```
In[•]:= Once[
         SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\ICERM-2305"];
         << KnotTheory`;
        ]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

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```
In[•]:= SetAttributes[Bndry, Orderless];
        CF[b_Bndry] := RotateLeft[#, First@Ordering[#] - 1] & /@ b
```

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```
In[•]:= CF[{}] = {};
        CF[rs_List] := Module[{ηs = Union@Cases[rs, η_, ∞], η},
          If[ηs === {}, {}, DeleteCases[
            RowReduce[Table[Coefficient[r, η], {r, rs}, {η, ηs}]].ηs,
            0
           ]]
         ]
```

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```
In[•]:= RuleOf[η_i_ + rest_.] := (η_i → Expand[-rest]);
        CF[PQ[rs_, q_]] := Module[{nrs = CF[rs]},
          PQ[nrs, Expand[q /. (RuleOf /@ nrs)]]
         ]
```

```
In[•]:= CF[{η1 - η2, η1 - η3}]
```

```
Out[•]=
```

$$\{\eta_1 - \eta_3,\ \eta_2 - \eta_3\}$$

```
In[•]:= RuleOf /@ CF[{η1 - η2, η1 - η3, η4}]
```

```
Out[•]=
```

$$\{\eta_1 \to \eta_3,\ \eta_2 \to \eta_3,\ \eta_4 \to 0\}$$

```
In[•]:= RuleOf[η1 + η2 + η3]
```

```
Out[•]=
```

$$\eta_1 \to -\eta_2 - \eta_3$$

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```
In[•]:= CF[Kas[b_, σ_, pq_]] := Kas[CF[b], σ, CF[pq]]
```

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```
In[•]:= Kas[P[i_, j_]] := Kas[CF@Bndry[{-i, j}], 0, PQ[{}, 0]]
```

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In[•]:=
```
Kas[X[i_, j_, k_, l_]] := If[PositiveQ@X[i, j, k, l],
  Kas[CF@Bndry[{-i, j, k, -l}], 0, PQ[{},
    η_i^2 + 2 u η_-i η_j + v η_j^2 + 2 η_-i η_k + 2 u η_j η_k + η_k^2 + 2 u η_-i η_-l + 2 η_j η_-l + 2 u η_k η_-l + v η_-l^2]],
  Kas[CF@Bndry[{-i, -j, k, l}], 0, PQ[{},
    -v η_-i^2 - 2 u η_-i η_-j - η_-j^2 - 2 η_-i η_k - 2 u η_-j η_k - v η_k^2 - 2 u η_-i η_l - 2 η_-j η_l - 2 u η_k η_l - η_l^2]]
 ]
```

The disjoint union in the world of multi-tangles.

pdf

In[•]:=
```
PQ[rs1_, q1_] ⊕ PQ[rs2_, q2_] := PQ[CF[rs1 ⋃ rs2], q1 + q2]
```

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In[•]:=
```
Kas /: Kas[b1_, σ1_, pq1_] ⋃ Kas[b2_, σ2_, pq2_] :=
  Kas[CF@Join[b1, b2], σ1 + σ2, pq1 ⊕ pq2];
```

In[•]:=
```
Kas[P[1, 2]] ⋃ Kas[P[3, 4]]
```

Out[•]=
```
Kas[Bndry[{1, 2}, {3, 4}], 0, PQ[{}, 0]]
```

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In[•]:=
```
(* FM for FaceMerge *)
FM_i_,j_ @Kas[Bndry[{li___, i_, ri___}, {lj___, j_, rj___}, bs___], σ_, PQ[rs_, q_]] :=
 Module[{},
  Kas[CF@Bndry[{ri, li, i, rj, lj, j}, bs], σ, CF@PQ[rs ⋃ {η_i - η_j}, q]]
 ]
```

In[•]:=
```
Kas[P[1, 2]] ⋃ Kas[P[3, 4]] // FM_1,4
```

Out[•]=
```
Kas[Bndry[{1, 3, 4, 2}], 0, PQ[{η_1 - η_4}, 0]]
```

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In[•]:=
```
Cordon_i_@Kas[Bndry[{li___, i_, ri___}, bs___], σ_, PQ[rs_, q_]] :=
 Module[{bi, ai, ϕ, nσ, nrs, nq, qii, p},
  ai = First@{ri, li}; bi = Last@{ri, li};
  {nσ, nrs, nq} = {σ, rs, q};
  ϕ = ∂_{η_i} rs;
  If[And @@ ((# == 0) & /@ ϕ),
   nσ += Sign[∂_{η_i,η_i} q]; AppendTo[nrs, (∂_{η_i} q) /. η_i → 0],
   (*else*) {p} = FirstPosition[(# == 0) & /@ ϕ, False];
   nrs = Table[r - ∂_{η_i} r / ∂_{η_i} (rs[[p]]) rs[[p]], {r, rs}];
   nq = q /. η_i → -rs[[p]] /. η_i → 0 / ∂_{η_i} (rs[[p]])
  ];
  CF@Kas[Bndry[Rest@{ri, li}, bs], nσ, PQ[nrs, nq] /. η_ai → η_bi]
 ]
```

