A Poly-Time Knot Polynomial Via Solvable Approximation

Work in Progress!

Abstract. Rozansky [Ro2] and Overbay [Ov] described a spectacular knot polynomial that failed to attract the attention it deserved as the first poly-time-computable knot polynomial since Alexander's [Al, 1928] and (in my opinion) as the second most likely knot polynomial (after Alexander's) to carry topological information. With Roland van der Veen, I will explain how to compute the Rozansky polynomial using some new commutator-calculus techniques and a Lie algebra $\mathfrak{g}_1$ which is at the same time solvable and an approximation of the simple Lie algebra sl_2.

Rozansky

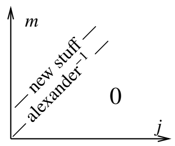
Overbay

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K, in the d-dimensional representation of sl_2. Writing

Melvin, Morton, Garoufalidis

$$\left.\frac{(q^{1/2}-q^{-1/2})J_d(K)}{q^{d/2}-q^{-d/2}}\right|_{q=e^\hbar}=\sum_{j,m\geq 0}a_{jm}(K)d^j\hbar^m,$$

"below diagonal" coefficients vanish, $a_{jm}(K)=0$ if $j>m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left(\sum_{m=0}^\infty a_{mm}(K)\hbar^m\right)\cdot A(K)(e^\hbar)=1$.



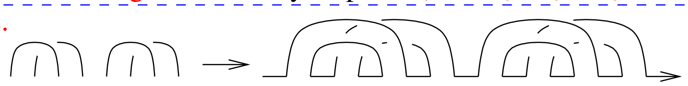
"Above diagonal" we have Rozansky's Theorem [Ro3, (1.2)]:

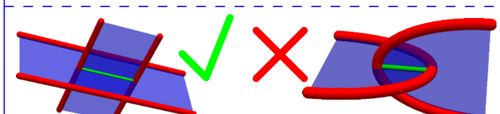
$$J_d(K)(q)=\frac{q^d-q^{-d}}{(q-q^{-1})A(K)(q^d)}\left(1+\sum_{k=1}^\infty\frac{(q-1)^kR_k(K)(q^d)}{A^{2k}(K)(q^d)}\right).$$

Why "spectacular"? Foremost reason: *OBVIOUSLY*. Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

Also, will bound genus and may disprove {ribbon} = {slice}.

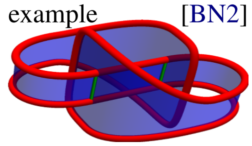
Genus.





a ribbon singularity a clasp singularity

example [BN2]

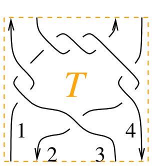


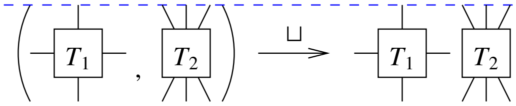
A bit about ribbon knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3=\partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

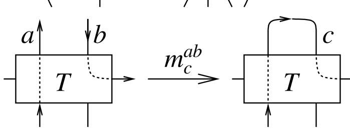
Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t)=f(t)f(1/t)$. (also for slice)

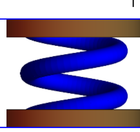
(v-)Tangles.

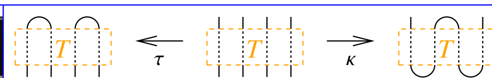






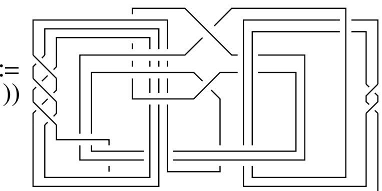
(meta-associativity: $m^{ab}_x /\!\!/ m^{xc}_y = m^{bc}_x /\!\!/ m^{ax}_y$)





Gompf, Scharlemann, Thompson [GST]

$$\begin{array}{ccccc} & U\in\mathcal{T}_n & & 1\in\mathcal{A}_n & \\ & \tau\nearrow & & \tau\nearrow & \\ \mathcal{T}_{2n} & \xrightarrow{z} & \mathcal{A}_{2n} & & \text{with } \mathcal{R}:=\kappa(\tau^{-1}(1)) \\ & \kappa\searrow & & \kappa\searrow & \\ & \text{ribbon } K\in\mathcal{T}_1 & & z(K)\in\mathcal{R}\subseteq\mathcal{A}_1 & \end{array}$$



Faster is better, leaner is meaner!

$A^+=-t^8+2t^7-t^6-2t^4+5t^3-2t^2-7t+13$

$\rho_1^+=5t^{15}-18t^{14}+33t^{13}-32t^{12}+2t^{11}+42t^{10}-62t^9-8t^8+166t^7-242t^6+108t^5+132t^4-226t^3+148t^2-11t-36$

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN1]. An S-component tangle T has $\Gamma(T)\in R_S\times M_{S\times S}(R_S)=\left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S:=\mathbb{Z}(\{t_a\colon a\in S\})$:

$$\left({}_a\!\nearrow_b,\ {}_b\!\nwarrow_a\right)\to\begin{array}{c|cc}1 & a & b\\ \hline a & 1 & 1-t_a^{\pm1}\\ b & 0 & t_a^{\pm1}\end{array}\qquad T_1\sqcup T_2\to\begin{array}{c|cc}\omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc}\omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array}\ \xrightarrow[t_a,t_b\to t_c]{m^{ab}_c}\ \left(\begin{array}{c|cc}(1-\beta)\omega & c & S\\ \hline c & \gamma+\frac{\alpha\delta}{1-\beta} & \epsilon+\frac{\delta\theta}{1-\beta}\\ S & \phi+\frac{\alpha\psi}{1-\beta} & \Xi+\frac{\psi\theta}{1-\beta}\end{array}\right)$$

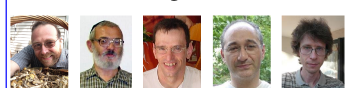
(Roland: "add to A the product of column b and row a, divide by $(1-A_{ab})$, delete column b and row a".)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

(There are also formulas for strand doubling and strand reversal).

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z\colon\left\{\begin{array}{c}S\text{-component}\\ (v/b\text{-})\text{tangles}\end{array}\right\}\to\mathcal{A}^v_S:=$$



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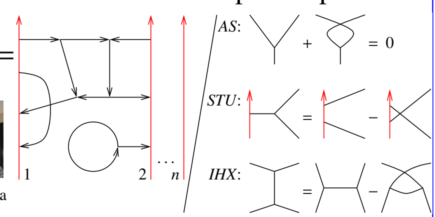
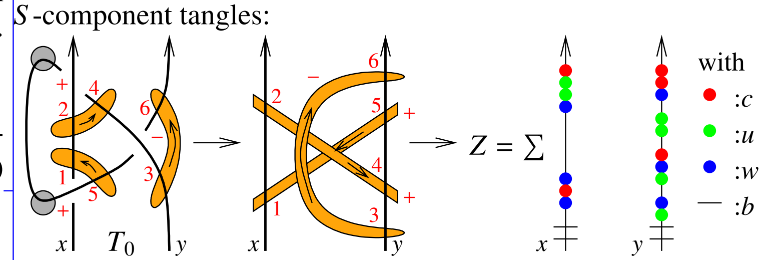


Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A\sim\hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R\in A\otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



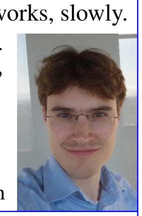
Good News. In theory, enough to know R, the cuaps, and stitching/multiplication $m^{ij}_k\colon A_i\otimes A_j\to A_k$.

Problem. Extract information out of Z.

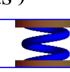
Textbook Solution. Use representation theory ...works, slowly.

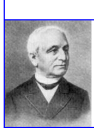
Today's Solution (with van der Veen). For some specific $\mathfrak{g}$'s, work in a space of "formulas of a specific type" for elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$:

$$\left\{\begin{array}{c}\text{ordered perturbed}\\ \text{Gaussian formulas}\end{array}\right\}\to\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$$



van der Veen





"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b,c,u,w\rangle$ over the ring $R=\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with b central and with $[w,c]=w$, $[c,u]=u$, and $[u,w]=b-2\epsilon c$, with CYBE $r_{ij} = (b_i-\epsilon c_i)c_j + u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle \supset \langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle \supset 0$. (note: $\deg(b,c,u,w,\epsilon)=(1,0,1,0,1)$)

0-Smidgen sl_2 ☺. Let $\mathfrak{g}_0$ be $\mathfrak{g}_1$ at $\epsilon=0$, or $\mathbb{Q}\langle b,c,u,w\rangle/([b,\cdot]=0,\ [c,u]=u,\ [c,w]=-w,\ [u,w]=b$ with $r_{ij}=b_ic_j+u_iw_j$. It is $\mathfrak{b}^*\rtimes\mathfrak{b}$ where $\mathfrak{b}$ is the 2D Lie algebra $\mathbb{Q}\langle c,w\rangle$ and (b,u) is the dual basis of (c,w). For topology, it is more valuable than $\mathfrak{g}_1$ / sl_2, but topology already got by other means almost everything $\mathfrak{g}_0$ gives.

