```
In[•]:= Once[
         SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\ICERM-2305"];
         << KnotTheory`;
        ]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

pdf

```
In[•]:= SetAttributes[Bndry, Orderless];
        CF[Bndry[]] = Bndry[];
        CF[b_Bndry] := RotateLeft[#, First@Ordering[#] - 1] & /@ DeleteCases[b, {}]
```

pdf

```
In[•]:= CF[{}] = {};
        CF[rs_List] := Module[{ηs = Union@Cases[rs, η_, ∞], η},
          If[ηs === {}, {}, DeleteCases[
            RowReduce[Table[Coefficient[r, η], {r, rs}, {η, ηs}]].ηs,
            0]] ]
```

pdf

```
In[•]:= RuleOf[η_i_ + rest_.] := (η_i → Expand[-rest]);
        CF[PQ[rs_, q_]] := Module[{nrs = CF[rs]},
          PQ[nrs, Expand[q /. (RuleOf /@ nrs)]] ]
```

```
In[•]:= CF[{η1 - η2, η1 - η3}]
Out[•]=
        {η1 - η3, η2 - η3}
```

```
In[•]:= RuleOf /@ CF[{η1 - η2, η1 - η3, η4}]
Out[•]=
        {η1 → η3, η2 → η3, η4 → 0}
```

```
In[•]:= RuleOf[η1 + η2 + η3]
Out[•]=
        η1 → -η2 - η3
```

pdf

```
In[•]:= CF[Kas[b_, σ_, pq_]] := Kas[CF[b], σ, CF[pq]]
```

The disjoint union in the world of multi-tangles.

pdf

```
In[•]:= Kas /: Kas[b1_, σ1_, PQ[rs1_, q1_]] ⋃ Kas[b2_, σ2_, PQ[rs2_, q2_]] :=
          CF@Kas[Join[b1, b2], σ1 + σ2, PQ[rs1 ⋃ rs2, q1 + q2]];
```

In[•]:=
```
Kas[P[1, 2]] ⋃ Kas[P[3, 4]]
```
Out[•]=
```
Kas[Bndry[{-3, 4}, {-1, 2}], 0, PQ[{}, 0]]
```

pdf

In[•]:=
```
(* FM for FaceMerge *)
FM_i_,j_@Kas[Bndry[{li___, i_, ri___}, {lj___, j_, rj___}, bs___], σ_, PQ[rs_, q_]] :=
 Module[{},
  Kas[CF@Bndry[{ri, li, i, rj, lj, j}, bs], σ, CF@PQ[rs ⋃ {η_i - η_j}, q]] ]
```

In[•]:=
```
Kas[P[1, 2]] ⋃ Kas[P[3, 4]] // FM_1,4
```
Out[•]=
```
FM_1,4[Kas[Bndry[{-3, 4}, {-1, 2}], 0, PQ[{}, 0]]]
```

pdf

In[•]:=
```
Cordon_i_@Kas[Bndry[{li___, i_, ri___}, bs___], σ_, PQ[rs_, q_]] :=
 Module[{bi, ai, ϕ, nσ, nrs, nq, qii, p},
  ai = First@{ri, li}; bi = Last@{ri, li};
  {nσ, nrs, nq} = {σ, rs, q};
  ϕ = ∂_η_i rs;
  If[And @@ ((# == 0) & /@ ϕ), qii = ∂_η_i,η_i q;
   If[TrueQ[qii == 0],
    AppendTo[nrs, ∂_η_i q]; nq = q /. η_i → 0,
    (*else*) nσ += Sign[qii]; nq = q /. η_i → -(∂_η_i q) /. η_i → 0 / qii],
   (*else*) {p} = FirstPosition[(# == 0) & /@ ϕ, False];
   {nrs, nq} = {rs, q} /. η_i → -rs[[p]] /. η_i → 0 / ∂_η_i (rs[[p]])];
  CF@Kas[Bndry[Rest@{ri, li}, bs], nσ, PQ[nrs, nq] /. η_ai → η_bi] ]
```

pdf

In[•]:=
```
c_i_,j_@Kas[Bndry[{li___, i_, ri___}, {lj___, j_, rj___}, bs___], σ_, pq_PQ] :=
  Module[{bi = Last@{ri, li}},
   Kas[Bndry[{li, i, ri}, {lj, j, rj}, bs], σ, pq] // FM_j,bi // Cordon_j ];
```

pdf

