ij}: A_i\otimes A_j\to A_k$.

Problem. Extract information out of Z.

Textbook Solution. Use representation theory ... works, slowly.

Today's Solution. (with van der Veen) For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$$\begin{Bmatrix}\text{ordered perturbed}\\\text{Gaussian formulas}\end{Bmatrix}\to\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$



van der Veen

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1=\langle b,c,u,w\rangle$ over the ring $R=\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with b central and with $[w,c]=w$, $[c,u]=u$, and $[u,w]=b-2\epsilon c$, with CYBE $r_{ij}=(b_i-\epsilon c_i)c_j+u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1\supset\langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset\langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle\supset 0$. (note: $\deg(b,c,u,w,\epsilon)=(1,0,1,0,1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon=0$, or $\mathbb{Q}\langle b,c,u,w\rangle/([b,\cdot]=0,[c,u]=u,[c,w]=-w,[u,w]=b$ with $r_{ij}=b_ic_j+u_iw_j$. It is $\mathfrak{b}^*\rtimes\mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c,w\rangle$ and (b,u) is the dual basis of (c,w). It is even more valuable than $\mathfrak{g}_1$, but topology already got by other means almost everything $\mathfrak{g}_0$ has to give.

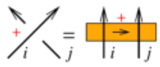
How did these arise? $sl_2=\mathfrak{b}^+\oplus\mathfrak{b}^-/\mathfrak{h}=:sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+=\langle c,w\rangle/[w,c]=w$ is a Lie bialgebra with $\delta:\mathfrak{b}^+\to\mathfrak{b}^+\otimes\mathfrak{b}^+$ by $\delta:(c,w)\mapsto(0,c\wedge w)$. Going back, $sl_2^+=\mathcal{D}(\mathfrak{b}^+)=(\mathfrak{b}^+)^*\oplus\mathfrak{b}^+=\langle b,u,c,w\rangle/\cdots$. **Idea.** Replace $\delta\to\epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1}=0)$. At $k=0$, get $\mathfrak{g}_0$. At $k=1$, get $[w,c]=w$, $[w,b']=-\epsilon w$, $[c,u]=u$, $[b',u]=-\epsilon u$, $[b',c]=0$, and $[u,w]=b'-\epsilon c$. Now note that $b'+\epsilon c$ is central, so switch to $b:=b'+\epsilon c$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly\mid specs)$ plants the variables of *poly* in $S(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\mathbb{O}\left(c_1^3u_1c_2e^{u_3}w_3^9\middle|x:w_3c_1,\ y:u_1u_3c_2\right)=w^9c^3\otimes ue^uc\in\mathcal{U}(\mathfrak{g})_x\otimes\mathcal{U}(\mathfrak{g})_y$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r=Id\in\mathfrak{b}^-\otimes\mathfrak{b}^+$ solves the CYBE $[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$$=R_{ij}=e^{r_{ij}}=e^{b_ic_j+u_iw_j}\in\mathcal{U}(\mathfrak{g}_{0,i}\oplus\mathfrak{g}_{0,j})$$ solves YB/R3.

Lemma. $R_{ij}=e^{b_ic_j+u_iw_j}=\mathbb{O}\left(\exp\left(b_ic_j+\frac{e^{b_i}-1}{b_i}u_iw_j\right)\middle|i:u_i,\ j:c_jw_j\right)$

Example. $Z(T_0)=$ $\quad=\sum_{m,n}\frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!}u^n\otimes c^mw^n$.

$$\mathbb{O}\left(\exp\left(b_5c_1+\frac{e^{b_5}-1}{b_5}u_5w_1+b_2c_4+\frac{e^{b_2}-1}{b_2}u_2w_4-b_3c_6+\frac{e^{-b_3}-1}{b_3}u_3w_6\right)\middle|\ x:c_1w_1u_2,\ y:u_3c_4w_4u_5c_6w_6\right)=\mathbb{O}\left(?\middle|x:c_xu_xw_x,\ y:c_yu_yw_y\right)$$

"cuw form"

Goal. Write ? as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S := \mathbb{Q}(b_i, e^{b_i})$, and $\omega \in R_S$.

