

```
In[*]:= Once[
  SetDirectory["C:\\drorbn\\AcademicPensieve\\Talks\\ICERM-2305"];
  << KnotTheory` ;
  << LinAlg.m
]
```

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```
In[*]:= SetAttributes[Bndry, Orderless];
```

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```
In[*]:= CF[b_Bndry] := RotateLeft[#, First@Ordering[#] - 1] & /@ b
```

```
In[*]:= CF[Bndry[{2, 1, 3, 4}, {6, 5, 7}]]
```

Out[\*]=

```
Bndry[{5, 7, 6}, {1, 3, 4, 2}]
```

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```
In[*]:= Kas[P[i_, j_]] := Kas[CF@Bndry[{i, j}], 0, PQ[Subspace[{y_i, y_j}, {y_i, y_j}], 0]]
```

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```
In[*]:= Kas[X[i_, j_, k_, L_]] := If[PositiveQ@X[i, j, k, L],
  Kas[CF@Bndry[{-i, j, k, -L}], 0, PQ[Subspace[{y_{-i}, y_j, y_k, y_{-L}}, {y_{-i}, y_j, y_k, y_{-L}}],
     $\frac{1}{2} (\eta_{-i}^2 + 2 u \eta_{-i} \eta_j + v \eta_j^2 + 2 \eta_{-i} \eta_k + 2 u \eta_j \eta_k + \eta_k^2 + 2 u \eta_{-i} \eta_{-L} + 2 \eta_j \eta_{-L} + 2 u \eta_k \eta_{-L} + v \eta_{-L}^2) ]],
  Kas[CF@Bndry[{-i, -j, k, L}], 0, PQ[Subspace[{y_{-j}, y_k, y_L, y_{-i}}, {y_{-j}, y_k, y_L, y_{-i}}],
     $\frac{1}{2} (-v \eta_{-i}^2 - 2 u \eta_{-i} \eta_{-j} - \eta_{-j}^2 - 2 \eta_{-i} \eta_k - 2 u \eta_{-j} \eta_k - v \eta_k^2 - 2 u \eta_{-i} \eta_L - 2 \eta_{-j} \eta_L - 2 u \eta_k \eta_L - \eta_L^2) ]],
]$$ 
```

The disjoint union in the world of multi-tangles.

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```
Kas /: Kas[b1_, σ1_, pq1_] ∪ Kas[b2_, σ2_, pq2_] :=
  Kas[CF@Join[b1, b2], σ1 + σ2, pq1 ⊕ pq2];
```

```
In[*]:= Kas[P[1, 2]] ∪ Kas[P[3, 4]]
```

Out[\*]=

```
Kas[Bndry[{1, 2}, {3, 4}], 0, PQ[Subspace[{y_1, y_2, y_3, y_4}, {y_1, y_2, y_3, y_4}], 0]]
```

In[\*]:= **Kas**[**X**[1, 5, 2, 6]] **U** **Kas**[**X**[12, 8, 13, 7]]

Out[\*]=

**Kas**[**Bndry** [{-12, 8, 13, -7}, {-5, 2, 6, -1}], **0**,  
**PQ**[**Subspace** [{y<sub>-12</sub>, y<sub>-7</sub>, y<sub>-5</sub>, y<sub>-1</sub>, y<sub>2</sub>, y<sub>6</sub>, y<sub>8</sub>, y<sub>13</sub>}, {y<sub>-12</sub>, y<sub>-7</sub>, y<sub>-5</sub>, y<sub>-1</sub>, y<sub>2</sub>, y<sub>6</sub>, y<sub>8</sub>, y<sub>13</sub>}],  
 $\frac{\eta_{-12}^2}{2} + u \eta_{-12} \eta_{-7} - \frac{\eta_{-5}^2}{2} - u \eta_{-5} \eta_{-1} - u \eta_{-5} \eta_2 - \eta_{-1} \eta_2 - \eta_{-5} \eta_6 - u \eta_{-1} \eta_6 -$   
 $u \eta_2 \eta_6 - \frac{\eta_6^2}{2} + u \eta_{-12} \eta_8 + \eta_{-7} \eta_8 + \eta_{-12} \eta_{13} + u \eta_{-7} \eta_{13} + u \eta_8 \eta_{13} + \frac{\eta_{13}^2}{2}$ ]]

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```
(* FM for FaceMerge *)
Kas[Bndry [{li___, i_, ri___}, {lj___, j_, rj___}, bs___],
  σ_, PQ[Subspace[vs_, gs_], Q]] // FMi,j := Module [{φ},
  φ = Echo@LT [{y0} U Complement [vs, {yi, yj}],
  vs, {y0 → yi + yj} U Table [v → v, {v, Complement [vs, {yi, yj]}]}];
  Kas[CF@Bndry [{ri, li, i, rj, lj, j}, bs], σ, PQ[Subspace[vs, gs], Q] // φ* // Echo // φ*]
]
```

In[\*]:= **Kas**[**P**[1, 2]] **U** **Kas**[**P**[3, 4]] // **FM**<sub>1,4</sub>

- » **LT** [{*y*<sub>0</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}, {*y*<sub>1</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>, *y*<sub>4</sub>}, {*y*<sub>0</sub> → *y*<sub>1</sub> + *y*<sub>4</sub>, *y*<sub>2</sub> → *y*<sub>2</sub>, *y*<sub>3</sub> → *y*<sub>3</sub>}]
- » **PQ**[**Subspace** [{*y*<sub>0</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}, {*y*<sub>0</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}], **0**]

