In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\ICERM-2305"];
<< Signatures`
```

In[∘]:= $\left\{\left(\left(2\,u-\omega+3\,\omega^{-1}\right)\bar\eta_1\,\eta_2\right)^*,\ \left(\eta_1\to\omega\,\eta_2\right)^+\right\}$

Out[∘]=

$$\left\{\left(2\,u-\frac{1}{\omega}+3\,\omega\right)\eta_1\,\bar\eta_2,\ \left\{\eta_1\to\omega\,\eta_2,\ \bar\eta_1\to\frac{\bar\eta_2}{\omega}\right\}\right\}$$

In[∘]:= $\mathrm{CF}[\{\eta_1-\eta_2,\ \eta_1-\eta_3\}]$

Out[∘]=

$$\{\eta_1-\eta_3,\ \eta_2-\eta_3\}$$

In[∘]:= $\mathrm{RulesOf}[\eta_1+\eta_2+\eta_3]$

Out[∘]=

$$\{\eta_1\to-\eta_2-\eta_3,\ \bar\eta_1\to-\bar\eta_2-\bar\eta_3\}$$

In[∘]:= $\Sigma_{B[\{-1,2\}]}[0,\mathrm{PQ}[\{\},0]]\bigcup\Sigma_{B[\{-3,4\}]}[0,\mathrm{PQ}[\{\},0]]$ // $\mathrm{FM}_{-1,4}$

Out[∘]=

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & 1 & 0 & -1 \\ & (\eta_{-3} & \eta_{-1} & \eta_2 & \eta_4) \\ \bar\eta_{-3} & 0 & 0 & 0 & 0 \\ \bar\eta_{-1} & 0 & 0 & 0 & 0 \\ \bar\eta_2 & 0 & 0 & 0 & 0 \\ \bar\eta_4 & 0 & 0 & 0 & 0 \end{array}$$

In[∘]:= Kas /@ $\{X_+[1,2,3,4],\ X_-[1,4,3,2]\}$

Out[∘]=

$$\left\{\begin{array}{ccccc} & & -1 & & \\ & (\eta_1 & \eta_2 & \eta_3 & \eta_4) \\ \bar\eta_1 & 2u^2-1 & u & 1 & u \\ \bar\eta_2 & u & 1 & u & 1 \\ \bar\eta_3 & 1 & u & 2u^2-1 & u \\ \bar\eta_4 & u & 1 & u & 1 \end{array},\ \begin{array}{ccccc} & & 1 & & \\ & (\eta_1 & \eta_4 & \eta_3 & \eta_2) \\ \bar\eta_1 & 1-2u^2 & -u & -1 & -u \\ \bar\eta_4 & -u & -1 & -u & -1 \\ \bar\eta_3 & -1 & -u & 1-2u^2 & -u \\ \bar\eta_2 & -u & -1 & -u & -1 \end{array}\right\}$$

In[∘]:= TL /@ $\{X[1,2,3,4],\ X[1,4,3,2]\}$

Out[∘]=

$$\left\{\begin{array}{ccccc} & & 0 & & \\ & (\eta_{-4} & \eta_{-1} & \eta_2 & \eta_3) \\ \bar\eta_{-4} & 0 & 1-\omega & 0 & \omega-1 \\ \bar\eta_{-1} & \frac{\omega-1}{\omega} & \frac{(\omega-1)^2}{\omega} & \omega-1 & -2\,(\omega-1) \\ \bar\eta_2 & 0 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} \\ \bar\eta_3 & -\frac{\omega-1}{\omega} & \frac{2\,(\omega-1)}{\omega} & 1-\omega & \frac{(\omega-1)^2}{\omega} \end{array},\ \begin{array}{ccccc} & & 0 & & \\ & (\eta_{-4} & \eta_3 & \eta_2 & \eta_{-1}) \\ \bar\eta_{-4} & -\frac{(\omega-1)^2}{\omega} & \omega-1 & -\frac{2\,(\omega-1)}{\omega} & \frac{\omega-1}{\omega} \\ \bar\eta_3 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 \\ \bar\eta_2 & 2\,(\omega-1) & 1-\omega & -\frac{(\omega-1)^2}{\omega} & -\frac{\omega-1}{\omega} \\ \bar\eta_{-1} & 1-\omega & 0 & \omega-1 & 0 \end{array}\right\}$$

In[∘]:= **Kas[Knot[3, 1]]**

Out[∘]=

$$4\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-4\,\theta\left(u-\frac{\sqrt{3}}{2}\right)$$

In[∘]:= **KasSig[Knot[3, 1]]**

Out[∘]=

$$-2\,\theta\left[-\frac{\sqrt{3}}{2}+\mathrm{u}\right]+2\,\theta\left[\frac{\sqrt{3}}{2}+\mathrm{u}\right]$$

In[∘]:= **TLSig[Knot[3, 1]]**

Out[∘]=

$$-2\,\theta\left[-\frac{\sqrt{3}}{2}+\mathrm{u}\right]+2\,\theta\left[\frac{\sqrt{3}}{2}+\mathrm{u}\right]$$

In[∘]:= **Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]]**

Out[∘]=

$$
\begin{array}{c|cccccccc}
 & & & & 0 & & & & \\
 & (\eta_{-5} & \eta_{3} & \eta_{6} & \eta_{-2}) & (\eta_{-4} & \eta_{-1} & \eta_{5} & \eta_{2}) \\
\overline{\eta}_{-5} & 1-2u^2 & -u & -1 & -u & 0 & 0 & 0 & 0 \\
\overline{\eta}_{3} & -u & -1 & -u & -1 & 0 & 0 & 0 & 0 \\
\overline{\eta}_{6} & -1 & -u & 1-2u^2 & -u & 0 & 0 & 0 & 0 \\
\overline{\eta}_{-2} & -u & -1 & -u & -1 & 0 & 0 & 0 & 0 \\
\overline{\eta}_{-4} & 0 & 0 & 0 & 0 & 1 & u & 1 & u \\
\overline{\eta}_{-1} & 0 & 0 & 0 & 0 & u & 2u^2-1 & u & 1 \\
\overline{\eta}_{5} & 0 & 0 & 0 & 0 & 1 & u & 1 & u \\
\overline{\eta}_{2} & 0 & 0 & 0 & 0 & u & 1 & u & 2u^2-1
\end{array}
$$

In[∘]:= **Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // $\mathrm{FM}_{5,-2}$**

Out[∘]=

$$
\begin{array}{c|cccccccc}
 & & & & 0 & & & & \\
 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
 & (\eta_{-5} & \eta_{3} & \eta_{6} & \eta_{5} & \eta_{2} & \eta_{-4} & \eta_{-1} & \eta_{-2}) \\
\overline{\eta}_{-5} & 1-2u^2 & -u & -1 & -u & 0 & 0 & 0 & 0 \\