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In[•]:=
```
c_{i_,j_}@Kas[Bndry[{li___, i_, ri___}, {lj___, j_, rj___}, bs___], σ_, pq_PQ] :=
  Module[{bi = Last@{ri, li}},
   Kas[Bndry[{li, i, ri}, {lj, j, rj}, bs], σ, pq] // FM_{j,bi} // Cordon_j
  ];
```

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In[•]:=
```
c_{i_,j_}@Kas[Bndry[{l___, i_, j_, r___}, bs___], σ_, pq_PQ] :=
  Cordon_i@Kas[Bndry[{l, i, j, r}, bs], σ, pq];
c_{i_,j_}@Kas[Bndry[{j_, m___, i_}, bs___], σ_, pq_PQ] :=
  Cordon_i@Kas[Bndry[{j, m, i}, bs], σ, pq];
c_{i_,j_}@Kas[Bndry[{l___, j_, i_, r___}, bs___], σ_, pq_PQ] :=
  Cordon_j@Kas[Bndry[{l, j, i, r}, bs], σ, pq];
c_{i_,j_}@Kas[Bndry[{i_, m___, j_}, bs___], σ_, pq_PQ] :=
  Cordon_j@Kas[Bndry[{i, m, j}, bs], σ, pq];
```

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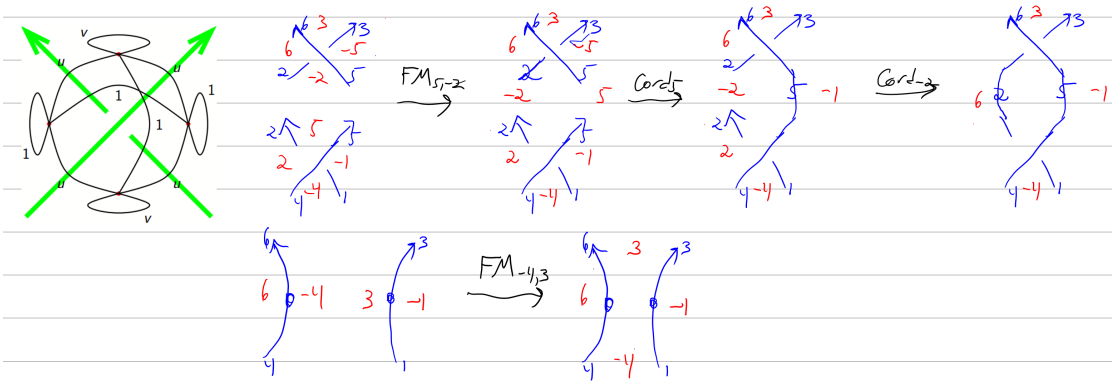
Reidemeister 2



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In[•]:= **Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]]**

Out[•]=
pdf

$$\text{Kas}\left[\text{Bndry}\left[\{-5,\,3,\,6,\,-2\},\,\{-4,\,-1,\,5,\,2\}\right],\,0,\,\text{PQ}\left[\{\},\,-\eta_{-5}^2+\text{v}\,\eta_{-4}^2-2\,\text{u}\,\eta_{-5}\,\eta_{-2}-\text{v}\,\eta_{-2}^2+2\,\text{u}\,\eta_{-4}\,\eta_{-1}+\eta_{-1}^2+2\,\text{u}\,\eta_{-4}\,\eta_2+2\,\eta_{-1}\,\eta_2+\eta_2^2-2\,\text{u}\,\eta_{-5}\,\eta_3-2\,\eta_{-2}\,\eta_3-\text{v}\,\eta_3^2+2\,\eta_{-4}\,\eta_5+2\,\text{u}\,\eta_{-1}\,\eta_5+2\,\text{u}\,\eta_2\,\eta_5+\text{v}\,\eta_5^2-2\,\eta_{-5}\,\eta_6-2\,\text{u}\,\eta_{-2}\,\eta_6-2\,\text{u}\,\eta_3\,\eta_6-\eta_6^2\right]\right]$$