The Big $\mathfrak{g}_0$ Lemma. Under $[c,u]=u$, $[c,w]=-w$, and $[u,w]=b$:

1. $N_k^{c_ic_j} := \mathbb{O}(\zeta|c_ic_j) \overset{\rightarrow}{=} \mathbb{O}(\zeta/(c_i,c_j\to c_k)|c_k)$

(Meaning, $N_k^{c_ic_j}: \zeta \mapsto (\zeta/(c_i,c_j\to c_k))$ and the diagram commutes. Trivial, also for b, u, w.)

$$\begin{tikzcd} S \arrow[r, "N_k^{c_ic_j}"] \arrow[dr, "\mathbb{O}(\cdot|c_ic_j)"'] & S \arrow[d, "\mathbb{O}(\cdot|c_k)"] \\ & \mathcal{U} \end{tikzcd}$$

2a. $N^{uc} := \mathbb{O}(e^{\gamma c+\beta u}|uc) \overset{\rightarrow}{=} \mathbb{O}(e^{\gamma c+e^{-\gamma}\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c} = e^{\gamma c}e^{e^{-\gamma}\beta u}$

2b. $N^{wc} := \mathbb{O}(e^{\gamma c+\alpha w}|wc) \overset{\rightarrow}{=} \mathbb{O}(e^{\gamma c+e^{\gamma}\alpha w}|cw)$ ...in the $\{ax+b\}$ group)

3. $\mathbb{O}(e^{\alpha w+\beta u}|wu) = \mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$ (the Weyl relations)

4. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u} = e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu = (1+b\delta)^{-1}$

(a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use "scatter and glow".)

5. $\mathbb{O}(e^{\delta uw}|wu) = \mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)

6. $N^{wu} := \mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu) \overset{\rightarrow}{=} \mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$

Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition

$$c_iu_i\overline{w_ic_j}\,u_jw_j \xrightarrow{N_k^{w_ic_j}} c_i\,\overline{u_ic_k}\,\overline{w_ku_j}\,w_j \xrightarrow{N_k^{u_ic_k}/\!\!/N_k^{w_ku_j}} \overline{c_ic_k}\,\overline{u_ku_k}\,\overline{w_kw_j} \xrightarrow{N_k^{c_ic_k}/\!\!/--/\!\!/N_k^{w_kw_j}} c_ku_kw_k$$

Golden again

Γ-calculus [BNS, BN1]. After re-packaging, especially setting $t_i := e^{b_i}$, an S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A \end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a : a \in S\})$:

$$\left({}_a\!\nearrow_b, {}_b\!\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm1} \\ b & 0 & t_a^{\pm1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a,t_b\to t_c]{m_c^{ab}} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta} \\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta} \end{array}\right)$$

there are also formulas for doubling & reversal.

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!

Dunfield: 1000-crossing fast.

1-Smidgen Invariants. Much is the same:

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 2 are the same, yet

6. $\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right) = \mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:

$$\begin{aligned}\Lambda = &-b\nu\left(\nu^2\alpha^2\beta^2+4\delta\nu\alpha\beta+2\delta^2\right)/2-\delta\nu^3(3b\delta+2)\beta^2u^2/2 \\ &-b\delta^4\nu^3u^2w^2/2-\delta^2\nu^3(2b\delta+1)\beta u^2w \\ &-\nu^2(2b\delta+1)(\nu\alpha\beta+2\delta)\beta u-2b\delta^2\nu^2(\nu\alpha\beta+\delta)uw \\ &+\delta\nu^3(b\delta+2)\alpha^2w^2/2+2(\nu\alpha\beta+\delta)c+2\delta\nu\beta cu+2\delta^2\nu cuw \\ &+2\delta\nu\alpha cw+\delta^2\nu^3\alpha uw^2+\nu^2(\nu\alpha\beta+2\delta)\alpha w.\end{aligned}$$

Proof. A brutal hell.

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c,u)e^{\gamma c+\beta u}|uc) = \mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+\beta u}|uc) = \mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+e^{-\gamma}\beta u}|cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u,w)e^{\alpha w+\beta u+\delta uw}|wu\right) = \mathbb{O}\left(\epsilon P(\partial_\beta,\partial_\alpha)\nu e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$$

Note. Strand stitching requires a tiny extra step.