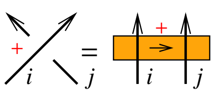
How did these arise? $sl_2=\mathfrak{b}^+\oplus\mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+=\langle c,w\rangle/[w,c]=w$ is a Lie bialgebra with $\delta\colon\mathfrak{b}^+\to\mathfrak{b}^+\otimes\mathfrak{b}^+$ by $\delta\colon(c,w)\mapsto(0,c\wedge w)$. Going back, $sl_2^+=\mathcal{D}(\mathfrak{b}^+)=(\mathfrak{b}^+)^*\oplus\mathfrak{b}^+=\langle b,u,c,w\rangle/\cdots$. **Idea.** Replace $\delta\to\epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1}=0)$. At $k=0$, get $\mathfrak{g}_0$. At $k=1$, get $[w,c]=w$, $[w,b']=-\epsilon w$, $[c,u]=u$, $[b',u]=-\epsilon u$, $[b',c]=0$, and $[u,w]=b'-\epsilon c$. Now note that $b'+\epsilon c$ is central, so switch to $b:=b'+\epsilon c$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly\mid specs)$ plants the variables of $poly$ in $\mathcal{S}(\oplus_i\mathfrak{g})$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to $specs$. E.g.,

$$\mathbb{O}\left(c_1^3u_1c_2e^{u_3}w_3^9 \mid x\colon w_3c_1,\ y\colon u_1u_3c_2\right)=w^9c^3\otimes ue^uc\in\mathcal{U}(\mathfrak{g})_x\otimes\mathcal{U}(\mathfrak{g})_y$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r=Id\in\mathfrak{b}^-\otimes\mathfrak{b}^+$ solves the CYBE $[r_{12},r_{13}]+[r_{12},r_{23}]+[r_{13},r_{23}]=0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,



$=R_{ij}=e^{r_{ij}}=e^{b_ic_j+u_iw_j}\in\mathcal{U}(\mathfrak{g}_{0,i}\oplus\mathfrak{g}_{0,j})$ solves YB/R3.

Lemma. $R_{ij}=e^{b_ic_j+u_iw_j}=\mathbb{O}\left(\exp\left(b_ic_j+\frac{e^{b_i}-1}{b_i}u_iw_j\right)\Big| i\colon u_i,\ j\colon c_jw_j\right)$

Example. $Z(T_0)=$ $=\sum_{m,n}\frac{b_i^{m-n}(e^{b_i}-1)^n}{m!n!}u^n\otimes c^mw^n$.

$$\mathbb{O}\Big(\exp\left(b_5c_1+\tfrac{e^{b_5}-1}{b_5}u_5w_1+b_2c_4+\tfrac{e^{b_2}-1}{b_2}u_2w_4-b_3c_6+\tfrac{e^{-b_3}-1}{b_3}u_3w_6\right)\Big|\ x\colon c_1w_1u_2,\ y\colon u_3c_4w_4u_5c_6w_6\Big)=\mathbb{O}\left(?\,|x\colon c_xu_xw_x,\ y\colon c_yu_yw_y\right)$$

"cuw form"

Goal. Write ? as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S:=\mathbb{Q}(b_i,e^{b_i})$, and $\omega\in R_S$.

The Big $\mathfrak{g}_0$ Lemma. Under $[c,u]=u$, $[c,w]=-w$, and $[u,w]=b$:

1a. $N^{uc}:=\mathbb{O}(e^{\gamma c+\beta u}|uc)\overset{\to}{=}\mathbb{O}(e^{\gamma c+e^{-\gamma}\beta u}|cu)$ (means $e^{\beta u}e^{\gamma c}=e^{\gamma c}e^{e^{-\gamma}\beta u}$

1b. $N^{wc}:=\mathbb{O}(e^{\gamma c+\alpha w}|wc)\overset{\to}{=}\mathbb{O}(e^{\gamma c+e^{\gamma}\alpha w}|cw)$ …in the $\{ax+b\}$ group)

2. $\mathbb{O}(e^{\alpha w+\beta u}|wu)=\mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw)$ (the Weyl relations)

3. $\mathbb{O}(e^{\delta uw}|wu)e^{\beta u}=e^{\nu\beta u}\mathbb{O}(e^{\delta uw}|wu)$, with $\nu=(1+b\delta)^{-1}$

(a. expand and crunch. b. use $w=b\hat{x}$, $u=\partial_x$. c. use "scatter and glow".)

4. $\mathbb{O}(e^{\delta uw}|wu)=\mathbb{O}(\nu e^{\nu\delta uw}|uw)$ (same techniques)

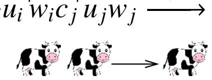
5. $N^{wu}:=\mathbb{O}(e^{\beta u+\alpha w+\delta uw}|wu)\overset{\to}{=}\mathbb{O}(\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\delta uw}|uw)$

6. $N_k^{c_ic_j}:=\mathbb{O}(\zeta|c_ic_j)\overset{\to}{=}\mathbb{O}(\zeta/(c_i,c_j\to c_k)|c_k)$

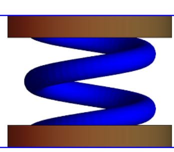
Sneaky. α may contain (other) u's, β may contain (other) w's.

Strand Stitching, m_k^{ij}, is defined as the composition

$$c_iu_i\overline{w_ic_j}u_jw_j\xrightarrow{N_k^{w_ic_j}}c_i\overline{u_ic_k}\,\overline{w_ku_j}\,w_j\xrightarrow{N_k^{u_ic_k}/\!/N_k^{w_ku_j}}\overline{c_ic_k}\,\overline{u_ku_k}\,\overline{w_kw_j}$$



$$\xrightarrow{N_k^{c_ic_k}/\!/-/\!/N_k^{w_kw_j}}c_ku_kw_k$$



1-Smidgen Invariants. Much is the same:

The Big $\mathfrak{g}_1$ Lemma. Parts 1 and 6 are the same, yet

5. $\mathbb{O}\left(e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\nu(1+\epsilon\nu\Lambda)e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$

Here Λ is for Λόγος, "a principle of order and knowledge", a balanced quartic in α, β, c, u, and w:

$$\begin{aligned}\Lambda=&-b\nu\left(\nu^2\alpha^2\beta^2+4\delta\nu\alpha\beta+2\delta^2\right)/2-\delta\nu^3(3b\delta+2)\beta^2u^2/2\\&-b\delta^4\nu^3u^2w^2/2-\delta^2\nu^3(2b\delta+1)\beta u^2w\\&-\nu^2(2b\delta+1)(\nu\alpha\beta+2\delta)\beta u-2b\delta^2\nu^2(\nu\alpha\beta+\delta)uw\\&+\delta\nu^3(b\delta+2)\alpha^2w^2/2+2(\nu\alpha\beta+\delta)c+2\delta\nu\beta cu+2\delta^2\nu cuw\\&+2\delta\nu\alpha cw+\delta^2\nu^3\alpha uw^2+\nu^2(\nu\alpha\beta+2\delta)\alpha w.\end{aligned}$$

Proof. A brutal hell.

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:

$$\mathbb{O}(\epsilon P(c,u)e^{\gamma c+\beta u}|uc)=\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+\beta u}|uc)=\mathbb{O}(\epsilon P(\partial_\gamma,\partial_\beta)e^{\gamma c+e^{-\gamma}\beta u}|cu),$$

and likewise

$$\mathbb{O}\left(\epsilon P(u,w)e^{\alpha w+\beta u+\delta uw}|wu\right)=\mathbb{O}\left(\epsilon P(\partial_\beta,\partial_\alpha)\nu e^{\nu(-b\alpha\beta+\alpha w+\beta u+\delta uw)}|cuw\right)$$

Note. Strand stitching requires a tiny extra step.

Finally, the values of the generators ⤡, ⤢, $\overrightarrow{n}$, $\overleftarrow{n}$, $\underset{\rightarrow}{u}$, and $\underset{\leftarrow}{u}$, are set by brutally solving many equations, non-uniquely.

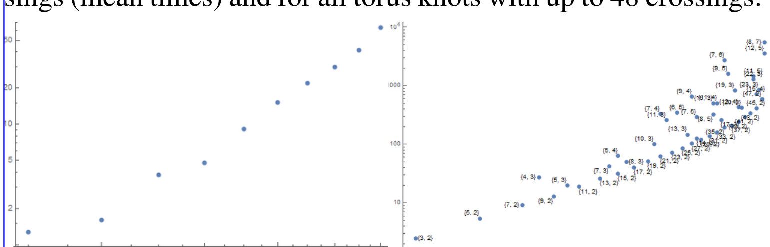
Pragmatic Simplifications. Get rid of $\zeta=(e^b-1)/b$ factors by rescaling $u\to\bar{u}=\zeta u$. Complement this with $\beta\to\bar{\beta}=\zeta^{-1}\beta$, $\delta\to\bar{\delta}=\zeta^{-1}\delta$, $\epsilon\to\bar{\epsilon}=\zeta^{-1}\epsilon$. Simplify further by naming $e^b\to t$; e.g., $\nu\to\bar{\nu}=(1+(t-1)\delta)^{-1}$. Get confused by renaming $(\bar{u},\bar{\beta},\bar{\delta},\bar{\nu})\to(u,\beta,\delta,\nu)$, and more confused by working with $\mu=\nu^{-1}$ and $\mathbb{E}(\omega,L,Q,P):=\omega^{-1}(1+\epsilon\omega^{-4}P)e^{L+\omega^{-1}Q}$, where $\omega\in R:=\mathbb{Q}(t_k)$, $L=\sum l_{ij}b_ic_j$ with $l_{ij}\in\mathbb{Z}$, $Q=\sum q_{ij}u_iw_j$ with $q_{ij}\in R$, and P is a balanced quartic polynomial in c_i, u_i, and w_i with coefficients in R. Magically, all coefficients are now Laurent polynomials in the t_k's.