pdf

In[•]:= **Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // FM$_{-2,5}$**

Out[•]=
pdf

$$\text{Kas}\left[\text{Bndry}\left[\{-5,\,3,\,6,\,-2,\,2,\,-4,\,-1,\,5\}\right],\,0,\,\text{PQ}\left[\{\eta_{-2}-\eta_5\},\,-\eta_{-5}^2+\text{v}\,\eta_{-4}^2+2\,\text{u}\,\eta_{-4}\,\eta_{-1}+\eta_{-1}^2+2\,\text{u}\,\eta_{-4}\,\eta_2+2\,\eta_{-1}\,\eta_2+\eta_2^2-2\,\text{u}\,\eta_{-5}\,\eta_3-\text{v}\,\eta_3^2-2\,\text{u}\,\eta_{-5}\,\eta_5+2\,\eta_{-4}\,\eta_5+2\,\text{u}\,\eta_{-1}\,\eta_5+2\,\text{u}\,\eta_2\,\eta_5-2\,\eta_3\,\eta_5-2\,\eta_{-5}\,\eta_6-2\,\text{u}\,\eta_3\,\eta_6-2\,\text{u}\,\eta_5\,\eta_6-\eta_6^2\right]\right]$$

pdf

In[•]:= **Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // FM$_{-2,5}$ // Cordon$_5$**

Out[•]=
pdf

$$\text{Kas}\left[\text{Bndry}\left[\{-4,\,-1,\,3,\,6,\,-2,\,2\}\right],\,0,\,\text{PQ}\left[\{\},\,\text{v}\,\eta_{-4}^2+2\,\eta_{-4}\,\eta_{-2}+2\,\text{u}\,\eta_{-4}\,\eta_{-1}+2\,\text{u}\,\eta_{-4}\,\eta_2+2\,\text{u}\,\eta_{-2}\,\eta_2+2\,\eta_{-1}\,\eta_2+\eta_2^2-2\,\eta_{-2}\,\eta_3-2\,\text{u}\,\eta_{-1}\,\eta_3-\text{v}\,\eta_3^2-2\,\text{u}\,\eta_{-2}\,\eta_6-2\,\eta_{-1}\,\eta_6-2\,\text{u}\,\eta_3\,\eta_6-\eta_6^2\right]\right]$$

pdf

In[•]:= **Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // FM$_{-2,5}$ // Cordon$_5$ // Cordon$_{-2}$**

Out[•]=
pdf

$$\text{Kas}\left[\text{Bndry}\left[\{-4,\,-1,\,3,\,6\}\right],\,0,\,\text{PQ}\left[\{\eta_{-4}-\eta_3\},\,0\right]\right]$$

pdf

In[•]:= **Kas[P[1, 3]] ⋃ Kas[P[4, 6]] // FM$_{-4,3}$**

Out[•]=
pdf

$$\text{Kas}\left[\text{Bndry}\left[\{-4,\,-1,\,3,\,6\}\right],\,0,\,\text{PQ}\left[\{\eta_{-4}-\eta_3\},\,0\right]\right]$$

pdf

In[•]:= **Kas[X[1, 5, 2, 4]] ⋃ Kas[X[2, 5, 3, 6]] // c$_{2,-2}$ // c$_{5,-5}$**

Out[•]=
pdf

$$\text{Kas}\left[\text{Bndry}\left[\{-4,\,-1,\,3,\,6\}\right],\,0,\,\text{PQ}\left[\{\eta_{-4}-\eta_3\},\,0\right]\right]$$

pdf

Reidemeister 3

In[•]:=
```
{u = 1 / 3, v = 2 u^2 - 1};
Kas[X[4, 2, 5, 1]] ⋃ Kas[X[7, 3, 8, 2]] ⋃ Kas[X[8, 6, 9, 5]] // c_{2,-2} // c_{5,-5} // c_{8,-8}
Kas[X[7, 5, 8, 4]] ⋃ Kas[X[8, 2, 9, 1]] ⋃ Kas[X[5, 3, 6, 2]] // c_{2,-2} // c_{5,-5} // c_{8,-8}
Clear[u, v]
```

Out[•]=

$$\text{Kas}\Big[\text{Bndry}[\{-7,3,6,9,-1,-4\}],-1,\ \text{PQ}\Big[\Big\{\eta_{-7}+\frac{2\eta_{-4}}{3}+\eta_{-1}+\frac{2\eta_3}{3}+\eta_6+\frac{2\eta_9}{3}\Big\},\ \frac{2\eta_{-4}^2}{9}+\frac{4}{3}\eta_{-4}\eta_{-1}+\frac{2\eta_{-1}^2}{9}+2\eta_{-4}\eta_3+\frac{2}{3}\eta_{-1}\eta_3+\frac{2\eta_3^2}{9}+\frac{2}{3}\eta_{-4}\eta_6+2\eta_{-1}\eta_6+\frac{4\eta_3\eta_6}{3}+\frac{2\eta_6^2}{9}-\frac{5\eta_8^2}{9}+\frac{22}{9}\eta_{-4}\eta_9+2\eta_{-1}\eta_9+\frac{22\eta_3\eta_9}{9}+2\eta_6\eta_9+\frac{22\eta_9^2}{9}\Big]\Big]$$