Finally, the values of the generators $\nearrow$, $\nwarrow$, $\overrightarrow{n}$, $\overleftarrow{n}$, $\underset{\rightarrow}{u}$, and $\underset{\leftarrow}{u}$, are set by brutally solving many equations, non-uniquely.

Pragmatic Simplifications. Get rid of $\zeta = (e^b-1)/b$ factors by rescaling $u \to \bar{u} = \zeta u$. Complement this with $\beta \to \bar{\beta} = \zeta^{-1}\beta$, $\delta \to \bar{\delta} = \zeta^{-1}\delta$, $\epsilon \to \bar{\epsilon} = \zeta^{-1}\epsilon$. Simplify further by naming $e^b \to t$; e.g., $\nu \to \bar{\nu} = (1+(t-1)\delta)^{-1}$. Get confused by renaming $(\bar{u},\bar{\beta},\bar{\delta},\bar{\nu}) \to (u,\beta,\delta,\nu)$, and more confused by working with $\mu = \nu^{-1}$ and $\mathbb{E}(\omega,L,Q,P) := \omega^{-1}(1+\epsilon\omega^{-4}P)e^{L+\omega^{-1}Q}$, where $\omega \in R := \mathbb{Q}(t_k)$, $L = \sum l_{ij}b_ic_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij}u_iw_j$ with $q_{ij} \in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

Rough complexity estimate, after $t_k \to t$. n: xing number; w: width, maybe $\sim\sqrt{n}$.

$$\underbrace{n}_{A}\overbrace{\sum_{d=0}^{4}\underbrace{w^{4-d}}_{E}\underbrace{w^d}_{F}}^{B}\underbrace{n^2}_{G} = n^3w^4 \in [n^5, n^7]$$

A: go over stitchings in order. B: multiplication ops per $N^{u_iw_j}$. d: deg of u_i, w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

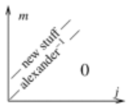
Expectation. Our invariant is the "1-higher diagonal" in the MMR expansion of the coloured Jones polynomial J_λ.

Melvin, Morton, Garoufalidis

Theorem ([BNG], conjectured [MM], ~~elucidated [Ro1]~~). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of $sl(2)$. Writing

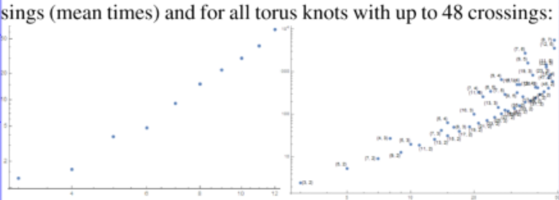
$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar} = \sum_{j,m\geq0} a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar) = 1$.



include specifics about higher lines?

Experimental Analysis (ωεβ/Exp). Log-log plot of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial a, there is a polynomial e (the "essence" of P) such that

global e→p

$$P = a^2\left((t-2+t^{-1})e + taa'\left(\frac{(4+t-t^2)(uw+(t-1)c)}{2(t-1)}-1\right)\right).$$

Furthermore, a and e are symmetric under $t \to t^{-1}$, so let a_+ and e_+ be their "positive parts", so e.g., $e(t) = e_+(t)+e_+(t^{-1})-e_+(0)$.

Power. On the 250 knots with at most 10 crossings, the pair (a,e) attains 250 distinct values, while (Khovanov, HOMFLYPT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 crossings, always $\deg e_+ \leq 2g-1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of e (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-crossing Alexander failures it does give the right answer.

Demo Programs for 0-Co.