```
In[•]:= c_{i_,j_}@Kas[Bndry[{l___, i_, j_, r___}, bs___], σ_, pq_PQ] :=
          Cordon_i@Kas[Bndry[{l, i, j, r}, bs], σ, pq];
        c_{i_,j_}@Kas[Bndry[{j_, m___, i_}, bs___], σ_, pq_PQ] :=
          Cordon_i@Kas[Bndry[{j, m, i}, bs], σ, pq];
        c_{i_,j_}@Kas[Bndry[{l___, j_, i_, r___}, bs___], σ_, pq_PQ] :=
          Cordon_j@Kas[Bndry[{l, j, i, r}, bs], σ, pq];
        c_{i_,j_}@Kas[Bndry[{i_, m___, j_}, bs___], σ_, pq_PQ] :=
          Cordon_j@Kas[Bndry[{i, m, j}, bs], σ, pq];
```

pdf

```
In[•]:= c@Kas[Bndry[{li___, i_, ri___}, {lj___, j_, rj___}, bs___], σ_, pq_PQ] /; j == -i :=
          c[c_{i,j}@Kas[Bndry[{li, i, ri}, {lj, j, rj}, bs], σ, pq]];
        c@Kas[Bndry[{l___, i_, j_, r___}, bs___], σ_, pq_PQ] /; j == -i :=
          c@Cordon_i@Kas[Bndry[{l, i, j, r}, bs], σ, pq];
        c@Kas[Bndry[{j_, m___, i_}, bs___], σ_, pq_PQ] /; j == -i :=
          c@Cordon_i@Kas[Bndry[{j, m, i}, bs], σ, pq];
        c@Kas[b_Bndry, σ_, pq_PQ] /; (Union@@b ⋂ (-Union@@b)) === {} := Kas[b, σ, pq]
```

pdf

```
In[•]:= Kas[P[i_, j_]] := Kas[CF@Bndry[{-i, j}], 0, PQ[{}, 0]]
```

pdf

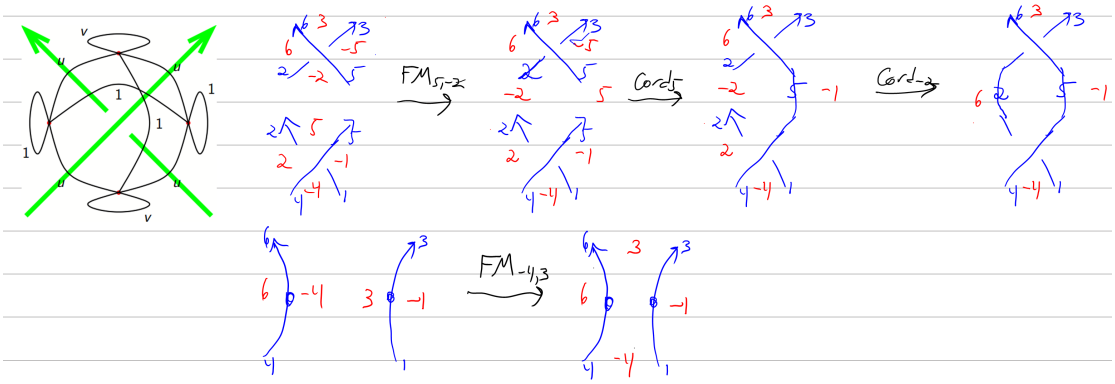
```
In[•]:= v := 2 u^2 - 1;
```

pdf