\overline{\eta}_{3} & -u & -1 & -u & -1 & 0 & 0 & 0 & 0 \\
\overline{\eta}_{6} & -1 & -u & 1-2u^2 & -u & 0 & 0 & 0 & 0 \\
\overline{\eta}_{5} & -u & -1 & -u & 0 & u & 1 & u & 0 \\
\overline{\eta}_{2} & 0 & 0 & 0 & u & 2u^2-1 & u & 1 & 0 \\
\overline{\eta}_{-4} & 0 & 0 & 0 & 1 & u & 1 & u & 0 \\
\overline{\eta}_{-1} & 0 & 0 & 0 & u & 1 & u & 2u^2-1 & 0 \\
\overline{\eta}_{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

In[∘]:=
```
Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // FM_{5,-2} // Cordon_{-2}
```

Out[∘]=

$$0$$

	η_{-5}	η_3	η_6	η_5	η_2	η_{-4}
$\overline{\eta}_{-5}$	0	$-u$	-1	0	1	u
$\overline{\eta}_3$	$-u$	-1	$-u$	-1	0	0
$\overline{\eta}_6$	-1	$-u$	$1-2u^2$	$-u$	0	0
$\overline{\eta}_5$	0	-1	$-u$	0	u	1
$\overline{\eta}_2$	1	0	0	u	$2u^2-1$	u
$\overline{\eta}_{-4}$	u	0	0	1	u	1

In[∘]:=
```
Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // FM_{-2,5} // Cordon_5 // Cordon_{-2}
```

Out[∘]=

$$0$$

	0	-1	0	1
	η_{-5}	η_3	η_2	η_{-4}
$\overline{\eta}_{-5}$	0	0	0	0
$\overline{\eta}_3$	0	0	0	0
$\overline{\eta}_2$	0	0	0	0
$\overline{\eta}_{-4}$	0	0	0	0

In[∘]:=
```
lhs = Kas[X[4, 2, 5, 1]] ⋃ Kas[X[7, 3, 8, 2]] ⋃ Kas[X[8, 6, 9, 5]] // mc;
rhs = Kas[X[7, 5, 8, 4]] ⋃ Kas[X[8, 2, 9, 1]] ⋃ Kas[X[5, 3, 6, 2]] // mc
{lhs⟦1⟧, rhs⟦1⟧}
Simplify[lhs⟦2, 2⟧ == rhs⟦2, 2⟧]
```

Out[∘]=

$$2\,\Theta\left(u-\frac{1}{2}\right)-2\,\Theta\left(u+\frac{1}{2}\right)-2$$

	η_{-7}	η_3	η_6	η_9	η_{-1}	η_{-4}
$\overline{\eta}_{-7}$	$\frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u\ (4\ u}{(2\ u-1)}$
$\overline{\eta}_3$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2(2u^2-1)}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2\ u-1)}$
$\overline{\eta}_6$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2}{(2\ u-1)}$
$\overline{\eta}_9$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2u^2(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2\ u-1)}$
$\overline{\eta}_{-1}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2(2u^2-1)}{(2u-1)(2u+1)}$	$\frac{u\ (4\ u}{(2\ u-1)}$
$\overline{\eta}_{-4}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$-\frac{2u}{(2u-1)(2u+1)}$	$-\frac{1}{(2u-1)(2u+1)}$	$\frac{u(4u^2-3)}{(2u-1)(2u+1)}$	$\frac{2\ u^2\ (4}{(2\ u-1)}$

Out[∘]=

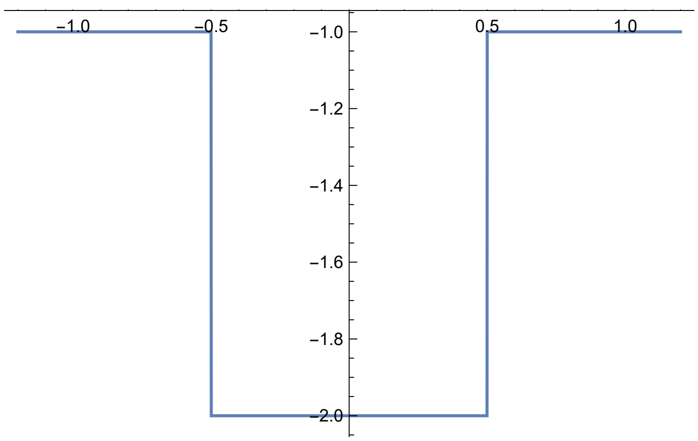
$$\left\{-2+2\,\Theta\left[-\frac{1}{2}+\mathrm{u}\right]-2\,\Theta\left[\frac{1}{2}+\mathrm{u}\right],\ -2+2\,\Theta\left[-\frac{1}{2}+\mathrm{u}\right]-2\,\Theta\left[\frac{1}{2}+\mathrm{u}\right]\right\}$$

Out[∘]=

True

In[∘]:=
```
f = KasSig@PD[X[4, 2, 5, 1], X[7, 3, 8, 2], X[8, 6, 9, 5]];
Plot[f, {u, -1.2, 1.2}]
```

Out[∘]=



In[∘]:=
```
lhs = TL[X[4, 2, 5, 1]] ⋃ TL[X[7, 3, 8, 2]] ⋃ TL[X[8, 6, 9, 5]] // mc;
rhs = TL[X[7, 5, 8, 4]] ⋃ TL[X[8, 2, 9, 1]] ⋃ TL[X[5, 3, 6, 2]] // mc
{lhs⟦1⟧, rhs⟦1⟧}
lhs⟦2, 2⟧ == rhs⟦2, 2⟧
```

Out[∘]=

$$2\,\Theta(u-1) - 2\,\Theta(u+1) + 1$$

$$\begin{array}{c|cccccc}
 & (\eta_{-7} & \eta_3 & \eta_6 & \eta_9 & \eta_{-1} & \eta_{-4}) \\
\hline
\overline{\eta}_{-7} & \frac{\omega^2+1}{\omega} & \omega-1 & -2\omega & 2 & 0 & -\frac{\omega+1}{\omega} \\
\overline{\eta}_3 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 & 0 & 0 \\
\overline{\eta}_6 & -\frac{2}{\omega} & 1-\omega & \frac{\omega^2+1}{\omega} & -\frac{\omega+1}{\omega} & 0 & \frac{2}{\omega} \\
\overline{\eta}_9 & 2 & 0 & -\omega-1 & \frac{\omega^2+1}{\omega} & -\frac{\omega-1}{\omega} & -\frac{2}{\omega} \\
\overline{\eta}_{-1} & 0 & 0 & 0 & \omega-1 & 0 & 1-\omega \\
\overline{\eta}_{-4} & -\omega-1 & 0 & 2\omega & -2\omega & \frac{\omega-1}{\omega} & \frac{\omega^2+1}{\omega}
\end{array}$$

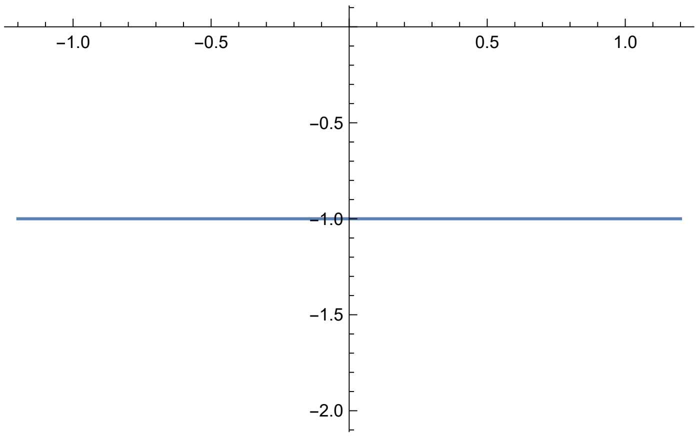
Out[∘]=

$\{1 + 2\,\Theta[-1 + u] - 2\,\Theta[1 + u],\ 1 + 2\,\Theta[-1 + u] - 2\,\Theta[1 + u]\}$

Out[∘]=

True

In[∘]:=
```
f = TLSig@PD[X[4, 2, 5, 1], X[7, 3, 8, 2], X[8, 6, 9, 5]];
Plot[f, {u, -1.2, 1.2}]
```
Out[∘]=



Kashaev for Knots

In[∘]:=
```
-KnotSignature /@ AllKnots[{3, 8}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