ωεβ/Demo

The R-matrices

$$R^+_{\theta,i\_,j\_} := \mathbb{E}\left[b_i c_j + b_i^{-1}\left(e^{b_i}-1\right) u_i w_j\right];$$
$$R^-_{\theta,i\_,j\_} := \mathbb{E}\left[-b_i c_j + b_i^{-1}\left(e^{-b_i}-1\right) u_i w_j\right];$$

Utilities

```
CF[ω_. 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

Normal Ordering Operators

$$N_{u_{i\_}c_{j\_}\to k\_}[\omega\_.\,\mathbb{E}[Q\_]] := \mathrm{CF}\big[\omega\,\mathbb{E}[e^{-\gamma}\beta u_k + \gamma c_k + (Q\ /.\ c_j \mid u_i \to 0)]\ /.\ \{\gamma\to\partial_{c_j}Q,\ \beta\to\partial_{u_i}Q\}\big];$$

$$N_{w_{i\_}c_{j\_}\to k\_}[\omega\_.\,\mathbb{E}[Q\_]] := \mathrm{CF}\big[\omega\,\mathbb{E}[e^{\gamma}\alpha w_k + \gamma c_k + (Q\ /.\ c_j \mid w_i \to 0)]\ /.\ \{\gamma\to\partial_{c_j}Q,\ \alpha\to\partial_{w_i}Q\}\big];$$

$$N_{w_{i\_}u_{j\_}\to k\_}[\omega\_.\,\mathbb{E}[Q\_]] := \mathrm{CF}\big[\nu\,\omega\,\mathbb{E}[-b_k\nu\alpha\beta + \nu\beta u_k + \nu\delta u_k w_k + \nu\alpha w_k + (Q\ /.\ w_i \mid u_j \to 0)]\ /.\ \nu\to(1+b_k\delta)^{-1}\ /.\ \{\alpha\to\partial_{w_i}Q\ /.\ u_j\to 0,\ \beta\to\partial_{u_j}Q\ /.\ w_i\to 0,\ \delta\to\partial_{w_i,u_j}Q\}\big];$$

Stitching

$$m_{i\_,j\_\to k\_}[\omega\_.\,\mathbb{E}[Q\_]] := \mathrm{CF}\big[\mathrm{Module}[\{x\},\ (\omega\,\mathbb{E}[Q]\ /.\ b_{i|j}\to b_k\ //\ N_{w_i c_j\to x}\ //\ N_{u_i c_x\to x}\ //\ N_{w_x u_j\to x})\ /.\ \{c_i\to c_k,\ w_j\to w_k,\ y\_{}_x :\!\to y_k\}]\big]$$

Some calculations for T_0

$$T_{\theta,\theta} = R^+_{\theta,5,1}\,R^+_{\theta,2,4}\,R^-_{\theta,3,6}$$

$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5})u_5 w_1}{b_5} + \frac{(-1+e^{b_2})u_2 w_4}{b_2} + \frac{(-1+e^{-b_3})u_3 w_6}{b_3}\right]$$

$$T_{\theta,1} = T_{\theta,\theta}\ //\ N_{u_3 c_4\to 4}$$

$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5})u_5 w_1}{b_5} + \frac{(-1+e^{b_2})u_2 w_4}{b_2} + \frac{e^{-b_2}(-1+e^{-b_3})u_4 w_6}{b_3}\right]$$

$$T_{\theta,2} = T_{\theta,1}\ //\ N_{w_4 u_5\to 4}$$

$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 + \frac{(-1+e^{b_5})\left(-(-1+e^{b_2})b_4 u_2 - b_2 u_4\right)w_1}{b_2 b_5} + \frac{(-1+e^{b_2})u_2 w_4}{b_2} - \frac{b_3^2 c_6 + e^{-b_2-b_3}(-1+e^{b_3})u_4 w_6}{b_3}\right]$$

$$T_{\theta,2}\ //\ N_{w_1 u_2\to 1}$$

$$\frac{1}{1-\frac{(-1+e^{b_2})(-1+e^{b_5})b_1 b_4}{b_2 b_5}}\,\mathbb{E}\Big[\frac{1}{b_3\left((-1+e^{b_2})(-1+e^{b_5})b_1 b_4 - b_2 b_5\right)}\Big(b_3 b_5\left((-1+e^{b_2})(-1+e^{b_5})b_1 b_4 - b_2 b_5\right)c_1 + b_2 b_3\left((-1+e^{b_2})(-1+e^{b_5})b_1 b_4 - b_2 b_5\right)c_4 + (-1+e^{b_2})(-1+e^{b_5})b_3 b_4 u_1 w_1 - (-1+e^{b_5})b_2 b_3 u_4 w_1 - (-1+e^{b_2})b_3 b_5 u_1 w_4 + (-1+e^{b_2})(-1+e^{b_5})b_1 b_3 u_4 w_4 - \left((-1+e^{b_2})(-1+e^{b_5})b_1 b_4 - b_2 b_5\right)\left(b_3^2 c_6 + e^{-b_2-b_3}(-1+e^{b_3})u_4 w_6\right)\Big)\Big]$$

$$T_{\theta,\theta}\ //\ m_{1,2\to 1}\ //\ m_{3,4\to 3}\ //\ m_{3,5\to 3}\ //\ m_{3,6\to 3}$$

$$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\,\mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{(-1+e^{b_1})(-1+e^{b_3})u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})b_1} - \frac{e^{-b_3}(-1+e^{b_1})\left(b_3 u_1 - e^{b_3}(-1+e^{b_3})b_1 u_3\right)w_3}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})b_1 b_3} + \frac{e^{-b_1}(-1+e^{b_3})u_3\left(-e^{b_1+b_3}w_1 + (e^{b_1}+e^{b_3}-e^{b_1+b_3})w_3\right)}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})b_3}\Big]$$

Verifying meta-associativity