Rough complexity estimate, after $t_k\to t$. n: xing number; w: width, maybe $\sim\sqrt{n}$.

$$\underbrace{n}_{A}\overbrace{\sum_{d=0}^{4}\underbrace{w^{4-d}}_{E}\underbrace{w^d}_{F}\underbrace{n^2}_{G}}^{B}=n^3w^4\in[n^5,n^7]$$

A: go over stitchings in order. B: multiplication ops per $N^{u_iw_j}$. d: deg of u_i,w_j in P. E: #terms of deg d in P. F: ops per term. G: cost per polynomial multiplication op.

Experimental Analysis (ωεβ/Exp). Log-log plot of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial A, there is a polynomial ρ_1 such that

$$P=A^2\left((t-2+t^{-1})\rho_1+tAA'\left(\frac{(4+t-t^2)(uw+(t-1)c)}{2(t-1)}-1\right)\right).$$

Furthermore, A and ρ_1 are symmetric under $t\to t^{-1}$, so let A^+ and ρ_1^+ be their "positive parts", so e.g., $\rho_1(t)=\rho_1^+(t)+\rho_1^+(t^{-1})-\rho_1^+(0)$.

Power. On the 250 knots with at most 10 crossings, the pair (A,ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 crossings, always $\deg\rho_1^+ \leq 2g-1$, where g is the 3-genus of K (equallity for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-crossing Alexander failures it does give the right answer.

Demo Programs for 0-Co.

ωεβ/Demo

The R-matrices

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R⁺0,i_,j_ := 𝔼[bᵢ cⱼ + bᵢ⁻¹ (e^bᵢ - 1) uᵢ wⱼ];
R⁻0,i_,j_ := 𝔼[-bᵢ cⱼ + bᵢ⁻¹ (e^-bᵢ - 1) uᵢ wⱼ];
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Utilities

```
CF[ω_. 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
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Normal Ordering Operators

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N_{uᵢ_ cⱼ_→k_}[ω_. 𝔼[Q_]] := CF[
   ω 𝔼[e^-γ β uₖ + γ cₖ + (Q /. cⱼ | uᵢ → 0)] /. {γ → ∂_{cⱼ}Q, β → ∂_{uᵢ}Q}];
N_{wᵢ_ cⱼ_→k_}[ω_. 𝔼[Q_]] := CF[
   ω 𝔼[e^γ α wₖ + γ cₖ + (Q /. cⱼ | wᵢ → 0)] /. {γ → ∂_{cⱼ}Q, α → ∂_{wᵢ}Q}];
N_{wᵢ_ uⱼ_→k_}[ω_. 𝔼[Q_]] := CF[
   ν ω 𝔼[-bₖ ν α β + ν β uₖ + ν δ uₖ wₖ + ν α wₖ + (Q /. wᵢ | uⱼ → 0)] /.
    ν → (1 + bₖ δ)^-1 /.
   {α → ∂_{wᵢ}Q /. uⱼ → 0, β → ∂_{uⱼ}Q /. wᵢ → 0, δ → ∂_{wᵢ,uⱼ}Q}];
```

Stitching

```
m_{i_,j_→k_}[ω_. 𝔼[Q_]] := CF[Module[{x},
   (ω 𝔼[Q] /. b_{i|j} → bₖ // N_{wᵢ cⱼ→x} // N_{uᵢ cₓ→x} // N_{wₓ uⱼ→x}) /.
    {cᵢ → cₖ, wⱼ → wₖ, y_ₓ ⧴ yₖ}]]
```

Some calculations for T_0

```
T0 = R⁺0,5,1 R⁺0,2,4 R⁻0,3,6
```

$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5})\, u_5 w_1}{b_5} + \frac{(-1+e^{b_2})\, u_2 w_4}{b_2} + \frac{(-1+e^{-b_3})\, u_3 w_6}{b_3}\right]$$

```
T0 // m1,2→1 // m3,4→3 // m3,5→3 // m3,6→3
```

$$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{(-1+e^{b_1})(-1+e^{b_3})\, u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\, b_1} - \frac{e^{-b_3}(-1+e^{b_1})\left(b_3 u_1 - e^{b_3}(-1+e^{b_3})\, b_1 u_3\right) w_3}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\, b_1 b_3} + \frac{e^{-b_1}(-1+e^{b_3})\, u_3 \left(-e^{b_1+b_3} w_1 + (e^{b_1}+e^{b_3}-e^{b_1+b_3})\, w_3\right)}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\, b_3}\Big]$$

Verifying meta-associativity

```
Q0 = 𝔼[Sum[fᵢ cᵢ, {i, 3}] + Sum[f_{i,j} uᵢ wⱼ, {i, 3}, {j, 3}]]
```

$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

```
(Q0 // m1,2→1 // m1,3→1) ≡ (Q0 // m2,3→2 // m1,2→1)
True
```

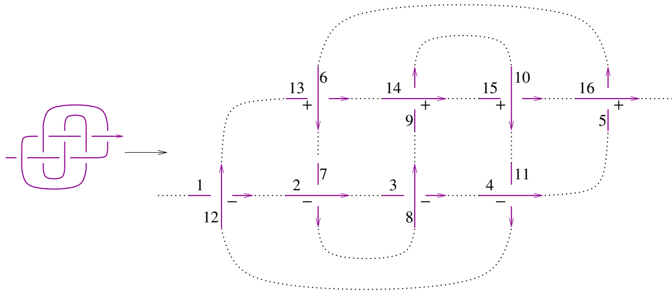
Testing R3

```
t1 = R⁺0,1,2 R⁺0,3,4 R⁺0,5,6 // m3,5→x // m1,6→y // m2,4→z
```

$$\mathbb{E}\left[b_x (c_y + c_z) + \frac{(-1+e^{b_x})\, u_x (w_y + w_z)}{b_x} + \frac{b_y^2 c_z + (-1+e^{b_y})\, u_y w_z}{b_y}\right]$$

```
t1 ≡ (R⁺0,1,2 R⁺0,3,4 R⁺0,5,6 // m1,3→x // m2,5→y // m4,6→z)
True
```

8_{17}




```
z1 = R⁻0,12,1 R⁻0,2,7 R⁻0,8,3 R⁻0,4,11 R⁺0,16,5 R⁺0,6,13 R⁺0,14,9 R⁺0,10,15;
Do[z1 = (z1 // m1,n→1) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$$

Demo Programs for 1-Co.

ωεβ/Demo

```
Λ[k_] := (1 - tₖ) (α² β² + 4 α β δ μ + 2 δ² μ²) / 2 + 2 μ² (α β + δ μ) cₖ -
   β (2 μ - 1) (α β + 2 δ μ) uₖ + 2 β δ μ² cₖ uₖ - β² δ (3 μ - 1) uₖ² / 2 +
   α (α β + 2 δ μ) wₖ + 2 α δ μ² cₖ wₖ - 2 (tₖ - 1) δ² (α β + δ μ) uₖ wₖ +
   2 δ² μ² cₖ uₖ wₖ - β δ² (2 μ - 1) uₖ² wₖ + α² δ (1 + μ) wₖ² / 2 +
   α δ² uₖ wₖ² - (tₖ - 1) δ⁴ uₖ² wₖ² / 2;
```

The Λόγος

The Generators

```
R⁺i_,j_ := 𝔼[1, bᵢ cⱼ, uᵢ wⱼ,
   -cᵢ (tᵢ - 1)² / 2 - cᵢ² (tᵢ - 1)² / 2 + cᵢ cⱼ (tⱼ² - tᵢ - 2) / 2 -
    cⱼ uᵢ wᵢ / 2 + cᵢ (1 - tᵢ) uᵢ wᵢ - uᵢ² wᵢ² / 2 + uᵢ wⱼ + cⱼ tᵢ uᵢ wⱼ / 2 +
    cᵢ (tᵢ - 2) tᵢ uᵢ wⱼ + cᵢ (1 + tⱼ) uⱼ wⱼ / 2 + (tᵢ - 1) uᵢ² wᵢ wⱼ -
    (tᵢ - 2) tᵢ uᵢ² wⱼ² / 2];
R⁻i_,j_ := 𝔼[1, -bᵢ cⱼ, -tᵢ⁻¹ uᵢ wⱼ,
   cᵢ (tᵢ - 1)² / 2 + cᵢ² (tᵢ - 1)² / 2 + cᵢ cⱼ (2 + tᵢ - tⱼ²) / 2 +
    cⱼ uᵢ wᵢ / 2 + cᵢ (tᵢ - 1) uᵢ wᵢ + uᵢ² wᵢ² / 2 + (1 - tᵢ⁻¹) uᵢ wⱼ / 2 +
    cᵢ (2 tᵢ - 5 + 3 tᵢ⁻¹) uᵢ wⱼ / 2 + cⱼ (tᵢ⁻¹ + 1 - tᵢ⁻¹ tⱼ²) uᵢ wⱼ / 2 -
    cᵢ (tⱼ + 1) uⱼ wⱼ / 2 + (2 - 3 tᵢ⁻¹) uᵢ² wᵢ wⱼ / 2 +
    (1 + 2 tᵢ⁻² - 3 tᵢ⁻¹) uᵢ² wⱼ² / 2 - tᵢ⁻¹ (1 + tⱼ) uᵢ uⱼ wⱼ² / 2];
urᵢ_ := 𝔼[tᵢ^-1/4, 0, 0, cᵢ tᵢ / 4 + uᵢ wᵢ / 8];
nrᵢ_ := 𝔼[tᵢ^1/4, 0, 0, -cᵢ tᵢ³ / 4 - tᵢ² uᵢ wᵢ / 8];
ulᵢ_ := 𝔼[tᵢ^1/4, 0, 0, cᵢ tᵢ (4 + tᵢ) / 4 - tᵢ² uᵢ wᵢ / 8];
nlᵢ_ := 𝔼[tᵢ^-1/4, 0, 0, -cᵢ (1 + 4 tᵢ⁻¹) / 4 + uᵢ wᵢ / 8];
```

Differential Polynomials

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := (* means P[∂α,∂β][f] *)
  Total[CoefficientRules[P, {x, y}] /.
    ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

Utilities