{2, 0, 4, 2, 0, 2, 0, 6, 2, -4, -2, 4, 2, 0, 0, 4,
 0, 2, -4, 2, -2, 0, 0, -2, 2, 0, 0, 2, 4, 2, 0, 0, -6, 0, 2}

In[∘]:=
```
(*u=0;*)
Kas[Knot[3, 1]]
Clear[u]
```
Out[∘]=

$$4\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-4\,\theta\left(u-\frac{\sqrt{3}}{2}\right)$$

In[∘]:=

$$\Sigma_{B[]}\left[\mathrm{sign}\left[\frac{1}{2}\left(3-4\,u^2\right)\right]+\mathrm{sign}\left[-2\left(-1+2\,u^2\right)\right]+\mathrm{sign}\left[-\frac{-3+4\,u^2}{-1+2\,u^2}\right],\ \mathrm{PQ}[\{\},0]\right]$$

Out[∘]=

$$-4\,\theta\left(u-\frac{\sqrt{3}}{2}\right)+4\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-3$$

In[∘]:= **Table[K → 2 KasSig[K], {K, AllKnots[{3, 7}]}] // Column**

Out[∘]=

$\text{Knot}[3,1] \to 2\left(-2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]+2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]\right)$

$\text{Knot}[4,1] \to 0$

$\text{Knot}[5,1] \to 2\left(2\,\theta[u-(-0.951\ldots)]+2\,\theta[u-(-0.588\ldots)]-2\,\theta[u-0.588\ldots]-2\,\theta[u-0.951\ldots]\right)$

$\text{Knot}[5,2] \to 2\left(-2\,\theta\left[-\frac{\sqrt{\frac{7}{2}}}{2}+u\right]+2\,\theta\left[\frac{\sqrt{\frac{7}{2}}}{2}+u\right]\right)$

$\text{Knot}[6,1] \to 0$

$\text{Knot}[6,2] \to 2\left(2\,\theta[u-(-0.772\ldots)]-2\,\theta[u-0.772\ldots]\right)$

$\text{Knot}[6,3] \to 0$

$\text{Knot}[7,1] \to 2\left(2\,\theta[u-(-0.975\ldots)]+2\,\theta[u-(-0.782\ldots)]+2\,\theta[u-(-0.434\ldots)]-2\,\theta[u-0.434\ldots]-2\,\theta[u-0.782\ldots]-2\,\theta[u-0.975\ldots]\right)$

$\text{Knot}[7,2] \to 2\left(-2\,\theta\left[-\frac{\sqrt{\frac{11}{3}}}{2}+u\right]+2\,\theta\left[\frac{\sqrt{\frac{11}{3}}}{2}+u\right]\right)$

$\text{Knot}[7,3] \to 2\left(-2\,\theta[u-(-0.972\ldots)]-2\,\theta[u-(-0.656\ldots)]+2\,\theta[u-0.656\ldots]+2\,\theta[u-0.972\ldots]\right)$

$\text{Knot}[7,4] \to 2\left(2\,\theta\left[-\frac{\sqrt{15}}{4}+u\right]-2\,\theta\left[\frac{\sqrt{15}}{4}+u\right]\right)$

$\text{Knot}[7,5] \to 2\left(2\,\theta[u-(-0.963\ldots)]+2\,\theta[u-(-0.757\ldots)]-2\,\theta[u-0.757\ldots]-2\,\theta[u-0.963\ldots]\right)$

$\text{Knot}[7,6] \to 2\left(2\,\theta[u-(-0.920\ldots)]-2\,\theta[u-0.920\ldots]\right)$

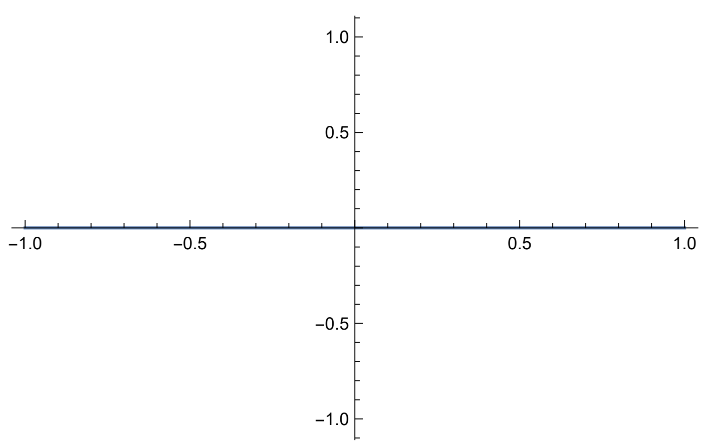
$\text{Knot}[7,7] \to 0$

In[∘]:= **f = KasSig[Knot[10, 1]]**
Plot[f, {u, -1, 1}]

Out[∘]=

0

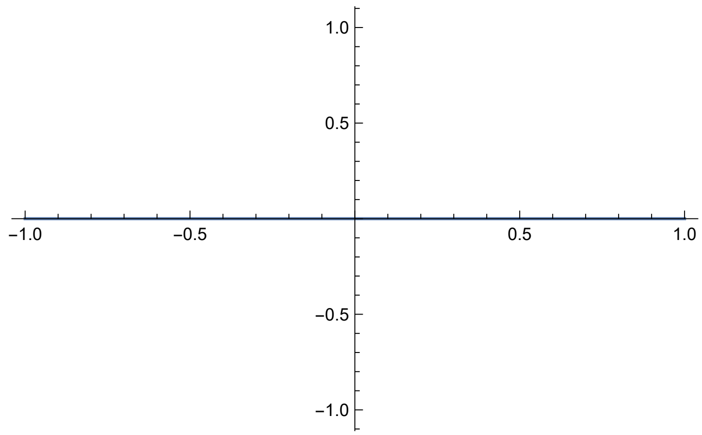
Out[∘]=



In[*]:=
```
f = TLSig[Knot[10, 1]]
Plot[f, {u, -1, 1}]
```
Out[*]=

0

Out[*]=



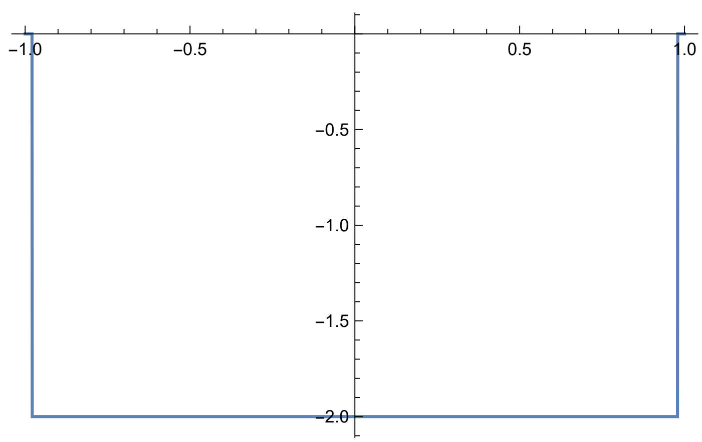
In[*]:=
```
(KasSig /@ AllKnots[{3, 8}]) /. u → 1 / 2
```
Out[*]=

{2, 0, 4, 2, 0, 2, 0, 4, 2, -4, -2, 4, 2, 0, 0, 4, 0, 2, -4, 2, -2, 0, 0, -2, 2, 0, 0, 2, 4, 2, 0, 0, -4, 0, 2}

In[*]:=
```
f = KasSig[Knot[9, 5]]
Plot[f, {u, -1, 1}]
```
Out[*]=

$$2\,\theta\left[-\frac{\sqrt{\frac{23}{6}}}{2}+u\right]-2\,\theta\left[\frac{\sqrt{\frac{23}{6}}}{2}+u\right]$$

Out[*]=

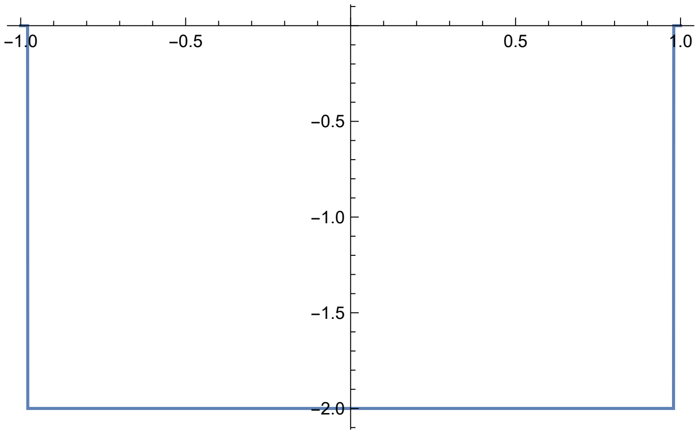


In[∘]:=
```
f = TLSig[Knot[9, 5]]
Plot[f, {u, -1, 1}]
```

Out[∘]=

$$2\,\theta\left[-\frac{\sqrt{\frac{23}{6}}}{2}+u\right]-2\,\theta\left[\frac{\sqrt{\frac{23}{6}}}{2}+u\right]$$