```
Q0 = 𝔼[Sum[f_i c_i, {i, 3}] + Sum[f_{i,j} u_i w_j, {i, 3}, {j, 3}]]
```

$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

$$(Q0\ //\ m_{1,2\to 1}\ //\ m_{1,3\to 1}) \equiv (Q0\ //\ m_{2,3\to 2}\ //\ m_{1,2\to 1})$$

True

Testing R3

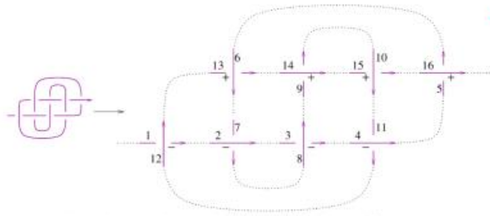
$$t1 = R^+_{\theta,1,2}\,R^+_{\theta,3,4}\,R^+_{\theta,5,6}\ //\ m_{3,5\to x}\ //\ m_{1,6\to y}\ //\ m_{2,4\to z}$$

$$\mathbb{E}\left[b_x(c_y + c_z) + \frac{(-1+e^{b_x})u_x(w_y+w_z)}{b_x} + \frac{b_y^2 c_z + (-1+e^{b_y})u_y w_z}{b_y}\right]$$

$$t1 \equiv (R^+_{\theta,1,2}\,R^+_{\theta,3,4}\,R^+_{\theta,5,6}\ //\ m_{1,3\to x}\ //\ m_{2,5\to y}\ //\ m_{4,6\to z})$$

True

8_{17}



$$z1 = R^-_{\theta,12,1}\,R^-_{\theta,2,7}\,R^-_{\theta,8,3}\,R^-_{\theta,4,11}\,R^+_{\theta,16,5}\,R^+_{\theta,6,13}\,R^+_{\theta,14,9}\,R^+_{\theta,10,15};$$

$$\mathrm{Do}[z1 = (z1\ //\ m_{1,n\to 1})\ /.\ b\_ \to b,\ \{n, 2, 16\}];$$

```
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$$

Demo Programs for 1-Co.

ωεβ/Demo

$$\Lambda[k\_] := (1-t_k)\left(\alpha^2\beta^2 + 4\alpha\beta\delta\mu + 2\delta^2\mu^2\right)/2 + 2\mu^2(\alpha\beta+\delta\mu)c_k - \beta(2\mu-1)(\alpha\beta+2\delta\mu)u_k + 2\beta\delta\mu^2 c_k u_k - \beta^2\delta(3\mu-1)u_k^2/2 + \alpha(\alpha\beta+2\delta\mu)w_k + 2\alpha\delta\mu^2 c_k w_k - 2(t_k-1)\delta^2(\alpha\beta+\delta\mu)u_k w_k + 2\delta^2\mu^2 c_k u_k w_k - \beta\delta^2(2\mu-1)u_k^2 w_k + \alpha^2\delta(1+\mu)w_k^2/2 + \alpha\delta^2 u_k w_k^2 - (t_k-1)\delta^4 u_k^2 w_k^2/2;$$