Out[\*]=

**Kas**[**Bndry** [{1, 3, 4, 2}], **0**, **PQ**[**Subspace** [{*y*<sub>1</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>, *y*<sub>4</sub>}, {*y*<sub>1</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>, *y*<sub>4</sub>}], **0**]]

In[\*]:= **LT** [{*y*<sub>0</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}, {*y*<sub>1</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>, *y*<sub>4</sub>}, {*y*<sub>0</sub> → *y*<sub>1</sub> + *y*<sub>4</sub>, *y*<sub>2</sub> → *y*<sub>2</sub>, *y*<sub>3</sub> → *y*<sub>3</sub>}]\_\* [**Subspace** [{*y*<sub>0</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}, {*y*<sub>0</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}]]

Out[\*]=

**Subspace** [{*y*<sub>1</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>, *y*<sub>4</sub>}, {*y*<sub>1</sub> + *y*<sub>4</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}]

In[\*]:= **LT** [{*y*<sub>0</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}, {*y*<sub>1</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>, *y*<sub>4</sub>}, {*y*<sub>0</sub> → *y*<sub>1</sub> + *y*<sub>4</sub>, *y*<sub>2</sub> → *y*<sub>2</sub>, *y*<sub>3</sub> → *y*<sub>3</sub>}]\_\* [**PQ**[**Subspace** [{*y*<sub>0</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}, {*y*<sub>0</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>}], **0**]]

Out[\*]=

**PQ**[**Subspace** [{*y*<sub>1</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>, *y*<sub>4</sub>}, {*y*<sub>1</sub>, *y*<sub>2</sub>, *y*<sub>3</sub>, *y*<sub>4</sub>}], **0**]

Dreams: The following should automatically evaluate.

In[\*]:= **Kas** [{**X**[1, 2, 5, 4], **X**[5, 3, 7, 6], **X**[4, 6, 9, 8]}]

Out[\*]=

**Kas** [{**X**[1, 2, 5, 4], **X**[5, 3, 7, 6], **X**[4, 6, 9, 8]}]

```
In[*]:= Kas /@ {X[1, 2, 5, 4], X[5, 3, 7, 6], X[4, 6, 9, 8]}
Out[*]= {Kas [Bndry [{-4, -1, 2, 5}], 0, PQ [Subspace [{y_{-1}, y_2, y_5, y_{-4}}, {y_{-1}, y_2, y_5, y_{-4}}],
      1/2 (v η_{-4}^2 + 2 u η_{-4} η_{-1} + η_{-1}^2 + 2 η_{-4} η_2 + 2 u η_{-1} η_2 + v η_2^2 + 2 u η_{-4} η_5 + 2 η_{-1} η_5 + 2 u η_2 η_5 + η_5^2) ]],
Kas [Bndry [{-6, -5, 3, 7}], 0, PQ [Subspace [{y_{-5}, y_3, y_7, y_{-6}}, {y_{-5}, y_3, y_7, y_{-6}}],
      1/2 (v η_{-6}^2 + 2 u η_{-6} η_{-5} + η_{-5}^2 + 2 η_{-6} η_3 + 2 u η_{-5} η_3 + v η_3^2 + 2 u η_{-6} η_7 + 2 η_{-5} η_7 + 2 u η_3 η_7 + η_7^2) ]],
Kas [Bndry [{-8, -4, 6, 9}], 0, PQ [Subspace [{y_{-4}, y_6, y_9, y_{-8}}, {y_{-4}, y_6, y_9, y_{-8}}],
      1/2 (v η_{-8}^2 + 2 u η_{-8} η_{-4} + η_{-4}^2 + 2 η_{-8} η_6 + 2 u η_{-4} η_6 + v η_6^2 + 2 u η_{-8} η_9 + 2 η_{-4} η_9 + 2 u η_6 η_9 + η_9^2) ]]}]
```

```
In[*]:= Union@@ (Kas /@ {X[1, 2, 5, 4], X[5, 3, 7, 6], X[4, 6, 9, 8]})
Out[*]= Kas [Bndry [{-8, -4, 6, 9}, {-6, -5, 3, 7}, {-4, -1, 2, 5}],
  0, PQ [Subspace [{y_{-8}, y_{-6}, y_{-5}, y_{-4}, y_{-1}, y_2, y_3, y_5, y_6, y_7, y_9},
    {y_{-8}, y_{-6}, y_{-5}, y_{-4}, y_{-1}, y_2, y_3, y_5, y_6, y_7, y_9}],
  u η_{-6} η_{-5} + η_{-5}^2/2 + u η_{-8} η_{-4} + η_{-4}^2/2 + u η_{-4} η_{-1} + η_{-1}^2/2 + η_{-4} η_2 + u η_{-1} η_2 + η_{-6} η_3 + u η_{-5} η_3 + u η_{-4} η_5 + η_{-1} η_5 +
  u η_2 η_5 + η_5^2/2 + η_{-8} η_6 + u η_{-4} η_6 + u η_{-6} η_7 + η_{-5} η_7 + u η_3 η_7 + η_7^2/2 + u η_{-8} η_9 + η_{-4} η_9 + u η_6 η_9 + η_9^2/2 ]]
```

**Union@@ (Kas /@ {X[1, 2, 5, 4], X[5, 3, 7, 6], X[4, 6, 9, 8]}) // C<sub>□</sub>**