```
In[•]:= Kas[X[i_, j_, k_, l_]] := If[PositiveQ@X[i, j, k, l],
          Kas[CF@Bndry[{-i, j, k, -l}], 0, PQ[{},
            η_{-i}^2 + 2 u η_{-i} η_j + v η_j^2 + 2 η_{-i} η_k + 2 u η_j η_k + η_k^2 + 2 u η_{-i} η_{-l} + 2 η_j η_{-l} + 2 u η_k η_{-l} + v η_{-l}^2]],
          Kas[CF@Bndry[{-i, -j, k, l}], 0, PQ[{},
            -v η_{-i}^2 - 2 u η_{-i} η_{-j} - η_{-j}^2 - 2 η_{-i} η_k - 2 u η_{-j} η_k - v η_k^2 - 2 u η_{-i} η_l - 2 η_{-j} η_l - 2 u η_k η_l - η_l^2]] ]
```

pdf

Reidemeister 2



In[•]:=
```
Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]]
```

Out[•]=
$$\text{Kas}\Big[\text{Bndry}[\{-5,3,6,-2\},\{-4,-1,5,2\}],0,\text{PQ}\Big[\{\},-\eta_{-5}^2-\eta_{-4}^2+2u^2\eta_{-4}^2-2u\eta_{-5}\eta_{-2}+\eta_{-2}^2-2u^2\eta_{-2}^2+2u\eta_{-4}\eta_{-1}+\eta_{-1}^2+2u\eta_{-4}\eta_2+2\eta_{-1}\eta_2+\eta_2^2-2u\eta_{-5}\eta_3-2\eta_{-2}\eta_3+\eta_3^2-2u^2\eta_3^2+2\eta_{-4}\eta_5+2u\eta_{-1}\eta_5+2u\eta_2\eta_5-\eta_5^2+2u^2\eta_5^2-2\eta_{-5}\eta_6-2u\eta_{-2}\eta_6-2u\eta_3\eta_6-\eta_6^2\Big]\Big]$$

In[•]:=
```
Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // FM_{-2,5}
```

Out[•]=
$$\text{Kas}\Big[\text{Bndry}[\{-5,3,6,-2,2,-4,-1,5\}],0,\text{PQ}\Big[\{\eta_{-2}-\eta_5\},-\eta_{-5}^2-\eta_{-4}^2+2u^2\eta_{-4}^2+2u\eta_{-4}\eta_{-1}+\eta_{-1}^2+2u\eta_{-4}\eta_2+2\eta_{-1}\eta_2+\eta_2^2-2u\eta_{-5}\eta_3+\eta_3^2-2u^2\eta_3^2-2u\eta_{-5}\eta_5+2\eta_{-4}\eta_5+2u\eta_{-1}\eta_5+2u\eta_2\eta_5-2\eta_3\eta_5-2\eta_{-5}\eta_6-2u\eta_3\eta_6-2u\eta_5\eta_6-\eta_6^2\Big]\Big]$$

In[•]:=
```
Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // FM_{-2,5} // Cordon_5
```

Out[•]=
$$\text{Kas}\Big[\text{Bndry}[\{-4,-1,3,6,-2,2\}],0,\text{PQ}\Big[\{\},-\eta_{-4}^2+2u^2\eta_{-4}^2+2\eta_{-4}\eta_{-2}+2u\eta_{-4}\eta_{-1}+2u\eta_{-4}\eta_2+2u\eta_{-2}\eta_2+2\eta_{-1}\eta_2+\eta_2^2-2\eta_{-2}\eta_3-2u\eta_{-1}\eta_3+\eta_3^2-2u^2\eta_3^2-2u\eta_{-2}\eta_6-2\eta_{-1}\eta_6-2u\eta_3\eta_6-\eta_6^2\Big]\Big]$$

In[•]:=
```
Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // FM_{-2,5} // Cordon_5 // Cordon_{-2}
```

Out[•]=
$$\text{Kas}[\text{Bndry}[\{-4,-1,3,6\}],0,\text{PQ}[\{\eta_{-4}-\eta_3\},0]]$$

pdf

In[•]:=
```
{Kas[P[1, 3]] ⋃ Kas[P[4, 6]] // FM_{-4,3}, Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // c_{2,-2} // c_{5,-5}}
```

Out[•]=
pdf

$$\{\text{Kas}[\text{Bndry}[\{-4,-1,3,6\}],0,\text{PQ}[\{\eta_{-4}-\eta_3\},0]],\ \text{Kas}[\text{Bndry}[\{-4,-1,3,6\}],0,\text{PQ}[\{\eta_{-4}-\eta_3\},0]]\}$$

pdf

Reidemeister 3

In[•]:=
```
{u = 7 / 29};
lhs = Kas[X[4, 2, 5, 1]] ⋃ Kas[X[7, 3, 8, 2]] ⋃ Kas[X[8, 6, 9, 5]] // c2,-2 // c5,-5 // c8,-8
rhs = Kas[X[7, 5, 8, 4]] ⋃ Kas[X[8, 2, 9, 1]] ⋃ Kas[X[5, 3, 6, 2]] // c2,-2 // c5,-5 // c8,-8
Clear[u]
```

Out[•]=