Out[•]=

$$\text{Kas}\Big[\text{Bndry}[\{-7,3,6,9,-1,-4\}],-1,\ \text{PQ}\Big[\Big\{\eta_{-7}+\frac{3\eta_{-4}}{2}+\eta_{-1}+\frac{3\eta_3}{2}+\eta_6+\frac{3\eta_9}{2}\Big\},\ -\frac{5\eta_{-8}^2}{9}+\frac{49\eta_{-4}^2}{18}+3\eta_{-4}\eta_{-1}+\frac{2\eta_{-1}^2}{9}+7\eta_{-4}\eta_3+\frac{7}{3}\eta_{-1}\eta_3+\frac{49\eta_3^2}{18}+\frac{7}{3}\eta_{-4}\eta_6+2\eta_{-1}\eta_6+3\eta_3\eta_6+\frac{2\eta_6^2}{9}+8\eta_{-4}\eta_9+\frac{11}{3}\eta_{-1}\eta_9+8\eta_3\eta_9+\frac{11\eta_6\eta_9}{3}+\frac{11\eta_9^2}{2}\Big]\Big]$$

In[•]:=
```
Kas[X[4, 2, 5, 1]] ⋃ Kas[X[7, 3, 8, 2]] ⋃ Kas[X[8, 6, 9, 5]] // c_{2,-2} // c_{5,-5} // c_{8,-8}
```

Out[•]=

$$\text{Kas}\Big[\text{Bndry}[\{-7,3,6,9,-1,-4\}],\text{Sign}[2+4v],\ \text{PQ}\Big[\{\eta_{-7}+2u\eta_{-4}+\eta_{-1}+2u\eta_3+\eta_6+2u\eta_9\},\ \eta_{-4}^2+v\eta_{-4}^2+4u\eta_{-4}\eta_{-1}+\eta_{-1}^2+v\eta_{-1}^2+2\eta_{-4}\eta_3+2u\eta_{-1}\eta_3+\eta_3^2+v\eta_3^2+2u\eta_{-4}\eta_6+2\eta_{-1}\eta_6+4u\eta_3\eta_6+\eta_6^2+v\eta_6^2+\eta_8^2+2v\eta_8^2+2\eta_{-4}\eta_9+4u^2\eta_{-4}\eta_9+6u\eta_{-1}\eta_9+2\eta_3\eta_9+4u^2\eta_3\eta_9+6u\eta_6\eta_9+2\eta_9^2+4u^2\eta_9^2\Big]\Big]$$

In[•]:=
```
Kas[X[7, 5, 8, 4]] ⋃ Kas[X[8, 2, 9, 1]] ⋃ Kas[X[5, 3, 6, 2]] // c_{2,-2} // c_{5,-5} // c_{8,-8}
```

Out[•]=

$$\text{Kas}\Big[\text{Bndry}[\{-7,3,6,9,-1,-4\}],\text{Sign}[2+4v],\ \text{PQ}\Big[\Big\{\eta_{-7}+\frac{\eta_{-4}}{2u}+\eta_{-1}+\frac{\eta_3}{2u}+\eta_6+\frac{\eta_9}{2u}\Big\},\ \eta_{-8}^2+2v\eta_{-8}^2-\eta_{-4}^2+\frac{\eta_{-4}^2}{2u^2}+v\eta_{-4}^2+\frac{\eta_{-4}\eta_{-1}}{u}+\eta_{-1}^2+v\eta_{-1}^2-2\eta_{-4}\eta_3+\frac{\eta_{-4}\eta_3}{u^2}+\frac{\eta_{-1}\eta_3}{u}-2u\eta_{-1}\eta_3-\eta_3^2+\frac{\eta_3^2}{2u^2}+v\eta_3^2+\frac{\eta_{-4}\eta_6}{u}-2u\eta_{-4}\eta_6+2\eta_{-1}\eta_6+\frac{\eta_3\eta_6}{u}+\eta_6^2+v\eta_6^2-\eta_{-4}\eta_9+\frac{\eta_{-4}\eta_9}{u^2}+\frac{\eta_{-1}\eta_9}{u}+2u\eta_{-1}\eta_9-\eta_3\eta_9+\frac{\eta_3\eta_9}{u^2}+\frac{\eta_6\eta_9}{u}+2u\eta_6\eta_9+\eta_9^2+\frac{\eta_9^2}{2u^2}\Big]\Big]$$