```
CF[𝔼[ω_, L_, Q_, P_]] := Expand /@ Together /@
   𝔼[ω /. b_ι_ ⧴ Log[t_ι], L, Q /. b_ι_ ⧴ Log[t_ι],
    P /. b_ι_ ⧴ Log[t_ι]];
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2⁴ P1 + ω1⁴ P2];
```

Normal Ordering Operators

```
N_{uᵢ_ cⱼ_→k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^-γ β uₖ + γ cₖ}, CF[
   𝔼[ω, γ cₖ + (L /. cⱼ → 0), ω e^-γ β uₖ + (Q /. uᵢ → 0),
     e^-q DP_{cⱼ→D_γ, uᵢ→D_β}[P][e^q]] /. {γ → ∂_{cⱼ}L, β → ω⁻¹ ∂_{uᵢ}Q}]];
N_{wᵢ_ cⱼ_→k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^γ α wₖ + γ cₖ}, CF[
   𝔼[ω, γ cₖ + (L /. cⱼ → 0), ω e^γ α wₖ + (Q /. wᵢ → 0),
     e^-q DP_{cⱼ→D_γ, wᵢ→D_α}[P][e^q]] /. {γ → ∂_{cⱼ}L, α → ω⁻¹ ∂_{wᵢ}Q}]];
N_{wᵢ_ uⱼ_→k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{q = (1 - tₖ) μ⁻¹ α β + μ⁻¹ β uₖ + μ⁻¹ δ uₖ wₖ + μ⁻¹ α wₖ}, CF[
   𝔼[μ ω, L, μ ω q + μ (Q /. wᵢ | uⱼ → 0),
      μ⁴ e^-q DP_{wᵢ→D_α, uⱼ→D_β}[P][e^q] + ω⁴ Λ[k]] /.
    μ → 1 + (tₖ - 1) δ /.
    {α → ω⁻¹ (∂_{wᵢ}Q /. uⱼ → 0), β → ω⁻¹ (∂_{uⱼ}Q /. wᵢ → 0),
     δ → ω⁻¹ ∂_{wᵢ,uⱼ}Q}]];
```

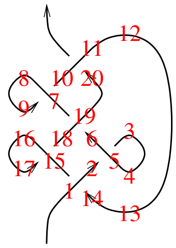
Stitching

```
m_{i_,j_→k_}[Z_] := Module[{x, y, z},
  Z // N_{wᵢ cⱼ→x} // N_{wₓ uⱼ→y} // ReplaceAll[{c_{x|y} → cₓ, wⱼ → w_y}] //
   N_{uᵢ cₓ→x} // ReplaceAll[z_{i|j|x|y} → zₖ] // CF]
```