$$\text{Kas}\left[\text{Bndry}[\{-7,3,6,9,-1,-4\}],-1,\text{PQ}\left[\{\},\ \frac{1486\,\eta_{-7}^2}{645}+\frac{32\,578\,\eta_{-7}\,\eta_{-4}}{18\,705}+\frac{228\,046\,\eta_{-4}^2}{542\,445}+\frac{1682}{645}\eta_{-7}\,\eta_{-1}+\frac{32\,578\,\eta_{-4}\,\eta_{-1}}{18\,705}+\frac{228\,046\,\eta_{-1}^2}{542\,445}+\frac{32\,578\,\eta_{-7}\,\eta_3}{18\,705}+\frac{1682}{645}\eta_{-4}\,\eta_3+\frac{812}{645}\eta_{-1}\,\eta_3+\frac{228\,046\,\eta_3^2}{542\,445}+\frac{1682}{645}\eta_{-7}\,\eta_6+\frac{812}{645}\eta_{-4}\,\eta_6+\frac{1682}{645}\eta_{-1}\,\eta_6+\frac{32\,578\,\eta_3\,\eta_6}{18\,705}+\frac{228\,046\,\eta_6^2}{542\,445}+\frac{812}{645}\eta_{-7}\,\eta_9+\frac{1682}{645}\eta_{-4}\,\eta_9+\frac{32\,578\,\eta_{-1}\,\eta_9}{18\,705}+\frac{1682\,\eta_3\,\eta_9}{645}+\frac{32\,578\,\eta_6\,\eta_9}{18\,705}+\frac{1486\,\eta_9^2}{645}\right]\right]$$

Out[•]=

$$\text{Kas}\left[\text{Bndry}[\{-7,3,6,9,-1,-4\}],-1,\text{PQ}\left[\{\},\ \frac{1486\,\eta_{-7}^2}{645}+\frac{32\,578\,\eta_{-7}\,\eta_{-4}}{18\,705}+\frac{228\,046\,\eta_{-4}^2}{542\,445}+\frac{1682}{645}\eta_{-7}\,\eta_{-1}+\frac{32\,578\,\eta_{-4}\,\eta_{-1}}{18\,705}+\frac{228\,046\,\eta_{-1}^2}{542\,445}+\frac{32\,578\,\eta_{-7}\,\eta_3}{18\,705}+\frac{1682}{645}\eta_{-4}\,\eta_3+\frac{812}{645}\eta_{-1}\,\eta_3+\frac{228\,046\,\eta_3^2}{542\,445}+\frac{1682}{645}\eta_{-7}\,\eta_6+\frac{812}{645}\eta_{-4}\,\eta_6+\frac{1682}{645}\eta_{-1}\,\eta_6+\frac{32\,578\,\eta_3\,\eta_6}{18\,705}+\frac{228\,046\,\eta_6^2}{542\,445}+\frac{812}{645}\eta_{-7}\,\eta_9+\frac{1682}{645}\eta_{-4}\,\eta_9+\frac{32\,578\,\eta_{-1}\,\eta_9}{18\,705}+\frac{1682\,\eta_3\,\eta_9}{645}+\frac{32\,578\,\eta_6\,\eta_9}{18\,705}+\frac{1486\,\eta_9^2}{645}\right]\right]$$

In[•]:=
```
Kas[Bndry[{-7, 3, 6, 9, -1, -4}], -1, PQ[{},
 1486 η-7^2/645 + 32578 η-7 η-4/18705 + 228046 η-4^2/542445 + 1682/645 η-7 η-1 + 32578 η-4 η-1/18705 + 228046 η-1^2/542445 + 32578 η-7 η3/18705 +
 1682/645 η-4 η3 + 812/645 η-1 η3 + 228046 η3^2/542445 + 1682/645 η-7 η6 + 812/645 η-4 η6 + 1682/645 η-1 η6 + 32578 η3 η6/18705 +
 228046 η6^2/542445 + 812/645 η-7 η9 + 1682/645 η-4 η9 + 32578 η-1 η9/18705 + 1682 η3 η9/645 + 32578 η6 η9/18705 + 1486 η9^2/645]]
```

Out[•]=

$$\text{Kas}\left[\text{Bndry}[\{-7,3,6,9,-1,-4\}],-1,\text{PQ}\left[\{\},\ \frac{1486\,\eta_{-7}^2}{645}+\frac{32\,578\,\eta_{-7}\,\eta_{-4}}{18\,705}+\frac{228\,046\,\eta_{-4}^2}{542\,445}+\frac{1682}{645}\eta_{-7}\,\eta_{-1}+\frac{32\,578\,\eta_{-4}\,\eta_{-1}}{18\,705}+\frac{228\,046\,\eta_{-1}^2}{542\,445}+\frac{32\,578\,\eta_{-7}\,\eta_3}{18\,705}+\frac{1682}{645}\eta_{-4}\,\eta_3+\frac{812}{645}\eta_{-1}\,\eta_3+\frac{228\,046\,\eta_3^2}{542\,445}+\frac{1682}{645}\eta_{-7}\,\eta_6+\frac{812}{645}\eta_{-4}\,\eta_6+\frac{1682}{645}\eta_{-1}\,\eta_6+\frac{32\,578\,\eta_3\,\eta_6}{18\,705}+\frac{228\,046\,\eta_6^2}{542\,445}+\frac{812}{645}\eta_{-7}\,\eta_9+\frac{1682}{645}\eta_{-4}\,\eta_9+\frac{32\,578\,\eta_{-1}\,\eta_9}{18\,705}+\frac{1682\,\eta_3\,\eta_9}{645}+\frac{32\,578\,\eta_6\,\eta_9}{18\,705}+\frac{1486\,\eta_9^2}{645}\right]\right]$$

pdf