Out[∘]=

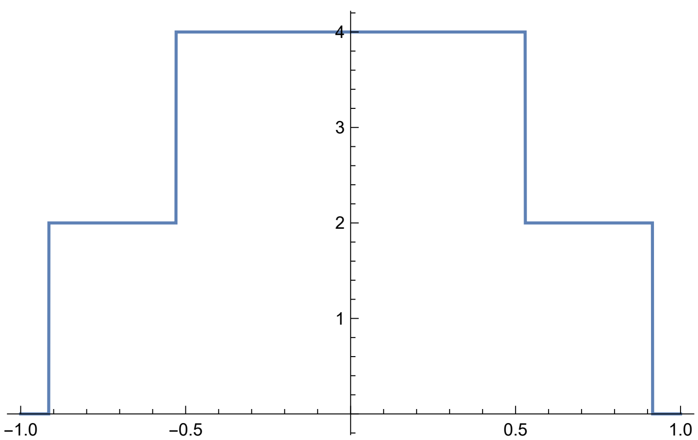


In[∘]:=
```
f = KasSig[Knot[8, 2]]
Plot[f, {u, -1, 1}]
```

Out[∘]=

$$2\,\theta[u-(-0.915\ldots)]+2\,\theta[u-(-0.529\ldots)]-2\,\theta[u-(0.529\ldots)]-2\,\theta[u-(0.915\ldots)]$$

Out[∘]=

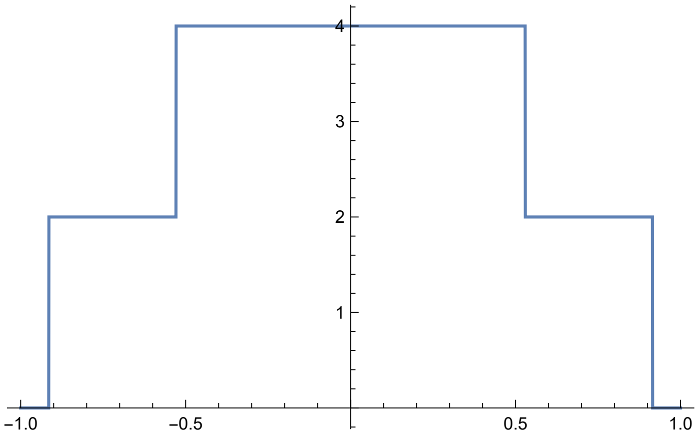


In[∘]:=
```
f = TLSig[Knot[8, 2]]
Plot[f, {u, -1, 1}]
```

Out[∘]=

$$2\,\theta[u - (-0.915\ldots)] + 2\,\theta[u - (-0.529\ldots)] - 2\,\theta[u - 0.529\ldots] - 2\,\theta[u - 0.915\ldots]$$

Out[∘]=



In[∘]:=
```
f = KasSig[Knot[12, Alternating, 422]]
Plot[f, {u, -1, 1}]
```

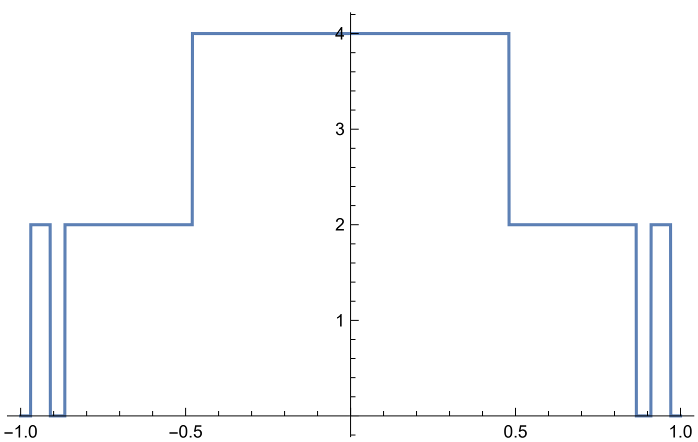
KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[∘]=

$$-2\,\theta\left[-\frac{\sqrt{3}}{2} + u\right] + 2\,\theta\left[\frac{\sqrt{3}}{2} + u\right] + 2\,\theta[u - (-0.970\ldots)] - 2\,\theta[u - (-0.910\ldots)] +$$
$$2\,\theta[u - (-0.480\ldots)] - 2\,\theta[u - 0.480\ldots] + 2\,\theta[u - 0.910\ldots] - 2\,\theta[u - 0.970\ldots]$$

Out[∘]=

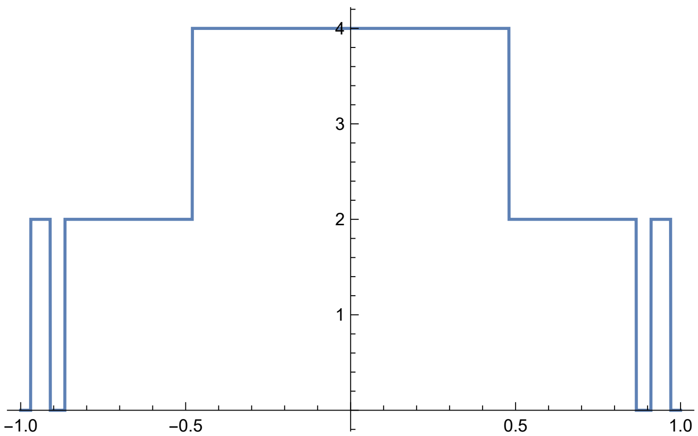


In[∘]:=
```
f = TLSig[Knot[12, Alternating, 422]]
Plot[f, {u, -1, 1}]
```

Out[∘]=

$$-2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]+2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]+2\,\theta\left[u-\boxed{-0.970\ldots}\right]-2\,\theta\left[u-\boxed{-0.910\ldots}\right]+$$
$$2\,\theta\left[u-\boxed{-0.480\ldots}\right]-2\,\theta\left[u-\boxed{0.480\ldots}\right]+2\,\theta\left[u-\boxed{0.910\ldots}\right]-2\,\theta\left[u-\boxed{0.970\ldots}\right]$$

Out[∘]=



Tristram-Levine for Knots

In[∘]:=
```
-KnotSignature /@ AllKnots[{3, 8}]
```

Out[∘]=
```
{2, 0, 4, 2, 0, 2, 0, 6, 2, -4, -2, 4, 2, 0, 0, 4,
 0, 2, -4, 2, -2, 0, 0, -2, 2, 0, 0, 2, 4, 2, 0, 0, -6, 0, 2}
```

In[∘]:=
```
TL[Knot[3, 1]]
```

Out[∘]=

$$2\,\theta\left(u+\frac{\sqrt{3}}{2}\right)-2\,\theta\left(u-\frac{\sqrt{3}}{2}\right)$$

In[∘]:=
```
TLSig /@ AllKnots[{3, 8}] /. u → 0
```

Out[∘]=
```
{2, 0, 4, 2, 0, 2, 0, 6, 2, -4, -2, 4, 2, 0, 0, 4,
 0, 2, -4, 2, -2, 0, 0, -2, 2, 0, 0, 2, 4, 2, 0, 0, -6, 0, 2}
```

In[∘]:=
```
TLSig /@ AllKnots[{3, 8}] /. u → 0.9999
```

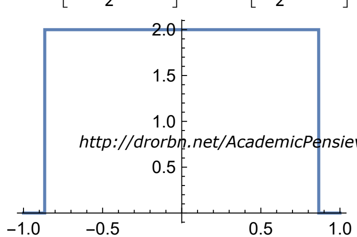
Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

In[∘]:=
```
Table[K → Column[{f = TLSig[K], Plot[f, {u, -1, 1}]}, Center],
  {K, AllKnots[{3, 8}]}] // Column
```

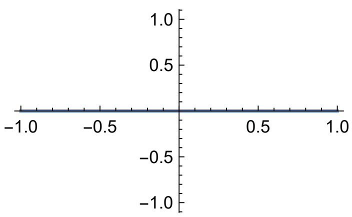
Out[∘]=

$$-2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]+2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]$$