The Λόγος

Differential Polynomials

$$\mathrm{DP}_{x\_\to D_{\alpha\_},\,y\_\to D_{\beta\_}}[P\_][f\_] := \text{(* means } P[\partial_\alpha,\partial_\beta][f] \text{ *)}\ \mathrm{Total}[\mathrm{CoefficientRules}[P, \{x, y\}]\ /.\ (\{m\_, n\_\}\to c\_) :\!\to c\,\mathrm{D}[f, \{\alpha, m\}, \{\beta, n\}]]$$

Utilities

$$\mathrm{CF}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathrm{Expand}\ /@\ \mathrm{Together}\ /@\ \mathbb{E}[\omega\ /.\ b_{l\_} :\!\to \mathrm{Log}[t_l],\ L,\ Q\ /.\ b_{l\_} :\!\to \mathrm{Log}[t_l],\ P\ /.\ b_{l\_} :\!\to \mathrm{Log}[t_l]];$$

$$\mathbb{E}\ /:\ \mathbb{E}[\omega1\_, L1\_, Q1\_, P1\_]\,\mathbb{E}[\omega2\_, L2\_, Q2\_, P2\_] := \mathrm{CF}@\mathbb{E}\left[\omega1\,\omega2,\ L1+L2,\ \omega2\,Q1+\omega1\,Q2,\ \omega2^4 P1 + \omega1^4 P2\right];$$

Normal Ordering Operators

$$N_{u_{i\_}c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathrm{With}\big[\{q = e^{-\gamma}\beta u_k + \gamma c_k\},\ \mathrm{CF}\big[\mathbb{E}\big[\omega,\ \gamma c_k + (L\ /.\ c_j\to 0),\ \omega e^{-\gamma}\beta u_k + (Q\ /.\ u_i\to 0),\ e^{-q}\,\mathrm{DP}_{c_j\to D_\gamma,\,u_i\to D_\beta}[P][e^q]\big]\ /.\ \{\gamma\to\partial_{c_j}L,\ \beta\to\omega^{-1}\partial_{u_i}Q\}\big]\big];$$

$$N_{w_{i\_}c_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathrm{With}\big[\{q = e^{\gamma}\alpha w_k + \gamma c_k\},\ \mathrm{CF}\big[\mathbb{E}\big[\omega,\ \gamma c_k + (L\ /.\ c_j\to 0),\ \omega e^{\gamma}\alpha w_k + (Q\ /.\ w_i\to 0),\ e^{-q}\,\mathrm{DP}_{c_j\to D_\gamma,\,w_i\to D_\alpha}[P][e^q]\big]\ /.\ \{\gamma\to\partial_{c_j}L,\ \alpha\to\omega^{-1}\partial_{w_i}Q\}\big]\big];$$

$$N_{w_{i\_}u_{j\_}\to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathrm{With}\big[\{q = (1-t_k)\mu^{-1}\alpha\beta + \mu^{-1}\beta u_k + \mu^{-1}\delta u_k w_k + \mu^{-1}\alpha w_k\},\ \mathrm{CF}\big[\mathbb{E}\big[\mu\omega,\ L,\ \mu\omega q + \mu(Q\ /.\ w_i \mid u_j\to 0),\ \mu^4 e^{-q}\,\mathrm{DP}_{w_i\to D_\alpha,\,u_j\to D_\beta}[P][e^q] + \omega^4\Lambda[k]\big]\ /.\ \mu\to 1+(t_k-1)\delta\ /.\ \{\alpha\to\omega^{-1}(\partial_{w_i}Q\ /.\ u_j\to 0),\ \beta\to\omega^{-1}(\partial_{u_j}Q\ /.\ w_i\to 0),\ \delta\to\omega^{-1}\partial_{w_i,u_j}Q\}\big]\big];$$