```
In[•]:= lhs = Kas[X[4, 2, 5, 1]] ⋃ Kas[X[7, 3, 8, 2]] ⋃ Kas[X[8, 6, 9, 5]] // c;
        rhs = Kas[X[7, 5, 8, 4]] ⋃ Kas[X[8, 2, 9, 1]] ⋃ Kas[X[5, 3, 6, 2]] // c;
        {lhs[[2]], rhs[[2]]}
        Simplify[lhs[[3, 2]] == rhs[[3, 2]]]
```

Out[•]=
pdf

$$\left\{\text{Sign}\left[-2+8\,u^2\right],\ \text{Sign}\left[-2+8\,u^2\right]\right\}$$

Out[•]=
pdf

True

Kashaev for Knots

```
In[•]:= -KnotSignature /@ AllKnots[{3, 8}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[•]=

```
{2, 0, 4, 2, 0, 2, 0, 6, 2, -4, -2, 4, 2, 0, 0, 4,
 0, 2, -4, 2, -2, 0, 0, -2, 2, 0, 0, 2, 4, 2, 0, 0, -6, 0, 2}
```

```
In[•]:= (*Kas[{k1_Kas, ks__Kas}]:=Module[{k2},
     k2=First@MaximalBy[{ks},Length[(-k1[[1,1]])⋂#[[1,1]]]&];
     Kas@Append[DeleteCases[{ks},k2],Echo@c@(k1⋃k2)]
    ]//First;*)
```

