Pensieve header: Derived from pensieve://Projects/SL2Invariant/Archive/nb/SL2Invariant-180811.pdf.

Program

Internal Utilities

Canonical Form:

```
CF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CF[x]];
```

The Kronecker δ:

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
```

Zip and Bind

Variables and their duals:

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};  (u_{i_})* := (u*)_i;
```

Finite Zips:

```
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P;  Zip_{ζ_,ζs___}[P_] :=
  (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :> ∂_{ζ*,d} f) /. ζ* → 0
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard the L variables as scalars.

All

In[∘]:=
```
QZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
  zs = Table[ζ*, {ζ, ζs}];
  c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
  ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
  ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
  qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
  zrule = Thread[zs → qt.(zs + ys)];
  Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
  CF /@ 𝔼[L, Q2, Det[qt] e^{-Q2} Zip_{ζs}[e^{Q1} (P /. zrule)]] ];
```

All

Upper to lower and lower to Upper:

All

In[∘]:=
```
U2l = {B_{i_}^{p_.} → e^{-p ħ γ b_i}, B^{p_.} → e^{-p ħ γ b}, T_{i_}^{p_.} → e^{p ħ t_i}, T^{p_.} → e^{p ħ t}, 𝒜_{i_}^{p_.} → e^{p γ α_i}, 𝒜^{p_.} → e^{p γ α}};
l2U = {e^{c_. b_{i_}+d_.} :→ B_i^{-c/(ħ γ)} e^d, e^{c_. b+d_.} :→ B^{-c/(ħ γ)} e^d,
  e^{c_. t_{i_}+d_.} :→ T_i^{c/ħ} e^d, e^{c_. t+d_.} :→ T^{c/ħ} e^d,
  e^{c_. α_{i_}+d_.} :→ 𝒜_i^{c/γ} e^d, e^{c_. α+d_.} :→ 𝒜^{c/γ} e^d,
  e^{ℰ_} :→ e^{Expand@ℰ}};
```

All

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^Q as a single"P". Here the z's are b and α and the ζ's are β and a.

All

In[∘]:=
```
LZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
  zs = Table[ζ*, {ζ, ζs}];
  c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
  ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
  ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
  lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
  zrule = Thread[zs → lt.(zs + ys)];
  L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
  Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
  CF /@ 𝔼[L2, Q2, Det[lt] e^{-L2-Q2} Zip_{ζs}[e^{L1+Q1} (P /. U2l /. zrule)]] //. l2U ];
```

All

In[∘]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n},
  Times[
    L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
    R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
  ] // LZip_{Join@@Table[{β_{n@i}, τ_{n@i}, a_{n@i}}, {i, {is}}]} // QZip_{Join@@Table[{ξ_{n@i}, y_{n@i}}, {i, {is}}]} ];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

All

In[∘]:=
$$m_{i\_,j\_\to k\_} := \mathbb{E}\Big[a_k\,\alpha_i + a_k\,\alpha_j + b_k\,\beta_i + b_k\,\beta_j,\ \frac{1}{\hbar\,\mathcal{A}_i\,\mathcal{A}_j}\left(\hbar\,y_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_i + \hbar\,y_k\,\mathcal{A}_j\,\eta_j + \hbar\,x_k\,\mathcal{A}_i\,\xi_i + \mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i - B_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i + \hbar\,x_k\,\mathcal{A}_i\,\mathcal{A}_j\,\xi_j\right), 1\Big];$$
$$R_{i\_,j\_} := \mathbb{E}[\hbar\,a_j\,b_i,\ \hbar\,x_j\,y_i,\ 1];$$
$$\overline{R}_{i\_,j\_} := \mathbb{E}\Big[-\hbar\,a_j\,b_i,\ -\frac{\hbar\,x_j\,y_i}{B_i},\ 1\Big];$$