Stitching

$$m_{i\_,j\_\to k\_}[Z\_] := \mathrm{Module}\big[\{x, y, z\},\ Z\ //\ N_{w_i c_j\to x}\ //\ N_{w_x u_j\to y}\ //\ \mathrm{ReplaceAll}[\{c_{x|y}\to c_x,\ w_j\to w_y\}]\ //\ N_{u_i c_x\to x}\ //\ \mathrm{ReplaceAll}[z\_{}_{i|j|x|y}\to z_k]\ //\ \mathrm{CF}\big]$$

The Generators

$$R^{+}_{i\_,j\_} := \mathbb{E}\Big[1,\ b_i c_j,\ u_i w_j,\ -c_i (t_i-1)^2/2 - c_i^2 (t_i-1)^2/2 + c_i c_j \left(t_j^2 - t_i - 2\right)/2 - c_j u_i w_i/2 + c_i (1-t_i) u_i w_i - u_i^2 w_i^2/2 + u_i w_j + c_j t_i u_i w_j/2 + c_i (t_i-2) t_i u_i w_j + c_i (1+t_j) u_j w_j/2 + (t_i-1) u_i^2 w_i w_j - (t_i-2) t_i u_i^2 w_j^2/2\Big];$$

$$R^{-}_{i\_,j\_} := \mathbb{E}\Big[1,\ -b_i c_j,\ -t_i^{-1} u_i w_j,\ c_i (t_i-1)^2/2 + c_i^2 (t_i-1)^2/2 + c_i c_j \left(2 + t_i - t_j^2\right)/2 + c_j u_i w_i/2 + c_i (t_i-1) u_i w_i + u_i^2 w_i^2/2 + \left(1 - t_i^{-1}\right) u_i w_j/2 + c_i \left(2 t_i - 5 + 3 t_i^{-1}\right) u_i w_j/2 + c_j \left(t_i^{-1} + 1 - t_i^{-1} t_j^2\right) u_i w_j/2 - c_i (t_j+1) u_j w_j/2 + \left(2 - 3 t_i^{-1}\right) u_i^2 w_i w_j/2 + \left(1 + 2 t_i^{-2} - 3 t_i^{-1}\right) u_i^2 w_j^2/2 - t_i^{-1} (1+t_j) u_i u_j w_j^2/2\Big];$$

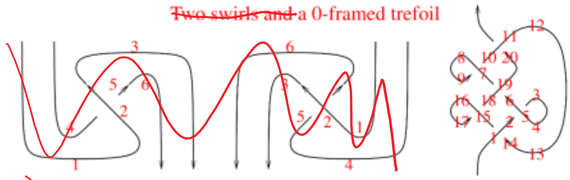
$$ur_{i\_} := \mathbb{E}\left[t_i^{-1/4}, 0, 0, c_i t_i/4 + u_i w_i/8\right];$$

$$nr_{i\_} := \mathbb{E}\left[t_i^{1/4}, 0, 0, -c_i t_i^3/4 - t_i^2 u_i w_i/8\right];$$

$$ul_{i\_} := \mathbb{E}\left[t_i^{1/4}, 0, 0, c_i t_i (4+t_i)/4 - t_i^2 u_i w_i/8\right];$$

$$nl_{i\_} := \mathbb{E}\left[t_i^{-1/4}, 0, 0, -c_i \left(1 + 4 t_i^{-1}\right)/4 + u_i w_i/8\right];$$

~~Two swirls and~~ a 0-framed trefoil



$$t2 = ur_1 R^{-}_{2,5}\, nr_3\, ur_4\, nr_6 \,//\, m_{1,2\to1} \,//\, m_{1,3\to1} \,//\, m_{4,5\to4} \,//\, m_{4,6\to4}$$