Knot[3, 1] →

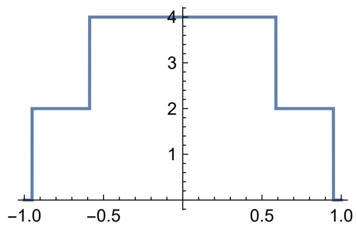


Knot[4, 1] → 0



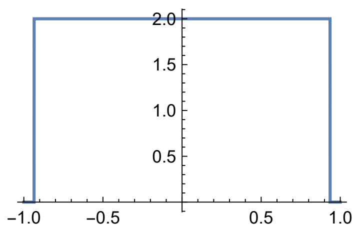
$2\,\theta\left[u - \boxed{-0.951\ldots}\right] + 2\,\theta\left[u - \boxed{-0.588\ldots}\right] - 2\,\theta\left[u - \boxed{0.588\ldots}\right] - 2\,\theta\left[u - \boxed{0.951\ldots}\right]$

Knot[5, 1] →

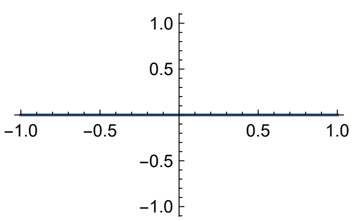


$-2\,\theta\left[-\frac{\sqrt{\frac{7}{2}}}{2} + u\right] + 2\,\theta\left[\frac{\sqrt{\frac{7}{2}}}{2} + u\right]$

Knot[5, 2] →

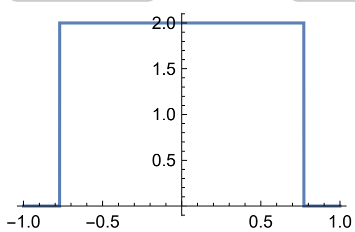


Knot[6, 1] → 0



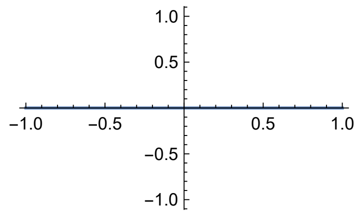
$2\,\theta\left[u - \boxed{-0.772\ldots}\right] - 2\,\theta\left[u - \boxed{0.772\ldots}\right]$

Knot[6, 2] →



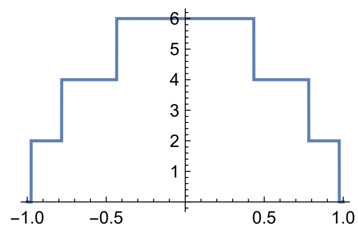
Knot[6, 3] →

0



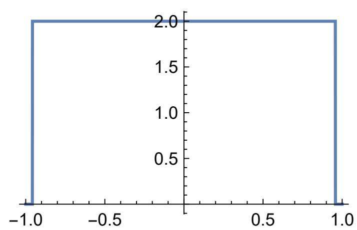
Knot[7, 1] →

$$2\,\theta[u - (-0.975\ldots)] + 2\,\theta[u - (-0.782\ldots)] + 2\,\theta[u - (-0.434\ldots)] - 2\,\theta[u - 0.434\ldots] - 2\,\theta[u - 0.782\ldots] - 2\,\theta[u - 0.975\ldots]$$



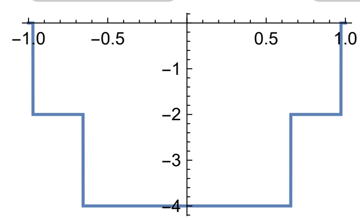
Knot[7, 2] →

$$-2\,\theta\left[-\frac{\sqrt{\frac{11}{3}}}{2} + u\right] + 2\,\theta\left[\frac{\sqrt{\frac{11}{3}}}{2} + u\right]$$

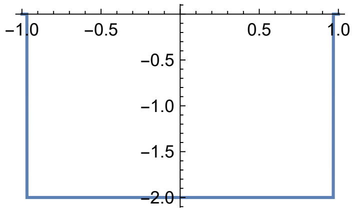


Knot[7, 3] →

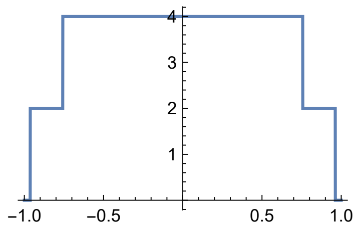
$$-2\,\theta[u - (-0.972\ldots)] - 2\,\theta[u - (-0.656\ldots)] + 2\,\theta[u - 0.656\ldots] + 2\,\theta[u - 0.972\ldots]$$



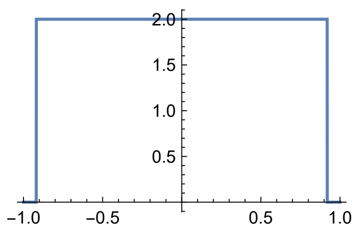
Knot[7, 4] → $2\,\theta\left[-\frac{\sqrt{15}}{4}+u\right]-2\,\theta\left[\frac{\sqrt{15}}{4}+u\right]$



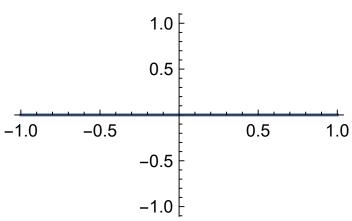
Knot[7, 5] → $2\,\theta[u-(-0.963\ldots)]+2\,\theta[u-(-0.757\ldots)]-2\,\theta[u-0.757\ldots]-2\,\theta[u-0.963\ldots]$



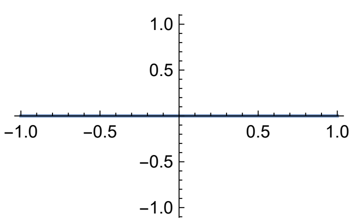
Knot[7, 6] → $2\,\theta[u-(-0.920\ldots)]-2\,\theta[u-0.920\ldots]$



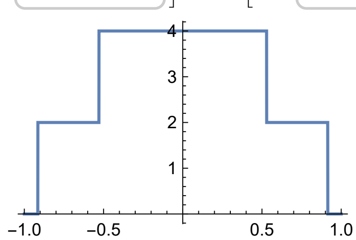
Knot[7, 7] → 0



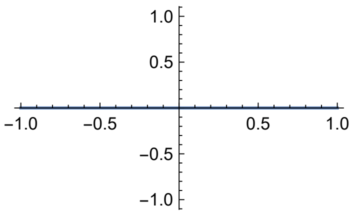
Knot[8, 1] → 0



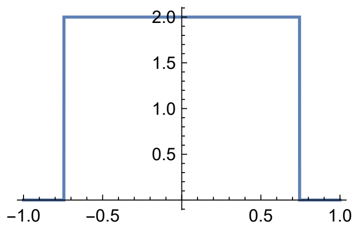
Knot[8, 2] → $2\,\theta[u-(-0.915\ldots)]+2\,\theta[u-(-0.529\ldots)]-2\,\theta[u-0.529\ldots]-2\,\theta[u-0.915\ldots]$



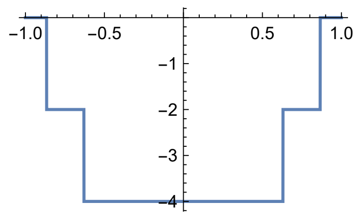
0

Knot[8, 3] → 

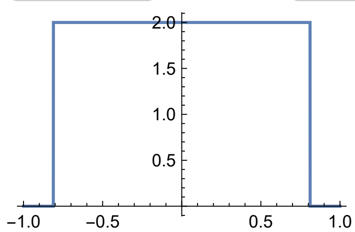
$2\,\theta\left[u - \boxed{-0.745\ldots}\right] - 2\,\theta\left[u - \boxed{0.745\ldots}\right]$