```
z2 = R⁺_{1,14} R⁻_{5,2} nr_3 ul_4 R⁺_{19,6} R⁻_{7,10} nl_8 ur_9 R⁺_{11,20}
    nr_12 ul_13 R⁻_{15,18} nl_16 ur_17;
(Do[z2 = z2 // m_{1,k→1}, {k, 2, 20}];
 z2 = z2 /. a_1 :→ a)
```

$$\mathbb{E}\Big[-1+\tfrac{1}{t}+t,\ 0,\ 0,\ -16+\tfrac{9c}{2}-\tfrac{2c}{t^4}+\tfrac{1}{t^3}+\tfrac{11c}{2t^3}-\tfrac{4}{t^2}-\tfrac{8c}{t^2}+\tfrac{10}{t}+\tfrac{4c}{t}+18t-10ct-14t^2+8ct^2+7t^3-\tfrac{3ct^3}{2}-2t^4-2ct^4+2ct^5-\tfrac{ct^6}{2}-4uw+\tfrac{2uw}{t^4}-\tfrac{7uw}{2t^3}+\tfrac{9uw}{2t^2}+\tfrac{uw}{2t}+6tuw-2t^2uw-\tfrac{1}{2}t^3uw+\tfrac{3}{2}t^4uw-\tfrac{1}{2}t^5uw\Big]$$

The 0-Framed Trefoil



Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods. Help Needed!

Disclaimer. This is all quite new. The overall picture is correct, but many pieces are certainly not in their final form yet.

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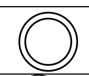
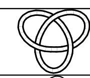
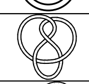
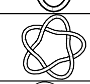
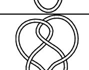
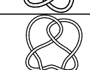
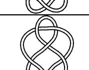
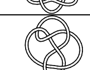
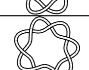
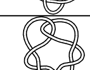
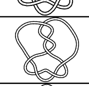
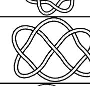
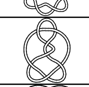
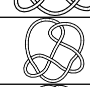
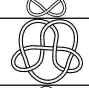
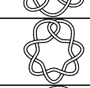
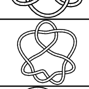
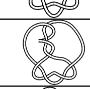
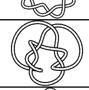
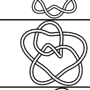
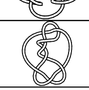
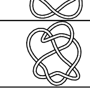
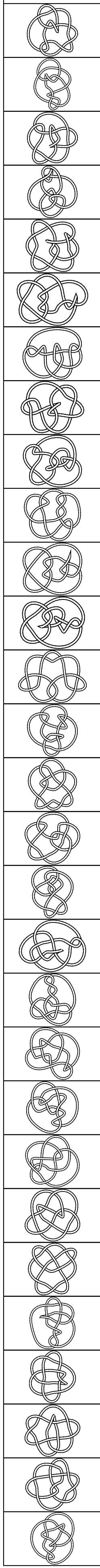
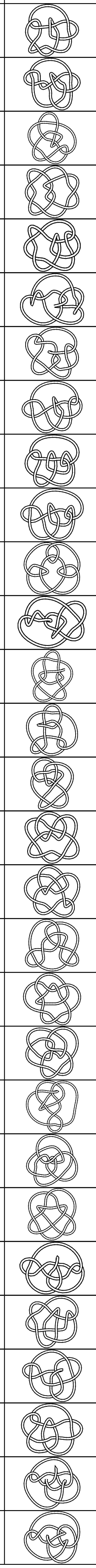
diagram	n_k^t Today's / Rozansky's ρ_1^+	Alexander's A_+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t Today's / Rozansky's ρ_1^+	Alexander's A_+	genus / ribbon unknotting number / amphicheiral
	0_1^a 0	1	0 / ✔ 0 / ✔		3_1^a t	$t-1$	1 / ✘ 1 / ✘
	4_1^a 0	$3-t$	1 / ✘ 1 / ✔		5_1^a $2t^3+3t$	t^2-t+1	2 / ✘ 2 / ✘
	5_2^a $5t-4$	$2t-3$	1 / ✘ 1 / ✘		6_1^a $t-4$	$5-2t$	1 / ✔ 1 / ✘
	6_2^a t^3-4t^2+4t-4	$-t^2+3t-3$	2 / ✘ 1 / ✘		6_3^a 0	t^2-3t+5	2 / ✘ 1 / ✔
	7_1^a $3t^5+5t^3+6t$	t^3-t^2+t-1	3 / ✘ 3 / ✘		7_2^a $14t-16$	$3t-5$	1 / ✘ 1 / ✘
	7_3^a $-9t^3+8t^2-16t+12$	$2t^2-3t+3$	2 / ✘ 2 / ✘		7_4^a $32-24t$	$4t-7$	1 / ✘ 2 / ✘
	7_5^a $9t^3-16t^2+29t-28$	$2t^2-4t+5$	2 / ✘ 2 / ✘		7_6^a $t^3-8t^2+19t-20$	$-t^2+5t-7$	2 / ✘ 1 / ✘
	7_7^a $8-3t$	t^2-5t+9	2 / ✘ 1 / ✘		8_1^a $5t-16$	$7-3t$	1 / ✘ 1 / ✘
	8_2^a $2t^5-8t^4+10t^3-12t^2+13t-12$	$-t^3+3t^2-3t+3$	3 / ✘ 2 / ✘		8_3^a 0	$9-4t$	1 / ✘ 2 / ✔
	8_4^a $3t^3-8t^2+6t-4$	$-2t^2+5t-5$	2 / ✘ 2 / ✘		8_5^a $-2t^5+8t^4-13t^3+20t^2-22t+24$	$-t^3+3t^2-4t+5$	3 / ✘ 2 / ✘
	8_6^a $5t^3-20t^2+28t-32$	$-2t^2+6t-7$	2 / ✘ 2 / ✘		8_7^a $-t^5+4t^4-10t^3+12t^2-13t+12$	t^3-3t^2+5t-5	3 / ✘ 1 / ✘

diagram	n_k^t	Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t	Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral
	8_8^a	$2t^2-6t+9$ $-t^3+4t^2-12t+16$	2 / ✔ 2 / ✘		8_9^a	$-t^3+3t^2-5t+7$ 0	3 / ✔ 1 / ✔
	8_{10}^a	t^3-3t^2+6t-7 $-t^5+4t^4-11t^3+16t^2-21t+20$	3 / ✘ 2 / ✘		8_{11}^a	$-2t^2+7t-9$ $5t^3-24t^2+39t-44$	2 / ✘ 1 / ✘
	8_{12}^a	$t^2-7t+13$ 0	2 / ✘ 2 / ✔		8_{13}^a	$2t^2-7t+11$ $-t^3+4t^2-14t+20$	2 / ✘ 1 / ✘
	8_{14}^a	$-2t^2+8t-11$ $5t^3-28t^2+57t-68$	2 / ✘ 1 / ✘		8_{15}^a	$3t^2-8t+11$ $21t^3-64t^2+120t-140$	2 / ✘ 2 / ✘
	8_{16}^a	t^3-4t^2+8t-9 $t^5-6t^4+17t^3-28t^2+35t-36$	3 / ✘ 2 / ✘		8_{17}^a	$-t^3+4t^2-8t+11$ 0	3 / ✘ 1 / ✔
	8_{18}^a	$-t^3+5t^2-10t+13$ 0	3 / ✘ 2 / ✔		8_{19}^n	t^3-t^2+1 $-3t^5-4t^2-3t$	3 / ✘ 3 / ✘
	8_{20}^n	t^2-2t+3 $4t-4$	2 / ✔ 1 / ✘		8_{21}^n	$-t^2+4t-5$ $t^3-8t^2+16t-20$	2 / ✘ 1 / ✘
	9_1^a	$t^4-t^3+t^2-t+1$ $4t^7+7t^5+9t^3+10t$	4 / ✘ 4 / ✘		9_2^a	$4t-7$ $30t-40$	1 / ✘ 1 / ✘
	9_3^a	$2t^3-3t^2+3t-3$ $-13t^5+12t^4-25t^3+20t^2-32t+24$	3 / ✘ 3 / ✘		9_4^a	$3t^2-5t+5$ $23t^3-28t^2+46t-44$	2 / ✘ 2 / ✘
	9_5^a	$6t-11$ $100-65t$	1 / ✘ 2 / ✘		9_6^a	$2t^3-4t^2+5t-5$ $13t^5-24t^4+45t^3-52t^2+68t-64$	3 / ✘ 3 / ✘
	9_7^a	$3t^2-7t+9$ $23t^3-56t^2+99t-108$	2 / ✘ 2 / ✘		9_8^a	$-2t^2+8t-11$ $3t^3-16t^2+29t-28$	2 / ✘ 2 / ✘
	9_9^a	$2t^3-4t^2+6t-7$ $13t^5-24t^4+55t^3-72t^2+98t-96$	3 / ✘ 3 / ✘		9_{10}^a	$4t^2-8t+9$ $-40t^3+72t^2-114t+120$	2 / ✘ 2, 3 / ✘
	9_{11}^a	$-t^3+5t^2-7t+7$ $-2t^5+16t^4-41t^3+52t^2-66t+64$	3 / ✘ 2 / ✘		9_{12}^a	$-2t^2+9t-13$ $5t^3-36t^2+84t-100$	2 / ✘ 1 / ✘
	9_{13}^a	$4t^2-9t+11$ $-40t^3+92t^2-154t+168$	2 / ✘ 2, 3 / ✘		9_{14}^a	$2t^2-9t+15$ $-t^3+8t^2-35t+60$	2 / ✘ 1 / ✘
	9_{15}^a	$-2t^2+10t-15$ $-5t^3+40t^2-108t+136$	2 / ✘ 2 / ✘		9_{16}^a	$2t^3-5t^2+8t-9$ $-13t^5+36t^4-80t^3+120t^2-161t+168$	3 / ✘ 3 / ✘
	9_{17}^a	t^3-5t^2+9t-9 $t^5-8t^4+23t^3-32t^2+28t-24$	3 / ✘ 2 / ✘		9_{18}^a	$4t^2-10t+13$ $40t^3-108t^2+193t-220$	2 / ✘ 2 / ✘