pdf

```
KasSig[K_] := Module[{pd = PD[K]},
    c[Union @@ (Kas /@ pd)][[2]] - Sum[If[PositiveQ@x, 1, -1], {x, List @@ pd}] ] / 2
```

```
In[•]:= u = 0;
        c[Union @@ (Kas /@ PD@Knot[3, 1])]
        Clear[u]
```

Out[•]=

```
Kas[Bndry[], 1, PQ[{}, 0]]
```

```
In[•]:= u = 0;
        KasSig /@ AllKnots[{3, 8}]
        Clear[u]
```

Out[•]=

```
{2, 0, 4, 2, 0, 2, 0, 6, 2, -4, -2, 4, 2, 0, 0, 4,
 0, 2, -4, 2, -2, 0, 0, -2, 2, 0, 0, 2, 4, 2, 0, 0, -6, 0, 2}
```

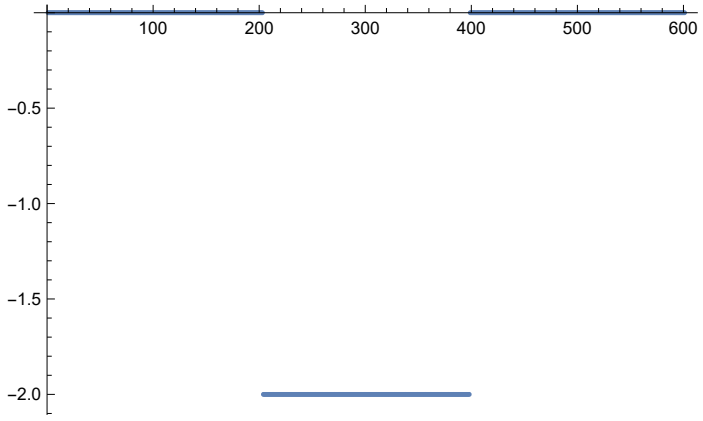
```
In[•]:= u = 1 / 2;
        KasSig /@ AllKnots[{3, 8}]
        Clear[u]
```

Out[•]=

```
{2, 0, 4, 2, 0, 2, 0, 4, 2, -4, -2, 4, 2, 0, 0, 4,
 0, 2, -4, 2, -2, 0, 0, -2, 2, 0, 0, 2, 4, 2, 0, 0, -4, 0, 2}
```

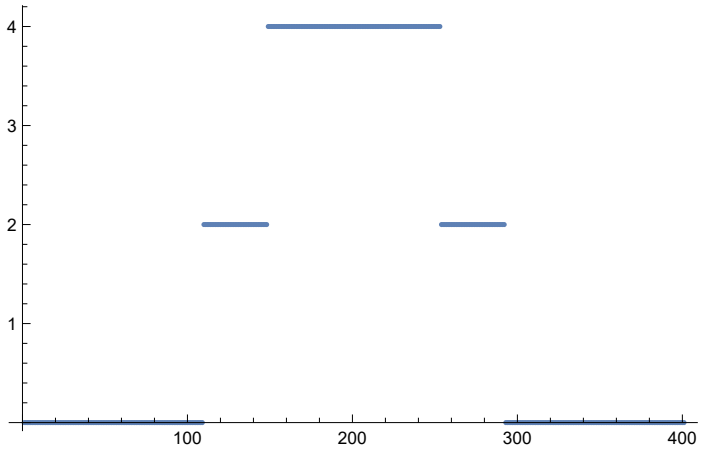
In[•]:=
```
ListPlot[Table[KasSig[Knot[9, 5]], {u, -3, 3, 1 / 100}]]
```