Knot[8, 4] → 

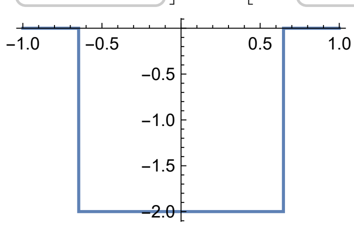
$2\,\theta\left[-\frac{\sqrt{3}}{2} + u\right] - 2\,\theta\left[\frac{\sqrt{3}}{2} + u\right] - 2\,\theta\left[u - \boxed{-0.630\ldots}\right] + 2\,\theta\left[u - \boxed{0.630\ldots}\right]$

Knot[8, 5] → 

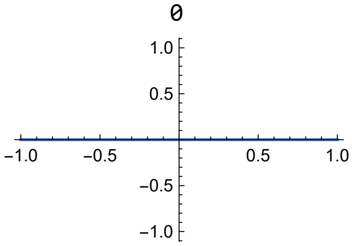
$2\,\theta\left[u - \boxed{-0.811\ldots}\right] - 2\,\theta\left[u - \boxed{0.811\ldots}\right]$

Knot[8, 6] → 

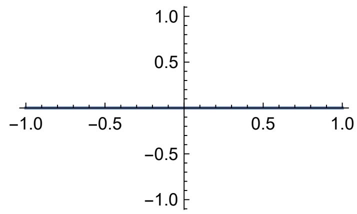
$-2\,\theta\left[u - \boxed{-0.647\ldots}\right] + 2\,\theta\left[u - \boxed{0.647\ldots}\right]$

Knot[8, 7] → 

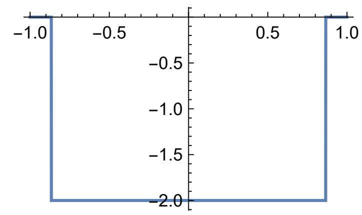
0

Knot[8, 8] → 

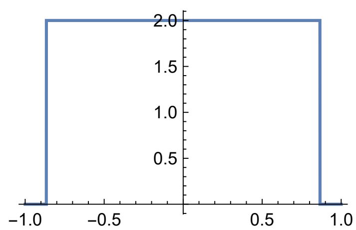
Knot[8, 9] → 0



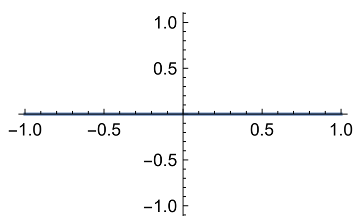
Knot[8, 10] → $2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]-2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]$



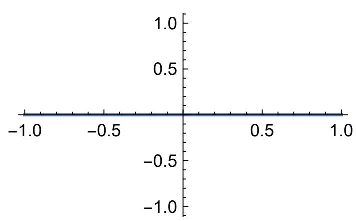
Knot[8, 11] → $-2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]+2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]$



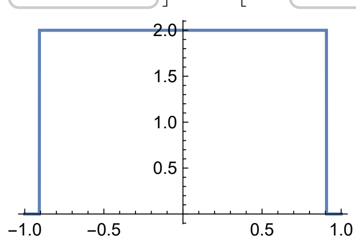
Knot[8, 12] → 0



Knot[8, 13] → 0

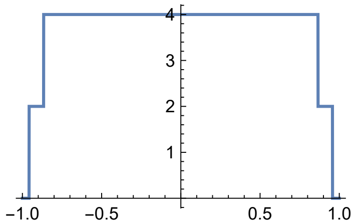


Knot[8, 14] → $2\,\theta\left[u-\boxed{-0.907\ldots}\right]-2\,\theta\left[u-\boxed{0.907\ldots}\right]$



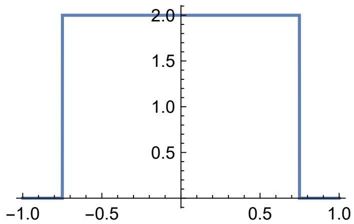
$$-2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]+2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]-2\,\theta\left[-\frac{\sqrt{\frac{11}{3}}}{2}+u\right]+2\,\theta\left[\frac{\sqrt{\frac{11}{3}}}{2}+u\right]$$

Knot[8, 15] →



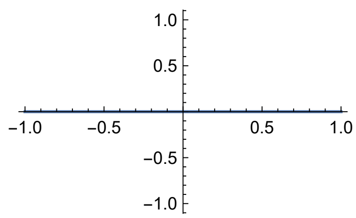
$$2\,\theta[u-\boxed{-0.749\ldots}]-2\,\theta[u-\boxed{0.749\ldots}]$$

Knot[8, 16] →



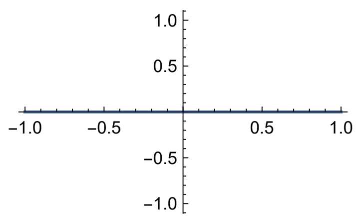
0

Knot[8, 17] →



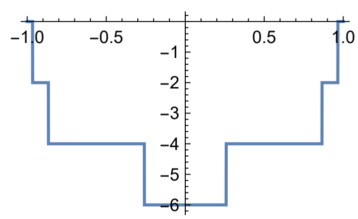
0

Knot[8, 18] →

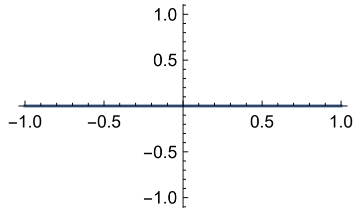


Knot[8, 19] →

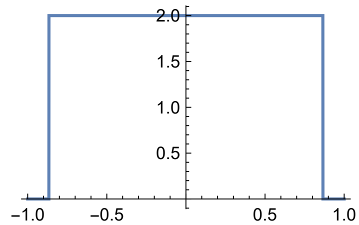
$$2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]-2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]-2\,\theta[u-\boxed{-0.966\ldots}]-2\,\theta[u-\boxed{-0.259\ldots}]+2\,\theta[u-\boxed{0.259\ldots}]+2\,\theta[u-\boxed{0.966\ldots}]$$



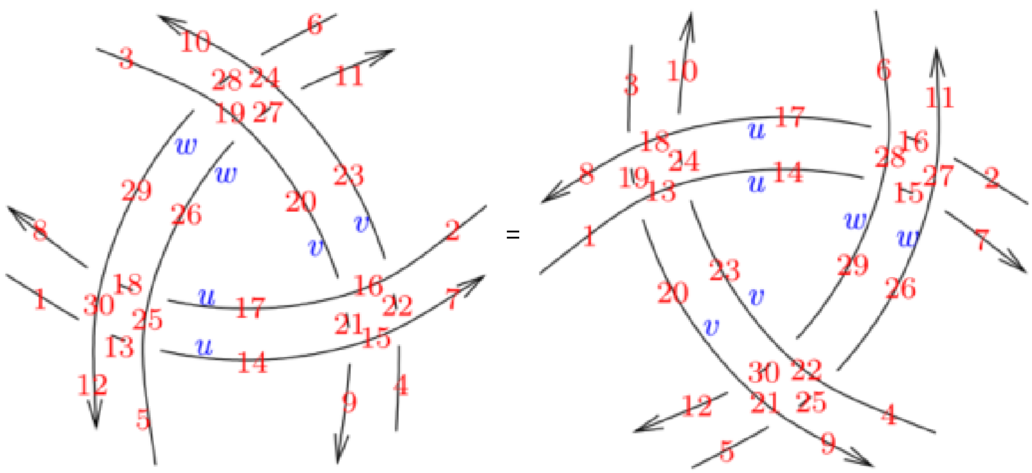
0

Knot[8, 20] → 

$-2\,\theta\left[-\frac{\sqrt{3}}{2}+u\right]+2\,\theta\left[\frac{\sqrt{3}}{2}+u\right]$