	9_{19}^a	$2t^2-10t+17$ $t^3-8t^2+20t-24$	2 / ✘ 1 / ✘		9_{20}^a	$-t^3+5t^2-9t+11$ $2t^5-16t^4+47t^3-84t^2+117t-124$	3 / ✘ 2 / ✘
	9_{21}^a	$-2t^2+11t-17$ $-5t^3+44t^2-127t+164$	2 / ✘ 1 / ✘		9_{22}^a	$t^3-5t^2+10t-11$ $-t^5+8t^4-24t^3+38t^2-40t+36$	3 / ✘ 1 / ✘
	9_{23}^a	$4t^2-11t+15$ $40t^3-128t^2+243t-288$	2 / ✘ 2 / ✘		9_{24}^a	$-t^3+5t^2-10t+13$ $-4t^2+16t-20$	3 / ✘ 1 / ✘
	9_{25}^a	$-3t^2+12t-17$ $12t^3-70t^2+153t-188$	2 / ✘ 2 / ✘		9_{26}^a	$t^3-5t^2+11t-13$ $-t^5+8t^4-31t^3+64t^2-85t+92$	3 / ✘ 1 / ✘
	9_{27}^a	$-t^3+5t^2-11t+15$ $t^3-8t^2+24t-32$	3 / ✔ 1 / ✘		9_{28}^a	$t^3-5t^2+12t-15$ $t^5-8t^4+30t^3-68t^2+105t-120$	3 / ✘ 1 / ✘
	9_{29}^a	$t^3-5t^2+12t-15$ $t^5-8t^4+26t^3-48t^2+59t-56$	3 / ✘ 2 / ✘		9_{30}^a	$-t^3+5t^2-12t+17$ $2t^3-10t^2+25t-32$	3 / ✘ 1 / ✘
	9_{31}^a	$t^3-5t^2+13t-17$ $t^5-8t^4+33t^3-80t^2+132t-152$	3 / ✘ 2 / ✘		9_{32}^a	$t^3-6t^2+14t-17$ $-t^5+10t^4-42t^3+94t^2-133t+148$	3 / ✘ 2 / ✘
	9_{33}^a	$-t^3+6t^2-14t+19$ $t^3-10t^2+30t-40$	3 / ✘ 1 / ✘		9_{34}^a	$-t^3+6t^2-16t+23$ $3t^3-18t^2+43t-56$	3 / ✘ 1 / ✘
	9_{35}^a	$7t-13$ $90t-144$	1 / ✘ 2, 3 / ✘		9_{36}^a	$-t^3+5t^2-8t+9$ $-2t^5+16t^4-44t^3+66t^2-87t+88$	3 / ✘ 2 / ✘
	9_{37}^a	$2t^2-11t+19$ $t^3-8t^2+22t-28$	2 / ✘ 2 / ✘		9_{38}^a	$5t^2-14t+19$ $62t^3-204t^2+382t-452$	2 / ✘ 2, 3 / ✘
	9_{39}^a	$-3t^2+14t-21$ $-12t^3+84t^2-210t+268$	2 / ✘ 1 / ✘		9_{40}^a	$t^3-7t^2+18t-23$ $t^5-12t^4+57t^3-144t^2+229t-264$	3 / ✘ 2 / ✘
	9_{41}^a	$3t^2-12t+19$ $3t^3-20t^2+70t-108$	2 / ✔ 2 / ✘		9_{42}^n	$-t^2+2t-1$ $-t^3+2t^2+t-4$	2 / ✘ 1 / ✘
	9_{43}^n	$-t^3+3t^2-2t+1$ $-2t^5+8t^4-7t^3+2t^2-5t+4$	3 / ✘ 2 / ✘		9_{44}^n	t^2-4t+7 $-2t^2+9t-12$	2 / ✘ 1 / ✘

diagram	n_k^t	Alexander's A_+	Today's / Rozansky's ρ_1^+	genus / ribbon	unknotting number / amphicheiral	diagram	n_k^t	Alexander's A_+	Today's / Rozansky's ρ_1^+	genus / ribbon	unknotting number / amphicheiral
	9^n_{45}	$-t^2+6t-9$	$t^3-14t^2+47t-60$	2 / ✘	1 / ✘		9^n_{46}	$5-2t$	$3t-12$	1 / ✔	2 / ✘
	9^n_{47}	t^3-4t^2+6t-5	$-t^5+6t^4-15t^3+16t^2-10t+12$	3 / ✘	2 / ✘		9^n_{48}	$-t^2+7t-11$	$-t^3+12t^2-42t+52$	2 / ✘	2 / ✘
	9^n_{49}	$3t^2-6t+7$	$-21t^3+38t^2-61t+60$	2 / ✘	3 / ✘		10^a_1	$9-4t$	$14t-40$	1 / ✘	1 / ✘
	10^a_2	$-t^4+3t^3-3t^2+3t-3$	$3t^7-12t^6+16t^5-20t^4+24t^3-24t^2+27t-24$	4 / ✘	3 / ✘		10^a_3	$13-6t$	$11t-28$	1 / ✔	2 / ✘
	10^a_4	$-3t^2+7t-7$	$4t^3-8t^2+t+8$	2 / ✘	2 / ✘		10^a_5	$t^4-3t^3+5t^2-5t+5$	$-2t^7+8t^6-20t^5+28t^4-36t^3+36t^2-39t+36$	4 / ✘	2 / ✘
	10^a_6	$-2t^3+6t^2-7t+7$	$9t^5-36t^4+56t^3-72t^2+81t-84$	3 / ✘	3 / ✘		10^a_7	$-3t^2+11t-15$	$14t^3-72t^2+135t-160$	2 / ✘	1 / ✘
	10^a_8	$-2t^3+5t^2-5t+5$	$7t^5-20t^4+23t^3-28t^2+26t-24$	3 / ✘	2 / ✘		10^a_9	$-t^4+3t^3-5t^2+7t-7$	$-t^7+4t^6-10t^5+20t^4-25t^3+28t^2-28t+28$	4 / ✘	1 / ✘
	10^a_{10}	$3t^2-11t+17$	$-5t^3+24t^2-71t+100$	2 / ✘	1 / ✘		10^a_{11}	$-4t^2+11t-13$	$16t^3-52t^2+68t-72$	2 / ✘	2, 3 / ✘
	10^a_{12}	$2t^3-6t^2+10t-11$	$-5t^5+20t^4-50t^3+72t^2-89t+92$	3 / ✘	2 / ✘		10^a_{13}	$2t^2-13t+23$	$t^3-12t^2+51t-84$	2 / ✘	2 / ✘
	10^a_{14}	$-2t^3+8t^2-12t+13$	$9t^5-52t^4+119t^3-180t^2+225t-236$	3 / ✘	2 / ✘		10^a_{15}	$2t^3-6t^2+9t-9$	$-3t^5+12t^4-24t^3+24t^2-17t+12$	3 / ✘	2 / ✘
	10^a_{16}	$-4t^2+12t-15$	$-16t^3+56t^2-76t+80$	2 / ✘	2 / ✘		10^a_{17}	$t^4-3t^3+5t^2-7t+9$	0	4 / ✘	1 / ✔
	10^a_{18}	$-4t^2+14t-19$	$16t^3-68t^2+121t-140$	2 / ✘	1 / ✘		10^a_{19}	$2t^3-7t^2+11t-11$	$3t^5-16t^4+35t^3-40t^2+30t-24$	3 / ✘	2 / ✘
	10^a_{20}	$-3t^2+9t-11$	$14t^3-56t^2+88t-104$	2 / ✘	2 / ✘		10^a_{21}	$-2t^3+7t^2-9t+9$	$9t^5-44t^4+80t^3-104t^2+121t-124$	3 / ✘	2 / ✘
	10^a_{22}	$-2t^3+6t^2-10t+13$	$-t^5+4t^4-10t^3+24t^2-37t+44$	3 / ✔	2 / ✘		10^a_{23}	$2t^3-7t^2+13t-15$	$-5t^5+24t^4-67t^3+108t^2-137t+144$	3 / ✘	1 / ✘
	10^a_{24}	$-4t^2+14t-19$	$24t^3-116t^2+221t-268$	2 / ✘	2 / ✘		10^a_{25}	$-2t^3+8t^2-14t+17$	$9t^5-52t^4+131t^3-232t^2+314t-344$	3 / ✘	2 / ✘
	10^a_{26}	$-2t^3+7t^2-13t+17$	$-t^5+4t^4-10t^3+28t^2-49t+60$	3 / ✘	1 / ✘		10^a_{27}	$2t^3-8t^2+16t-19$	$5t^5-28t^4+87t^3-164t^2+229t-252$	3 / ✘	1 / ✘
	10^a_{28}	$4t^2-13t+19$	$-8t^3+36t^2-100t+136$	2 / ✘	2 / ✘		10^a_{29}	$t^3-7t^2+15t-17$	$t^5-12t^4+52t^3-104t^2+124t-128$	3 / ✘	2 / ✘
	10^a_{30}	$-4t^2+17t-25$	$24t^3-148t^2+345t-440$	2 / ✘	1 / ✘		10^a_{31}	$4t^2-14t+21$	$-4t^2+9t-12$	2 / ✘	1 / ✘
	10^a_{32}	$-2t^3+8t^2-15t+19$	$t^5-4t^4+13t^3-40t^2+78t-96$	3 / ✘	1 / ✘		10^a_{33}	$4t^2-16t+25$	0	2 / ✘	1 / ✔
	10^a_{34}	$3t^2-9t+13$	$-5t^3+20t^2-52t+68$	2 / ✘	2 / ✘		10^a_{35}	$2t^2-12t+21$	$-t^3+12t^2-47t+76$	2 / ✔	2 / ✘
	10^a_{36}	$-3t^2+13t-19$	$14t^3-88t^2+208t-264$	2 / ✘	2 / ✘		10^a_{37}	$4t^2-13t+19$	0	2 / ✘	2 / ✔
	10^a_{38}	$-4t^2+15t-21$	$24t^3-128t^2+270t-336$	2 / ✘	2 / ✘		10^a_{39}	$-2t^3+8t^2-13t+15$	$9t^5-52t^4+125t^3-204t^2+263t-280$	3 / ✘	2 / ✘
	10^a_{40}	$2t^3-8t^2+17t-21$	$-5t^5+28t^4-89t^3+176t^2-258t+288$	3 / ✘	2 / ✘		10^a_{41}	$t^3-7t^2+17t-21$	$t^5-12t^4+54t^3-120t^2+157t-164$	3 / ✘	2 / ✘
	10^a_{42}	$-t^3+7t^2-19t+27$	$2t^3-8t^2+11t-12$	3 / ✔	1 / ✘		10^a_{43}	$-t^3+7t^2-17t+23$	0	3 / ✘	2 / ✔