$$S_{i\_} := \mathbb{E}\Big[-a_i\,\alpha_i - b_i\,\beta_i,\ \frac{1}{\hbar\,B_i}\left(-\hbar\,y_i\,\mathcal{A}_i\,\eta_i - \hbar\,B_i\,x_i\,\mathcal{A}_i\,\xi_i + \mathcal{A}_i\,\eta_i\,\xi_i - B_i\,\mathcal{A}_i\,\eta_i\,\xi_i\right), 1\Big];$$
$$\Delta_{i\_\to j\_,k\_} := \mathbb{E}[a_j\,\alpha_i + a_k\,\alpha_i + b_j\,\beta_i + b_k\,\beta_i,\ y_j\,\eta_i + B_j\,y_k\,\eta_i + x_j\,\xi_i + x_k\,\xi_i,\ 1];$$
$$C_{i\_} := \mathbb{E}\big[0, 0, B_i^{1/2}\big];$$
$$\overline{C}_{i\_} := \mathbb{E}\big[0, 0, B_i^{-1/2}\big];$$

All

In[∘]:=
$$\text{Kink}_{i\_} := \left(R_{1,3}\,\overline{C}_2\right) \sim B_{1,2} \sim m_{1,2\to 1} \sim B_{1,3} \sim m_{1,3\to i};$$
$$\overline{\text{Kink}}_{i\_} := \left(\overline{R}_{1,3}\,C_2\right) \sim B_{1,2} \sim m_{1,2\to 1} \sim B_{1,3} \sim m_{1,3\to i};$$

All

Testing

All

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

All

(co)-associativity

All

In[∘]:=
$$\text{HL} /@ \{(\Delta_{1\to 1,2} \sim B_2 \sim \Delta_{2\to 2,3}) \equiv (\Delta_{1\to 1,3} \sim B_1 \sim \Delta_{1\to 1,2}),\ (m_{1,2\to 1} \sim B_1 \sim m_{1,3\to 1}) \equiv (m_{2,3\to 2} \sim B_2 \sim m_{1,2\text{->}1})\}$$

All

Out[∘]= {True, True}

All

Δ is an algebra morphism

All

In[∘]:=
$$\text{HL}[m_{1,2\to 1} \sim B_1 \sim \Delta_{1\to 1,2} \equiv (\Delta_{1\to 1,3}\,\Delta_{2\to 2,4}) \sim B_{1,2,3,4} \sim (m_{3,4\to 2}\,m_{1,2\to 1})]$$

All

Out[∘]= True

All

S_2 inverts R, but not S_1:

All

In[∘]:=
$$\{R_{1,2} \sim B_1 \sim S_1 \equiv \overline{R}_{1,2},\ \text{HL}[R_{1,2} \sim B_2 \sim S_2 \equiv \overline{R}_{1,2}]\}$$

All

Out[∘]= {True, True}

All

S is convolution inverse of id

All
In[∘]:=
```
HL[# ≡ 𝔼[0, 0, 1]] & /@ {(Δ_{1→1,2} ~ B_1 ~ S_1) ~ B_{1,2} ~ m_{1,2→1}, (Δ_{1→1,2} ~ B_2 ~ S_2) ~ B_{1,2} ~ m_{1,2→1}}
```
All
Out[∘]= {True, True}

All

S is a (co)-algebra anti-morphism

All
In[∘]:=
```
HL /@ Expand /@ {m_{1,2→1} ~ B_1 ~ S_1 ≡ (S_1 S_2) ~ B_{1,2} ~ m_{2,1→1}, S_1 ~ B_1 ~ Δ_{1→1,2} ≡ Δ_{1→2,1} ~ B_{1,2} ~ (S_1 S_2)}
```
All
Out[∘]= {True, True}

All

Quasi-triangular axiom 1:

All
In[∘]:=
```
HL[R_{1,2} ~ B_1 ~ Δ_{1->1,3} ≡ (R_{1,4} R_{3,2}) ~ B_{2,4} ~ m_{2,4→2}]
```
All
Out[∘]= True

All

Quasi-triangular axiom 2:

All
In[∘]:=
```
HL[((Δ_{1->1,2} R_{3,4}) ~ B_{1,2,3,4} ~ (m_{1,3→1} m_{2,4→2})) ≡ ((Δ_{1→2,1} R_{3,4}) ~ B_{1,2,3,4} ~ (m_{3,1→1} m_{4,2→2}))]
```
All
Out[∘]= True

All
In[∘]:=
```
HL /@ {(C_i C̄_j) ~ B_{i,j} ~ m_{i,j→i} ≡ 𝔼[0, 0, 1], (C̄_i C_j) ~ B_{i,j} ~ m_{i,j→i} ≡
  (((R_{1,2} ~ B_1 ~ S_1 ~ B_{1,2} ~ m_{2,1→i}) ~ B_i ~ S_i) (R_{1,2} ~ B_2 ~ S_2 ~ B_2 ~ S_2 ~ B_{1,2} ~ m_{2,1→j})) ~ B_{i,j} ~ m_{i,j→i}}
```
All
Out[∘]= {True, True}

All

Reidemeister 2:

All
In[∘]:=
```
HL[# ≡ 𝔼[0, 0, 1]] & /@ {(R̄_{1,2} R_{3,4}) ~ B_{1,2,3,4} ~ (m_{1,3→1} m_{2,4→2}), (R_{1,2} R̄_{3,4}) ~ B_{1,2,3,4} ~ (m_{1,3→1} m_{2,4→2})}
```
All
Out[∘]= {True, True}

All

Cyclic Reidemeister 2:

All
In[∘]:=
```
HL[(R_{1,4} R̄_{5,2} C̄_3) ~ B_{2,4} ~ m_{2,4→2} ~ B_{1,3} ~ m_{1,3→1} ~ B_{1,5} ~ m_{1,5→1} ≡ C̄_1]
```
All
Out[∘]= True

All

Reidemeister 3:

All
In[∘]:=
```
HL[((R_{1,2} R_{4,3} R_{5,6}) ~ B_{1,4} ~ m_{1,4→1} ~ B_{2,5} ~ m_{2,5→2} ~ B_{3,6} ~ m_{3,6→3}) ≡
  ((R_{1,6} R_{2,3} R_{4,5}) ~ B_{1,4} ~ m_{1,4→1} ~ B_{2,5} ~ m_{2,5→2} ~ B_{3,6} ~ m_{3,6→3})]
```
All
Out[∘]= True

All

All

The Trefoil

All

In[ ]:=
```
Z = R_{1,5} R_{6,2} R_{3,7} C̄_4 K̄ink_8 K̄ink_9 K̄ink_10;
Do[Z = Z ~ B_{1,r} ~ m_{1,r→1}, {r, 2, 10}];
Simplify /@ Z
```

All

Out[ ]= $\mathbb{E}\left[0, 0, \frac{B_1}{1 - B_1 + B_1^2}\right]$