Knot[8, 21] → 

The Naik-Stanford Double Delta Move



In[∘]:=

$$\text{pd} = \text{PD}\left[X_{6,10,28,24},\ \overline{X}_{28,3,29,19},\ X_{26,20,27,19},\ \overline{X}_{27,23,11,24},\ X_{1,12,13,30},\ \overline{X}_{13,5,14,25},\ X_{17,26,18,25},\ \overline{X}_{18,29,8,30},\ X_{4,7,22,15},\ \overline{X}_{22,2,23,16},\ X_{20,17,21,16},\ \overline{X}_{21,14,9,15}\right]\ /.\ \left\{X_{i\_,j\_,k\_,l\_} \mapsto X_{-i,j,k,-l},\ \overline{X}_{i\_,j\_,k\_,l\_} \mapsto \overline{X}_{-j,k,l,-i}\right\}$$

$$\text{lhs} = \text{TL}[\text{pd}]$$

Out[∘]=

$$\text{PD}\left[X_{-6,10,28,-24},\ \overline{X}_{-3,29,19,-28},\ X_{-26,20,27,-19},\ \overline{X}_{-23,11,24,-27},\ X_{-1,12,13,-30},\ \overline{X}_{-5,14,25,-13},\ X_{-17,26,18,-25},\ \overline{X}_{-29,8,30,-18},\ X_{-4,7,22,-15},\ \overline{X}_{-2,23,16,-22},\ X_{-20,17,21,-16},\ \overline{X}_{-14,9,15,-21}\right]$$

Out[∘]=

$$2\,\Theta\left(u-\frac{1}{2}\right) - 2\,\Theta\left(u+\frac{1}{2}\right) + 1$$

$$\begin{array}{c|cccccccccc}
 & (\eta_{-6} & \eta_{10} & \eta_{-3} & \eta_{8} & \eta_{-1} & \eta_{12} & \eta_{-5} & \eta_{9} & \eta_{-4} & \eta_{7} \\
\overline{\eta}_{-6} & -\frac{2\omega}{\omega^2+\omega+1} & \omega-1 & \frac{2}{\omega^2+\omega+1} & 0 & -\frac{2\omega^2}{\omega^2+\omega+1} & 0 & \frac{2\omega}{\omega^2+\omega+1} & 0 & -\frac{2}{\omega^2+\omega+1} & 0 \\
\overline{\eta}_{10} & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{\eta}_{-3} & \frac{2\omega^2}{\omega^2+\omega+1} & 1-\omega & -\frac{2\omega}{\omega^2+\omega+1} & \omega-1 & \frac{2}{\omega^2+\omega+1} & 0 & -\frac{2\omega^2}{\omega^2+\omega+1} & 0 & \frac{2\omega}{\omega^2+\omega+1} & 0 \\
\overline{\eta}_{8} & 0 & 0 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 & 0 & 0 & 0 & 0 \\
\overline{\eta}_{-1} & -\frac{2}{\omega^2+\omega+1} & 0 & \frac{2\omega^2}{\omega^2+\omega+1} & 1-\omega & -\frac{2\omega}{\omega^2+\omega+1} & \omega-1 & \frac{2}{\omega^2+\omega+1} & 0 & -\frac{2\omega^2}{\omega^2+\omega+1} & 0 \\
\overline{\eta}_{12} & 0 & 0 & 0 & 0 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 & 0 & 0 \\
\overline{\eta}_{-5} & \frac{2\omega}{\omega^2+\omega+1} & 0 & -\frac{2}{\omega^2+\omega+1} & 0 & \frac{2\omega^2}{\omega^2+\omega+1} & 1-\omega & -\frac{2\omega}{\omega^2+\omega+1} & \omega-1 & \frac{2}{\omega^2+\omega+1} & 0 \\
\overline{\eta}_{9} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\omega-1}{\omega} & 0 & \frac{\omega-1}{\omega} & 0 \\
\overline{\eta}_{-4} & -\frac{2\omega^2}{\omega^2+\omega+1} & 0 & \frac{2\omega}{\omega^2+\omega+1} & 0 & -\frac{2}{\omega^2+\omega+1} & 0 & \frac{2\omega^2}{\omega^2+\omega+1} & 1-\omega & -\frac{2\omega}{\omega^2+\omega+1} & \omega - \\
\overline{\eta}_{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\omega-1}{\omega} & 0 \\
\overline{\eta}_{-2} & \frac{2}{\omega^2+\omega+1} & 0 & -\frac{2\omega^2}{\omega^2+\omega+1} & 0 & \frac{2\omega}{\omega^2+\omega+1} & 0 & -\frac{2}{\omega^2+\omega+1} & 0 & \frac{2\omega^2}{\omega^2+\omega+1} & 1 - \\
\overline{\eta}_{11} & \frac{\omega-1}{\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

In[∘]:=
```
pd = PD[X_{5,9,25,21}, X̄_{25,4,26,22}, X_{29,23,30,22}, X̄_{30,20,12,21}, X_{2,11,16,27}, X̄_{16,6,17,28}, X_{14,29,15,28}, X̄_{15,26,7,27},
    X_{3,8,19,18}, X̄_{19,1,20,13}, X_{23,14,24,13}, X̄_{24,17,10,18}] /. {X_{i_,j_,k_,l_} :→ X_{-i,j,k,-l}, X̄_{i_,j_,k_,l_} :→ X̄_{-j,k,l,-i}}
rhs = TL[pd]
```

Out[∘]=

$\mathrm{PD}\left[X_{-5,9,25,-21}, \overline{X}_{-4,26,22,-25}, X_{-29,23,30,-22}, \overline{X}_{-20,12,21,-30}, X_{-2,11,16,-27}, \overline{X}_{-6,17,28,-16},\right.$
$\left. X_{-14,29,15,-28}, \overline{X}_{-26,7,27,-15}, X_{-3,8,19,-18}, \overline{X}_{-1,20,13,-19}, X_{-23,14,24,-13}, \overline{X}_{-17,10,18,-24}\right]$

Out[∘]=

$$2\,\Theta\left(u-\frac{1}{2}\right)-2\,\Theta\left(u+\frac{1}{2}\right)+1$$

	$(\eta_{-6}$	η_{10}	η_{-3}	η_{8}	η_{-1}	η_{12}	η_{-5}	η_{9}	η_{-4}	η_{7}
$\overline{\eta}_{-6}$	$-\frac{2\omega}{\omega^2+\omega+1}$	$\omega-1$	$\frac{2}{\omega^2+\omega+1}$	0	$-\frac{2\omega^2}{\omega^2+\omega+1}$	0	$\frac{2\omega}{\omega^2+\omega+1}$	0	$-\frac{2}{\omega^2+\omega+1}$	0
$\overline{\eta}_{10}$	$-\frac{\omega-1}{\omega}$	0	$\frac{\omega-1}{\omega}$	0	0	0	0	0	0	0
$\overline{\eta}_{-3}$	$\frac{2\omega^2}{\omega^2+\omega+1}$	$1-\omega$	$-\frac{2\omega}{\omega^2+\omega+1}$	$\omega-1$	$\frac{2}{\omega^2+\omega+1}$	0	$-\frac{2\omega^2}{\omega^2+\omega+1}$	0	$\frac{2\omega}{\omega^2+\omega+1}$	0
$\overline{\eta}_{8}$	0	0	$-\frac{\omega-1}{\omega}$	0	$\frac{\omega-1}{\omega}$	0	0	0	0	0
$\overline{\eta}_{-1}$	$-\frac{2}{\omega^2+\omega+1}$	0	$\frac{2\omega^2}{\omega^2+\omega+1}$	$1-\omega$	$-\frac{2\omega}{\omega^2+\omega+1}$	$\omega-1$	$\frac{2}{\omega^2+\omega+1}$	0	$-\frac{2\omega^2}{\omega^2+\omega+1}$	0
$\overline{\eta}_{12}$	0	0	0	0	$-\frac{\omega-1}{\omega}$	0	$\frac{\omega-1}{\omega}$	0	0	0
$\overline{\eta}_{-5}$	$\frac{2\omega}{\omega^2+\omega+1}$	0	$-\frac{2}{\omega^2+\omega+1}$	0	$\frac{2\omega^2}{\omega^2+\omega+1}$	$1-\omega$	$-\frac{2\omega}{\omega^2+\omega+1}$	$\omega-1$	$\frac{2}{\omega^2+\omega+1}$	0
$\overline{\eta}_{9}$	0	0	0	0	0	0	$-\frac{\omega-1}{\omega}$	0	$\frac{\omega-1}{\omega}$	0
$\overline{\eta}_{-4}$	$-\frac{2\omega^2}{\omega^2+\omega+1}$	0	$\frac{2\omega}{\omega^2+\omega+1}$	0	$-\frac{2}{\omega^2+\omega+1}$	0	$\frac{2\omega^2}{\omega^2+\omega+1}$	$1-\omega$	$-\frac{2\omega}{\omega^2+\omega+1}$	$\omega -$
$\overline{\eta}_{7}$	0	0	0	0	0	0	0	0	$-\frac{\omega-1}{\omega}$	0
$\overline{\eta}_{-2}$	$\frac{2}{\omega^2+\omega+1}$	0	$-\frac{2\omega^2}{\omega^2+\omega+1}$	0	$\frac{2\omega}{\omega^2+\omega+1}$	0	$-\frac{2}{\omega^2+\omega+1}$	0	$\frac{2\omega^2}{\omega^2+\omega+1}$	$1 -$
$\overline{\eta}_{11}$	$\frac{\omega-1}{\omega}$	0	0	0	0	0	0	0	0	0