	10^a_{44}	$t^3-7t^2+19t-25$	$t^5-12t^4+56t^3-140t^2+220t-248$	3 / ✘	1 / ✘		10^a_{45}	$-t^3+7t^2-21t+31$	0	3 / ✘	2 / ✔
	10^a_{46}	$-t^4+3t^3-4t^2+5t-5$	$-3t^7+12t^6-21t^5+34t^4-43t^3+52t^2-55t+56$	4 / ✘	3 / ✘		10^a_{47}	$t^4-3t^3+6t^2-7t+7$	$-2t^7+8t^6-23t^5+38t^4-56t^3+60t^2-68t+64$	4 / ✘	2, 3 / ✘
	10^a_{48}	$t^4-3t^3+6t^2-9t+11$	$t^5-2t^4+2t^3-3t+4$	4 / ✔	2 / ✘		10^a_{49}	$3t^3-8t^2+12t-13$	$30t^5-94t^4+196t^3-292t^2+372t-392$	3 / ✘	3 / ✘
	10^a_{50}	$-2t^3+7t^2-11t+13$	$-9t^5+44t^4-94t^3+150t^2-186t+200$	3 / ✘	2 / ✘		10^a_{51}	$2t^3-7t^2+15t-19$	$-5t^5+24t^4-73t^3+134t^2-194t+212$	3 / ✘	2, 3 / ✘
	10^a_{52}	$2t^3-7t^2+13t-15$	$-3t^5+16t^4-37t^3+50t^2-49t+44$	3 / ✘	2 / ✘		10^a_{53}	$6t^2-18t+25$	$93t^3-346t^2+680t-828$	2 / ✘	2, 3 / ✘





n_k^t	Alexander's A_+ / Today's / Rozansky's ρ_1^+	genus / ribbon; unknotting number / amphicheiral
10^a_{54}	$2t^3 - 6t^2 + 10t - 11$	3 / ✘
	$-3t^5 + 12t^4 - 24t^3 + 26t^2 - 21t + 16$	2, 3 / ✘
10^a_{55}	$5t^2 - 15t + 21$	2 / ✘
	$66t^3 - 246t^2 + 488t - 596$	2 / ✘
10^a_{56}	$-2t^3 + 8t^2 - 14t + 17$	3 / ✘
	$-9t^5 + 52t^4 - 133t^3 + 234t^2 - 312t + 340$	2 / ✘
10^a_{57}	$2t^3 - 8t^2 + 18t - 23$	3 / ✘
	$-5t^5 + 28t^4 - 93t^3 + 194t^2 - 300t + 340$	2 / ✘
10^a_{58}	$3t^2 - 16t + 27$	2 / ✘
	$3t^3 - 28t^2 + 94t - 140$	2 / ✘
10^a_{59}	$t^3 - 7t^2 + 18t - 23$	3 / ✘
	$-t^5 + 12t^4 - 55t^3 + 128t^2 - 181t + 196$	1 / ✘
10^a_{60}	$-t^3 + 7t^2 - 20t + 29$	3 / ✘
	$5t^3 - 40t^2 + 122t - 176$	1 / ✘
10^a_{61}	$-2t^3 + 5t^2 - 6t + 7$	3 / ✘
	$-7t^5 + 20t^4 - 27t^3 + 36t^2 - 35t + 36$	2, 3 / ✘
10^a_{62}	$t^4 - 3t^3 + 6t^2 - 8t + 9$	4 / ✘
	$-2t^7 + 8t^6 - 23t^5 + 40t^4 - 63t^3 + 76t^2 - 89t + 88$	2 / ✘
10^a_{63}	$5t^2 - 14t + 19$	2 / ✘
	$66t^3 - 220t^2 + 416t - 496$	2 / ✘
10^a_{64}	$-t^4 + 3t^3 - 6t^2 + 10t - 11$	4 / ✘
	$-t^7 + 4t^6 - 11t^5 + 24t^4 - 37t^3 + 52t^2 - 60t + 64$	2 / ✘
10^a_{65}	$2t^3 - 7t^2 + 14t - 17$	3 / ✘
	$-5t^5 + 24t^4 - 71t^3 + 124t^2 - 169t + 180$	2 / ✘
10^a_{66}	$3t^3 - 9t^2 + 16t - 19$	3 / ✘
	$30t^5 - 112t^4 + 279t^3 - 480t^2 + 662t - 724$	3 / ✘
10^a_{67}	$-4t^2 + 16t - 23$	2 / ✘
	$24t^3 - 140t^2 + 312t - 392$	2 / ✘
10^a_{68}	$4t^2 - 14t + 21$	2 / ✘
	$8t^3 - 40t^2 + 117t - 164$	2 / ✘
10^a_{69}	$t^3 - 7t^2 + 21t - 29$	3 / ✘
	$-t^5 + 12t^4 - 68t^3 + 212t^2 - 397t + 476$	2 / ✘
10^a_{70}	$t^3 - 7t^2 + 16t - 19$	3 / ✘
	$-t^5 + 12t^4 - 53t^3 + 114t^2 - 146t + 152$	2 / ✘
10^a_{71}	$-t^3 + 7t^2 - 18t + 25$	3 / ✘
	$t^3 - 2t^2 - t + 4$	1 / ✘
10^a_{72}	$-2t^3 + 9t^2 - 16t + 19$	3 / ✘
	$-9t^5 + 60t^4 - 167t^3 + 298t^2 - 410t + 448$	2 / ✘
10^a_{73}	$t^3 - 7t^2 + 20t - 27$	3 / ✘
	$t^5 - 12t^4 + 65t^3 - 194t^2 + 350t - 416$	1 / ✘
10^a_{74}	$-4t^2 + 16t - 23$	2 / ✘
	$24t^3 - 136t^2 + 290t - 360$	2 / ✘
10^a_{75}	$-t^3 + 7t^2 - 19t + 27$	3 / ✔ (ribbon)
	$-4t^3 + 36t^2 - 117t + 172$	2 / ✘
10^a_{76}	$-2t^3 + 7t^2 - 12t + 15$	3 / ✘
	$-9t^5 + 44t^4 - 104t^3 + 184t^2 - 245t + 272$	2, 3 / ✘
10^a_{77}	$2t^3 - 7t^2 + 14t - 17$	3 / ✘
	$-5t^5 + 24t^4 - 71t^3 + 132t^2 - 189t + 208$	2, 3 / ✘
10^a_{78}	$-t^3 + 7t^2 - 16t + 21$	3 / ✘
	$2t^5 - 24t^4 + 105t^3 - 244t^2 + 390t - 448$	2 / ✘
10^a_{79}	$t^4 - 3t^3 + 7t^2 - 12t + 15$	4 / ✘
	0	2, 3 / ✔ (amphicheiral)
10^a_{80}	$3t^3 - 9t^2 + 15t - 17$	3 / ✘
	$30t^5 - 112t^4 + 260t^3 - 426t^2 + 568t - 616$	3 / ✘
10^a_{81}	$-t^3 + 8t^2 - 20t + 27$	3 / ✘
	0	2 / ✔ (amphicheiral)
10^a_{82}	$-t^4 + 4t^3 - 8t^2 + 12t - 13$	4 / ✘
	$t^7 - 6t^6 + 19t^5 - 42t^4 + 64t^3 - 78t^2 + 84t - 84$	1 / ✘
10^a_{83}	$2t^3 - 9t^2 + 19t - 23$	3 / ✘
	$-5t^5 + 34t^4 - 110t^3 + 214t^2 - 301t + 332$	2 / ✘
10^a_{84}	$2t^3 - 9t^2 + 20t - 25$	3 / ✘
	$-5t^5 + 34t^4 - 116t^3 + 246t^2 - 373t + 424$	1 / ✘
10^a_{85}	$t^4 - 4t^3 + 8t^2 - 10t + 11$	4 / ✘
	$2t^7 - 12t^6 + 36t^5 - 68t^4 + 101t^3 - 124t^2 + 138t - 140$	2 / ✘
10^a_{86}	$-2t^3 + 9t^2 - 19t + 25$	3 / ✘
	$-t^5 + 6t^4 - 21t^3 + 58t^2 - 105t + 128$	2 / ✘
10^a_{87}	$-2t^3 + 9t^2 - 18t + 23$	3 / ✔ (ribbon)
	$-t^5 + 6t^4 - 23t^3 + 66t^2 - 125t + 152$	2 / ✘
10^a_{88}	$-t^3 + 8t^2 - 24t + 35$	3 / ✘
	0	1 / ✔ (amphicheiral)
10^a_{89}	$t^3 - 8t^2 + 24t - 33$	3 / ✘
	$t^5 - 14t^4 + 83t^3 - 264t^2 + 495t - 596$	2 / ✘
10^a_{90}	$-2t^3 + 8t^2 - 17t + 23$	3 / ✘
	$-t^5 + 6t^4 - 21t^3 + 54t^2 - 93t + 112$	2 / ✘
10^a_{91}	$t^4 - 4t^3 + 9t^2 - 14t + 17$	4 / ✘
	$t^5 - 2t^4 + 2t^3 - 3t + 4$	1 / ✘
10^a_{92}	$-2t^3 + 10t^2 - 20t + 25$	3 / ✘
	$-9t^5 + 68t^4 - 216t^3 + 428t^2 - 622t + 696$	2 / ✘
10^a_{93}	$2t^3 - 8t^2 + 15t - 17$	3 / ✘
	$3t^5 - 18t^4 + 43t^3 - 58t^2 + 55t - 48$	2 / ✘
10^a_{94}	$-t^4 + 4t^3 - 9t^2 + 14t - 15$	4 / ✘
	$-t^7 + 6t^6 - 20t^5 + 46t^4 - 76t^3 + 102t^2 - 115t + 120$	2 / ✘
10^a_{95}	$2t^3 - 9t^2 + 21t - 27$	3 / ✘
	$-5t^5 + 32t^4 - 114t^3 + 248t^2 - 384t + 436$	1 / ✘
10^a_{96}	$-t^3 + 7t^2 - 22t + 33$	3 / ✘
	$-7t^3 + 50t^2 - 147t + 212$	2 / ✘
10^a_{97}	$-5t^2 + 22t - 33$	2 / ✘
	$-37t^3 + 242t^2 - 603t + 788$	2 / ✘
10^a_{98}	$-2t^3 + 9t^2 - 18t + 23$	3 / ✘
	$9t^5 - 60t^4 + 177t^3 - 348t^2 + 501t - 564$	2 / ✘
