Post mortem

July 21, 2016 8:16 AM

Dror Bar-Natan: Talks: Greece-1607: ωεβ:=http://drorbn.net/Greece-1607/

Work in Progress!

The Brute and the Hidden Paradise

Abstract. There is expected to be a hidden paradise of poly-time computable knot polynomials lying just beyond the Alexander polynomial. I will describe my brute attempts to gain entry.

Why "expected"? Gauss diagram formulas [PV, GPV] show that $v_{d,f}(K) = \sum_{Y\subset X(K),\ |Y|=d} f(Y)$ finite-type invariants are all poly-time, and tempt to conjecture that there are no others. But Alexander shows it nonsense:

d	2	3	4	5	6	7	8	$\cdots$
known invts in* $O(n^d)$	1	1	∞	3	4	8	11	$\cdots$

This is an unreasonable picture! **Fresh, numerical, no cheating.*

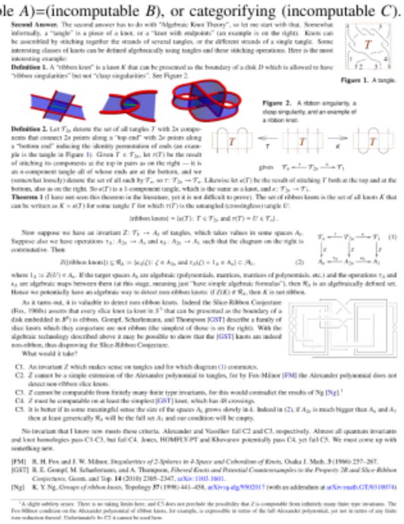
So there ought to be further poly-time invariants.

Also. • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.

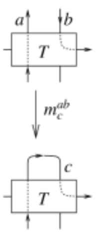
Rozansky

Why "paradise"? Foremost answer: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

ωεβ/K17: (extend to tangles, perhaps detect non-slice ribbon knots)



Moral. Need "stitching":



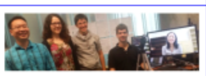
Why "brute"? Cause it's the only thing I know, for now. There may be better ways in, and it's fair to hope that sooner or later they will be found.

The Gold Standard is set by the formulas [BNS, BN] for Alexander. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A \end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a\colon a \in S\})$:

$$\left({}_a\!\nearrow_b,\ {}_b\!\nwarrow_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm 1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}\right)$$

Help Needed! Disorganized videos of talks in a private seminar are at ωεβ/PP.



Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

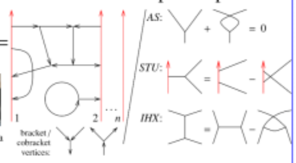
For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!

Dunfield: 1000-crossing fast.

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z\colon \left\{\begin{matrix} S\text{-component} \\ (v/b\text{-})\text{tangles} \end{matrix}\right\} \to \mathcal{A}^v_S :=$$

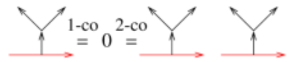
Etingof Kazhdan Haviv Enriquez Ševera

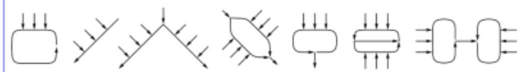


(it is enough to know Z on ⤫ and have disjoint union and stitching formulas) ...exponential and too hard!

Idea. Look for "ideal" quotients of $\mathcal{A}^v_S$ that have poly-sized descriptions; ...specifically, limit the co-brackets.

1-co and 2-co, aka TC and TC^2, on the right. The primitives that remain are:

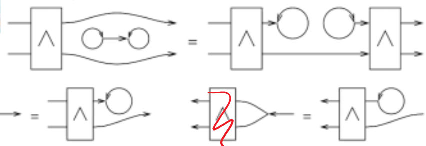




...manageable but still exponential!

Jones

The $2D$ relations come from the relation with 2D Lie bialgebras:

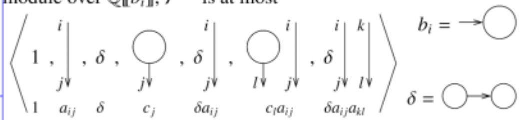


No need for these antisymmetrization boxes.

We let $\mathcal{A}^{2,2}$ be $\mathcal{A}^v$ modulo 2-co and $2D$, and $z^{2,2}$ be the projection of $\log Z$ to $\mathcal{P}^{2,2} := \pi\mathcal{P}^v$, where $\mathcal{P}^v$ are the primitives of $\mathcal{A}^v$.

Main Claim. $z^{2,2}$ is poly-time computable.

Main Point. $\mathcal{P}^{2,2}$ is poly-size, so how hard can it be? Indeed, as a module over $\mathbb{Q}[\![b_i]\!]$, $\mathcal{P}^{2,2}$ is at most



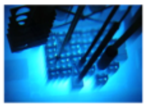
Claim. $R_{jk} = e^{a_{jk}} e^{\rho_{jk}}$ is a solution of the Yang-Baxter / R3 equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in $\exp\mathcal{P}^{2,2}$, with $\rho_{jk} :=$

$$\psi(b_j)\left(-c_k + \frac{c_k a_{jk}}{b_j} - \frac{\delta a_{jk} a_{jk}}{b_j^2}\right) + \frac{\phi(b_j)\psi(b_k)}{b_k\phi(b_k)}\left(c_k a_{kk} - \frac{\delta a_{jk} a_{kk}}{b_j}\right),$$

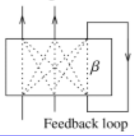
and with $\phi(x) := e^{-x} - 1 = -x + x^2/2 - \ldots$, and $\psi(x) := ((x+2)e^{-x} - 2 + x)/(2x) = x^2/12 - x^3/24 + \ldots$. (This already gives some new (v-)braid group representations, as below).