In[∘]:=
```
lhs == rhs
```

Out[∘]=

True

In[∘]:=

$$\mathbf{pd} = \mathrm{PD}\big[X_{6,10,28,24},\ \overline{X}_{28,3,29,19},\ X_{26,20,27,19},\ \overline{X}_{27,23,11,24},\ X_{1,12,13,30},\ \overline{X}_{13,5,14,25},\ X_{17,26,18,25},\ \overline{X}_{18,29,8,30},\ X_{4,7,22,15},\ \overline{X}_{22,2,23,16},\ X_{20,17,21,16},\ \overline{X}_{21,14,9,15}\big]\ /.\ \big\{X_{i\_,j\_,k\_,l\_} \mapsto X_{-i,j,k,-l},\ \overline{X}_{i\_,j\_,k\_,l\_} \mapsto \overline{X}_{-j,k,l,-i}\big\}$$

$$\mathbf{lhs} = \mathrm{Kas}[\mathbf{pd}]$$

Out[∘]=

$$\mathrm{PD}\big[X_{-6,10,28,-24},\ \overline{X}_{-3,29,19,-28},\ X_{-26,20,27,-19},\ \overline{X}_{-23,11,24,-27},\ X_{-1,12,13,-30},\ \overline{X}_{-5,14,25,-13},\ X_{-17,26,18,-25},\ \overline{X}_{-29,8,30,-18},\ X_{-4,7,22,-15},\ \overline{X}_{-2,23,16,-22},\ X_{-20,17,21,-16},\ \overline{X}_{-14,9,15,-21}\big]$$

Out[∘]=

$$\begin{array}{cccccccccc}
 & & & & & -1 & & & & \\
 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
 & (\eta_{-6} & \eta_{10} & \eta_{-3} & \eta_{8} & \eta_{-1} & \eta_{12} & \eta_{-5} & \eta_{9} & \eta_{-4} \\
\overline{\eta}_{-6} & \frac{1}{2} & u & \frac{1}{2} & 0 & -\frac{1}{2} & -u & -\frac{1}{2} & 0 & -\frac{1}{2} \\
\overline{\eta}_{10} & u & 2 & u & 4u^2-3 & 0 & -2\left(2u^2-1\right) & -u & 0 & -u \\
\overline{\eta}_{-3} & \frac{1}{2} & u & \frac{1}{2} & u & \frac{1}{2} & -u & -\frac{1}{2} & 0 & -\frac{1}{2} \\
\overline{\eta}_{8} & 0 & 4u^2-3 & u & 2 & u & -1 & 0 & 0 & -u \\
\overline{\eta}_{-1} & -\frac{1}{2} & 0 & \frac{1}{2} & u & \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
\overline{\eta}_{12} & -u & -2\left(2u^2-1\right) & -u & -1 & 0 & 2u^2 & u & 0 & u \\
\overline{\eta}_{-5} & -\frac{1}{2} & -u & -\frac{1}{2} & 0 & \frac{1}{2} & u & \frac{1}{2} & 0 & \frac{1}{2} \\
\overline{\eta}_{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{\eta}_{-4} & -\frac{1}{2} & -u & -\frac{1}{2} & -u & -\frac{1}{2} & u & \frac{1}{2} & 0 & \frac{1}{2} \\
\overline{\eta}_{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{\eta}_{-2} & \frac{1}{2} & 0 & -\frac{1}{2} & -u & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
\overline{\eta}_{11} & 0 & -1 & -u & -2\left(2u^2-1\right) & -u & 2u^2-1 & 0 & 0 & u
\end{array}$$

In[∘]:=
pd = **PD**$\left[X_{5,9,25,21}, \overline{X}_{25,4,26,22}, X_{29,23,30,22}, \overline{X}_{30,20,12,21}, X_{2,11,16,27}, \overline{X}_{16,6,17,28}, X_{14,29,15,28}, \overline{X}_{15,26,7,27}, X_{3,8,19,18}, \overline{X}_{19,1,20,13}, X_{23,14,24,13}, \overline{X}_{24,17,10,18}\right]$ /. $\left\{X_{i\_,j\_,k\_,l\_} \mapsto X_{-i,j,k,-l}, \overline{X}_{i\_,j\_,k\_,l\_} \mapsto \overline{X}_{-j,k,l,-i}\right\}$
rhs = **Kas**[**pd**]

Out[∘]=
$\mathrm{PD}\left[X_{-5,9,25,-21}, \overline{X}_{-4,26,22,-25}, X_{-29,23,30,-22}, \overline{X}_{-20,12,21,-30}, X_{-2,11,16,-27}, \overline{X}_{-6,17,28,-16}, X_{-14,29,15,-28}, \overline{X}_{-26,7,27,-15}, X_{-3,8,19,-18}, \overline{X}_{-1,20,13,-19}, X_{-23,14,24,-13}, \overline{X}_{-17,10,18,-24}\right]$

Out[∘]=

-1

	0	0	0	1	0	-1	0	0	0
	0	1	0	0	0	-1	0	1	0
	$(\eta_{-6}$	η_{10}	η_{-3}	η_{8}	η_{-1}	η_{12}	η_{-5}	η_{9}	η_{-4}
$\overline{\eta}_{-6}$	$\frac{1}{2}$	u	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-u$	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\overline{\eta}_{10}$	u	2	u	$4u^2-3$	0	$-2\left(2u^2-1\right)$	$-u$	0	$-u$
$\overline{\eta}_{-3}$	$\frac{1}{2}$	u	$\frac{1}{2}$	u	$\frac{1}{2}$	$-u$	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\overline{\eta}_{8}$	0	$4u^2-3$	u	2	u	-1	0	0	$-u$
$\overline{\eta}_{-1}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	u	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
$\overline{\eta}_{12}$	$-u$	$-2\left(2u^2-1\right)$	$-u$	-1	0	$2u^2$	u	0	u
$\overline{\eta}_{-5}$	$-\frac{1}{2}$	$-u$	$-\frac{1}{2}$	0	$\frac{1}{2}$	u	$\frac{1}{2}$	0	$\frac{1}{2}$
$\overline{\eta}_{9}$	0	0	0	0	0	0	0	0	0
$\overline{\eta}_{-4}$	$-\frac{1}{2}$	$-u$	$-\frac{1}{2}$	$-u$	$-\frac{1}{2}$	u	$\frac{1}{2}$	0	$\frac{1}{2}$
$\overline{\eta}_{7}$	0	0	0	0	0	0	0	0	0
$\overline{\eta}_{-2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-u$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
$\overline{\eta}_{11}$	0	-1	$-u$	$-2\left(2u^2-1\right)$	$-u$	$2u^2-1$	0	0	u

In[∘]:= **lhs** == **rhs**

Out[∘]=
True