10^a_{99}	$t^4 - 4t^3 + 10t^2 - 16t + 19$	4 / ✔ (ribbon)
	0	2 / ✔ (amphicheiral)
10^a_{100}	$t^4 - 4t^3 + 9t^2 - 12t + 13$	4 / ✘
	$2t^7 - 12t^6 + 39t^5 - 80t^4 + 128t^3 - 164t^2 + 192t - 196$	2, 3 / ✘
10^a_{101}	$7t^2 - 21t + 29$	2 / ✘
	$-129t^3 + 480t^2 - 942t + 1148$	2, 3 / ✘
10^a_{102}	$-2t^3 + 8t^2 - 16t + 21$	3 / ✘
	$-t^5 + 6t^4 - 19t^3 + 50t^2 - 89t + 108$	1 / ✘
10^a_{103}	$2t^3 - 8t^2 + 17t - 21$	3 / ✘
	$5t^5 - 30t^4 + 93t^3 - 178t^2 + 254t - 280$	3 / ✘
10^a_{104}	$t^4 - 4t^3 + 9t^2 - 15t + 19$	4 / ✘
	$t^5 - 2t^4 + 2t^3 - 3t + 4$	1 / ✘
10^a_{105}	$t^3 - 8t^2 + 22t - 29$	3 / ✘
	$-t^5 + 14t^4 - 71t^3 + 184t^2 - 292t + 332$	2 / ✘
10^a_{106}	$-t^4 + 4t^3 - 9t^2 + 15t - 17$	4 / ✘
	$-t^7 + 6t^6 - 20t^5 + 48t^4 - 82t^3 + 114t^2 - 134t + 140$	2 / ✘
10^a_{107}	$-t^3 + 8t^2 - 22t + 31$	3 / ✘
	$2t^3 - 8t^2 + 13t - 16$	1 / ✘
10^a_{108}	$2t^3 - 8t^2 + 14t - 15$	3 / ✘
	$-3t^5 + 18t^4 - 41t^3 + 50t^2 - 40t + 32$	2 / ✘
10^a_{109}	$t^4 - 4t^3 + 10t^2 - 17t + 21$	4 / ✘
	0	2 / ✔ (amphicheiral)
10^a_{110}	$t^3 - 8t^2 + 20t - 25$	3 / ✘
	$t^5 - 14t^4 + 69t^3 - 160t^2 + 219t - 236$	2 / ✘
10^a_{111}	$-2t^3 + 9t^2 - 17t + 21$	3 / ✘
	$-9t^5 + 60t^4 - 171t^3 + 316t^2 - 436t + 480$	2 / ✘

diagram	n_k^t	Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral	diagram	n_k^t	Alexander's A_+ Today's / Rozansky's ρ_1^+	genus / ribbon unknotting number / amphicheiral
	10^a_{112}	$-t^4+5t^3-11t^2+17t-19$ $t^7-8t^6+29t^5-68t^4+115t^3-152t^2+175t-180$	4 / ✘ 2 / ✘		10^a_{113}	$2t^3-11t^2+26t-33$ $-5t^5+42t^4-167t^3+394t^2-623t+720$	3 / ✘ 1 / ✘
	10^a_{114}	$-2t^3+10t^2-21t+27$ $t^5-8t^4+30t^3-78t^2+140t-168$	3 / ✘ 1 / ✘		10^a_{115}	$-t^3+9t^2-26t+37$ 0	3 / ✘ 2 / ✔
	10^a_{116}	$-t^4+5t^3-12t^2+19t-21$ $t^7-8t^6+30t^5-74t^4+132t^3-184t^2+217t-228$	4 / ✘ 2 / ✘		10^a_{117}	$2t^3-10t^2+24t-31$ $-5t^5+38t^4-144t^3+330t^2-522t+600$	3 / ✘ 2 / ✘
	10^a_{118}	$t^4-5t^3+12t^2-19t+23$ 0	4 / ✘ 1 / ✔		10^a_{119}	$-2t^3+10t^2-23t+31$ $-t^5+6t^4-26t^3+86t^2-175t+220$	3 / ✘ 1 / ✘
	10^a_{120}	$8t^2-26t+37$ $166t^3-692t^2+1433t-1788$	2 / ✘ 2, 3 / ✘		10^a_{121}	$2t^3-11t^2+27t-35$ $5t^5-42t^4+167t^3-396t^2+634t-732$	3 / ✘ 2 / ✘
	10^a_{122}	$-2t^3+11t^2-24t+31$ $-t^5+8t^4-34t^3+104t^2-211t+264$	3 / ✘ 2 / ✘		10^a_{123}	$t^4-6t^3+15t^2-24t+29$ 0	4 / ✔ 2 / ✔
	10^n_{124}	t^4-t^3+t-1 $-4t^7-6t^4-4t^2-6t$	4 / ✘ 4 / ✘		10^n_{125}	t^3-2t^2+2t-1 $-t^5+2t^4-2t^3+3t-4$	3 / ✘ 2 / ✘
	10^n_{126}	t^3-2t^2+4t-5 $t^5-2t^4+10t^3-12t^2+22t-20$	3 / ✘ 2 / ✘		10^n_{127}	$-t^3+4t^2-6t+7$ $2t^5-14t^4+32t^3-52t^2+67t-72$	3 / ✘ 2 / ✘
	10^n_{128}	$2t^3-3t^2+t+1$ $-13t^5+12t^4-3t^3-10t^2-9t+12$	3 / ✘ 3 / ✘		10^n_{129}	$2t^2-6t+9$ $-t^3-2t^2+14t-20$	2 / ✔ 1 / ✘
	10^n_{130}	$2t^2-4t+5$ $t^3-2t^2+19t-24$	2 / ✘ 2 / ✘		10^n_{131}	$-2t^2+8t-11$ $5t^3-38t^2+87t-112$	2 / ✘ 1 / ✘
	10^n_{132}	t^2-t+1 $2t^2+5t-4$	2 / ✘ 1 / ✘		10^n_{133}	$-t^2+5t-7$ $t^3-14t^2+37t-48$	2 / ✘ 1 / ✘
	10^n_{134}	$2t^3-4t^2+4t-3$ $-13t^5+24t^4-33t^3+30t^2-41t+40$	3 / ✘ 3 / ✘		10^n_{135}	$3t^2-9t+13$ $t^3-6t^2+18t-24$	2 / ✘ 2 / ✘
	10^n_{136}	$-t^2+4t-5$ $-t^3+4t^2-2t-4$	2 / ✘ 1 / ✘		10^n_{137}	$t^2-6t+11$ $-4t^2+24t-44$	2 / ✔ 1 / ✘
	10^n_{138}	t^3-5t^2+8t-7 $-t^5+8t^4-22t^3+24t^2-11t+8$	3 / ✘ 2 / ✘		10^n_{139}	t^4-t^3+2t-3 $-4t^7-12t^4+5t^3-4t^2-16t+12$	4 / ✘ 4 / ✘
	10^n_{140}	t^2-2t+3 $8t-8$	2 / ✔ 2 / ✘		10^n_{141}	$-t^3+3t^2-4t+5$ $t^3-8t^2+16t-20$	3 / ✘ 1 / ✘
	10^n_{142}	$2t^3-3t^2+2t-1$ $-13t^5+12t^4-13t^3+4t^2-17t+12$	3 / ✘ 3 / ✘		10^n_{143}	t^3-3t^2+6t-7 $t^5-4t^4+15t^3-28t^2+45t-48$	3 / ✘ 1 / ✘
	10^n_{144}	$-3t^2+10t-13$ $10t^3-44t^2+80t-96$	2 / ✘ 2 / ✘		10^n_{145}	t^2+t-3 $2t^3+8t^2+6t-8$	2 / ✘ 2 / ✘
	10^n_{146}	$2t^2-8t+13$ $t^3-8t^2+21t-28$	2 / ✘ 1 / ✘		10^n_{147}	$-2t^2+7t-9$ $-3t^3+12t^2-15t+12$	2 / ✘ 1 / ✘
	10^n_{148}	t^3-3t^2+7t-9 $t^5-4t^4+18t^3-36t^2+62t-68$	3 / ✘ 2 / ✘		10^n_{149}	$-t^3+5t^2-9t+11$ $2t^5-18t^4+55t^3-104t^2+149t-164$	3 / ✘ 2 / ✘
	10^n_{150}	$-t^3+4t^2-6t+7$ $-2t^5+12t^4-26t^3+38t^2-45t+44$	3 / ✘ 2 / ✘		10^n_{151}	$t^3-4t^2+10t-13$ $-t^5+6t^4-21t^3+42t^2-66t+72$	3 / ✘ 2 / ✘
	10^n_{152}	$t^4-t^3-t^2+4t-5$ $4t^7-7t^5+18t^4-7t^3-12t^2+45t-52$	4 / ✘ 4 / ✘		10^n_{153}	t^3-t^2-t+3 $t^5-2t^4+t^3+2t^2-t$	3 / ✔ 2 / ✘
	10^n_{154}	t^3-4t+7 $-3t^5-6t^4+13t^3-47t+68$	3 / ✘ 3 / ✘		10^n_{155}	$-t^3+3t^2-5t+7$ $-2t^3+12t^2-22t+28$	3 / ✔ 2 / ✘
	10^n_{156}	t^3-4t^2+8t-9 $t^5-6t^4+19t^3-30t^2+33t-32$	3 / ✘ 1 / ✘		10^n_{157}	$-t^3+6t^2-11t+13$ $-2t^5+22t^4-78t^3+148t^2-218t+240$	3 / ✘ 2 / ✘
	10^n_{158}	$-t^3+4t^2-10t+15$ $2t^2-7t+12$	3 / ✘ 2 / ✘		10^n_{159}	$t^3-4t^2+9t-11$ $t^5-6t^4+26t^3-60t^2+98t-112$	3 / ✘ 1 / ✘
	10^n_{160}	$-t^3+4t^2-4t+3$ $-2t^5+12t^4-20t^3+14t^2-16t+12$	3 / ✘ 2 / ✘		10^n_{161}	t^3-2t+3 $3t^5+6t^4-3t^3+4t^2+14t-12$	3 / ✘ 3 / ✘
	10^n_{162}	$-3t^2+9t-11$ $10t^3-38t^2+58t-68$	2 / ✘ 2 / ✘		10^n_{163}	$t^3-5t^2+12t-15$ $-t^5+8t^4-30t^3+62t^2-89t+96$	3 / ✘ 1, 2 / ✘
	10^n_{164}	$3t^2-11t+17$ $t^3-10t^2+29t-40$	2 / ✘ 1 / ✘		10^n_{165}	$-2t^2+10t-15$ $-5t^3+50t^2-146t+196$	2 / ✘ 2 / ✘