$$\mathbb{E}\Big[1,\ -b_1 c_4,\ -\frac{u_1 w_4}{t_1},\ \frac{c_1}{2} + \frac{c_1^2}{2} + c_1 c_4 - c_1 t_1 - c_1^2 t_1 + \frac{1}{2} c_1 c_4 t_1 + \frac{1}{2} c_1 t_1^2 + \frac{1}{2} c_1^2 t_1^2 - \frac{1}{2} c_1 c_4 t_4^2 - c_1 u_1 w_1 + \frac{1}{2} c_4 u_1 w_1 + c_1 t_1 u_1 w_1 + \frac{1}{2} u_1^2 w_1^2 + \frac{3 u_1 w_4}{8} - \frac{5}{2} c_1 u_1 w_4 + \frac{1}{2} c_4 u_1 w_4 - \frac{u_1 w_4}{2 t_1} + \frac{3 c_1 u_1 w_4}{2 t_1} + \frac{c_4 u_1 w_4}{2 t_1} - \frac{1}{8} t_1 u_1 w_4 + c_1 t_1 u_1 w_4 + \frac{t_4 u_1 w_4}{8 t_1} + \frac{t_4^2 u_1 w_4}{8 t_1} - \frac{c_4 t_4^2 u_1 w_4}{2 t_1} - \frac{1}{2} c_1 u_4 w_4 - \frac{1}{2} c_1 t_4 u_4 w_4 + u_1^2 w_1 w_4 - \frac{3 u_1^2 w_1 w_4}{2 t_1} + \frac{1}{2} u_1^2 w_4^2 + \frac{u_1^2 w_4^2}{t_1^2} - \frac{3 u_1^2 w_4^2}{2 t_1} - \frac{u_1 u_4 w_4^2}{2 t_1} - \frac{t_4 u_1 u_4 w_4^2}{2 t_1}\Big]$$

$$\cancel{t2 == (ul_1 R^{-}_{2,5}\, nl_3\, ul_4\, nl_6 \,//\, m_{1,2\to1} \,//\, m_{1,3\to1} \,//\, m_{4,5\to4} \,//\, m_{4,6\to4})}$$

~~True~~

$$z2 = R^{+}_{1,14} R^{-}_{5,2}\, nr_3\, ul_4\, R^{+}_{19,6} R^{-}_{7,10}\, nl_8\, ur_9\, R^{+}_{11,20}\, nr_{12}\, ul_{13}\, R^{-}_{15,18}\, nl_{16}\, ur_{17};$$

$$(\mathrm{Do}[z2 = z2 \,//\, m_{1,k\to1},\ \{k, 2, 20\}];\ z2 = z2\ /.\ a\_{}_1 :\to a)$$

$$\mathbb{E}\Big[-1 + \frac{1}{t} + t,\ 0,\ 0,\ -16 + \frac{9c}{2} - \frac{2c}{t^4} + \frac{1}{t^3} + \frac{11c}{2t^3} - \frac{4}{t^2} - \frac{8c}{t^2} + \frac{10}{t} + \frac{4c}{t} + 18t - 10ct - 14t^2 + 8ct^2 + 7t^3 - \frac{3ct^3}{2} - 2t^4 - 2ct^4 + 2ct^5 - \frac{ct^6}{2} - 4uw + \frac{2uw}{t^4} - \frac{7uw}{2t^3} + \frac{9uw}{2t^2} + \frac{uw}{2t} + 6tuw - 2t^2 uw - \frac{1}{2} t^3 uw + \frac{3}{2} t^4 uw - \frac{1}{2} t^5 uw\Big]$$

```
FromCoefficientRules[
 CoefficientRules[z2[[4]], {c, u, w}] /.
  {(e_ → a_) :→ (e → Simplify[a])}, {c, u, w}]
```

$$-\frac{(1-t+t^2)^2 (1+2t-3t^2+2t^3)}{t^3} - \frac{c (1-t+t^2)^3 (4+t-5t^2-t^3+t^4)}{2t^4} - \frac{(1-t+t^2)^3 (-4+5t+t^3)\, uw}{2t^4}$$

Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

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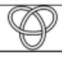
1 column

Disclaimer. This is all quite new. The overall picture is correct, yet some details might be somewhat off. Many pieces are certainly not in their final form yet. **Help Needed!**

at least two lines.

diagram	n^t_k Alexander a_+ / *P-essence* e_+	genus / ribbon, unknotting number / amphicheiral	diagram	n^t_k Alexander a_+ / *P-essence* e_+	genus / ribbon, unknotting number / amphicheiral
	0^a_1 1 0	0 / ✔ 0 / ✔		3^a_1 $t-1$ t	1 / ✘ 1 / ✘