Out[•]=



In[•]:=
```
ListPlot[Table[KasSig[Knot[8, 2]], {u, -2, 2, 1 / 100}]]
```

Out[•]=

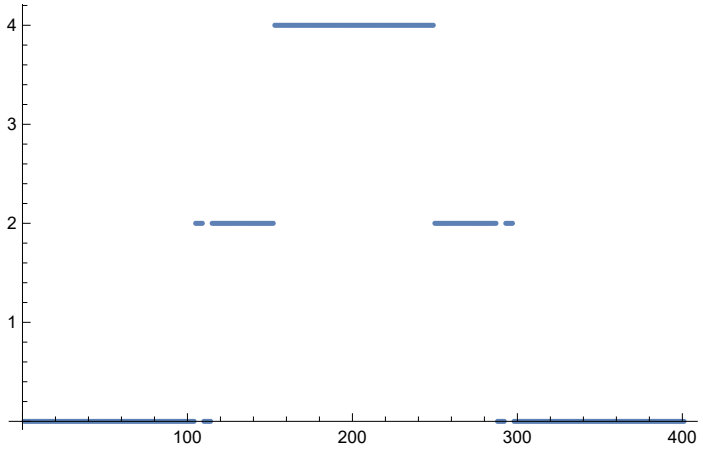


In[•]:=
```
ListPlot[Table[KasSig[Knot[12, Alternating, 422]], {u, -2, 2, 1 / 100}]]
```

KnotTheory: Loading precomputed data in KnotTheory/12A.dts.

KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

Out[•]=



In[•]:=
$$\text{Kas}\left[\text{Bndry}[\{-15,-10,12,1\}],-1,\text{PQ}\left[\{\},-\frac{\eta_{-15}^2}{2}+2\,\eta_{-10}^2+4\,\eta_{-10}\,\eta_1+2\,\eta_1^2+\eta_{-15}\,\eta_{12}-\frac{\eta_{12}^2}{2}\right]\right]\cup\text{Kas}\left[\text{Bndry}[\{-12,10,15,-1\}],1,\text{PQ}\left[\{\},\frac{11\,\eta_{-12}^2}{13}-\frac{13\,\eta_{-1}^2}{11}+\frac{26}{11}\,\eta_{-1}\,\eta_{10}-\frac{13\,\eta_{10}^2}{11}+\frac{22}{13}\,\eta_{-12}\,\eta_{15}+\frac{11\,\eta_{15}^2}{13}\right]\right]$$

Out[•]=
$$\text{Kas}\left[\text{Bndry}[\{-15,-10,12,1\},\{-12,10,15,-1\}],0,\text{PQ}\left[\{\},-\frac{\eta_{-15}^2}{2}+\frac{11\,\eta_{-12}^2}{13}+2\,\eta_{-10}^2-\frac{13\,\eta_{-1}^2}{11}+4\,\eta_{-10}\,\eta_1+2\,\eta_1^2+\frac{26}{11}\,\eta_{-1}\,\eta_{10}-\frac{13\,\eta_{10}^2}{11}+\eta_{-15}\,\eta_{12}-\frac{\eta_{12}^2}{2}+\frac{22}{13}\,\eta_{-12}\,\eta_{15}+\frac{11\,\eta_{15}^2}{13}\right]\right]$$

In[•]:=
```
Kas[Bndry[{-15, -10, 12, 1}, {-12, 10, 15, -1}], 0,
  PQ[{}, -η[-15]^2/2 + 11 η[-12]^2/13 + 2 η[-10]^2 - 13 η[-1]^2/11 + 4 η[-10] η[1] +
    2 η[1]^2 + 26/11 η[-1] η[10] - 13 η[10]^2/11 + η[-15] η[12] - η[12]^2/2 +
    22/13 η[-12] η[15] + 11 η[15]^2/13]] // c[15,-15] // c[10,-10] // c[12,-12] // c[1,-1]
```

Out[•]=
```
Kas[Bndry[], 0, PQ[{}, 0]]
```