Problem. How do we multiply in $\exp(\mathcal{P}^{2,2})$? How do we stitch? BCH is a theoretical dream. Instead, use "scatter and glow" and "feedback loops":

The Euler trick:
With $Ef := (\deg f)f$ get $Ee^x = xe^x$ and $E(e^x e^y e^z) = xe^x e^y e^z + e^x y e^y e^z + e^x e^y z e^z$.



Cherenkov radiation



Feedback loop

Work in Progress!

The Brute and the Hidden Paradise

van der Veen

Local Algebra (with van der Veen) Much can be re-formulated as (non-standard) "quantum algebra" for the 4D Lie algebra $\mathfrak{g} = \langle b, c, u, w\rangle$ over $\mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$. The key: $a_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g})^{\otimes\{i,j\}}$.

Some (new) representationss of the (v-)braid groups.

ωεβ/Reps

Burau (old)

```
B_{i_,j_}[ℰ_] := ℰ /. v_j ↦ (1 - t) v_i + t v_j
```

…testing R3

```
Column@{lhs = {v1, v2, v3} // B1,2 // B1,3 // B2,3,
  rhs = {v1, v2, v3} // B2,3 // B1,3 // B1,2 ,
  lhs - rhs // Expand}

{v1, (1 - t) v1 + t v2, (1 - t) v1 + t ((1 - t) v2 + t v3)}
{v1, (1 - t) v1 + t v2,
 (1 - t) ((1 - t) v1 + t v2) + t ((1 - t) v1 + t v3)}
{0, 0, 0}
```

Gassner (old)

```
G_{i_,j_}[ℰ_] := ℰ /. v_j ↦ (1 - t_i) v_i + t_i v_j
```

…Overcrossings Commute (OC):

```
Column@{lhs = {v1, v2, v3} // G1,2 // G1,3,
  Expand[lhs - ({v1, v2, v3} // G1,3 // G1,2)]}

{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}
{0, 0, 0}
```

…Undercrossings Commute (UC):

```
Column@{lhs = {v1, v2, v3} // G1,3 // G2,3,
  rhs = {v1, v2, v3} // G2,3 // G1,3 ,
  lhs - rhs // Expand}

{v1, v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3)}
{v1, v2, (1 - t2) v2 + t2 ((1 - t1) v1 + t1 v3)}
{0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}
```

Gassner Plus (new?)

```
GP_{i_,j_}[ℰ_] := Expand[ℰ /. {u_j ↦ (1 - t_i) u_i + t_i u_j,
    f_. v_j ↦ f (1 - t_i) v_i + f t_i v_j + (t_i - 1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) u_i +
      f t_i u_i }];
bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3,
   u1, u2, u3};
```

…R3 (left)

```
Short[lhs = bas // GP1,2 // GP1,3 // GP2,3, 2]

{f[t1, t2, t3] v1, f[t1, t2, t3] t1 u1 + f[t1, t2, t3] v1 -
  f[t1, t2, t3] t1 v1 + <<6>> + t1^2 u1 f^(1,0,0)[t1, t2, t3],
 <<1>> + <<19>> + <<1>>, <<1>>, u1 - t1 u1 + t1 u2,
 u1 - t1 u1 + t1 u2 - t1 t2 u2 + t1 t2 u3}
```

…R3 (rest)

```
(bas // GP2,3 // GP1,3 // GP1,2) - lhs

{0, 0, 0, 0, 0, 0}
```

…OC

```
(bas // GP1,2 // GP1,3) - (bas // GP1,3 // GP1,2)

{0, 0, 0, 0, 0, 0}
```

Question. Does Gassner Plus factor through Gassner?

Turbo-Gassner (new!)

```
Kδ_{i_,j_} := KroneckerDelta[i, j];
TG_{i_,j_}[ℰ_] := Expand[ℰ /. {
    f_. v_{k_} ↦ Plus[f v_k /. v_j → (1 - t_i) v_i + t_i v_j,
      (1 - t_i^-1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) *
       (u_k /. u_j → (1 - t_i) u_i + t_i u_j) * u_i w_j,
      Kδ_{k,i} f (u_j - u_i) u_i w_j],
    u_j → (1 - t_i) u_i + t_i u_j,
    w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3,
   u1, u2, u3, w1, w2, w3};
```

Satisfies R3…

…OC

```
(bas // TG1,2 // TG1,3) - (bas // TG1,3 // TG1,2)

{0, -f[t1, t2, t3] u1 u2 w3 + f[t1, t2, t3] t1 u1 u2 w3 +
  f[t1, t2, t3] u1 u3 w3 - f[t1, t2, t3] t1 u1 u3 w3,
 -f[t1, t2, t3] u1 u2 w2 + f[t1, t2, t3] t1 u1 u2 w2 +
  f[t1, t2, t3] u1 u3 w2 -
  f[t1, t2, t3] t1 u1 u3 w2, 0, 0, 0, 0, 0, 0}
```

Turbo-Burau (new!)

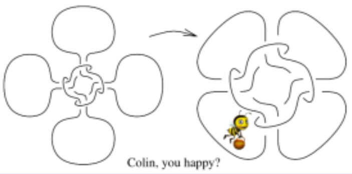
```
η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
TB_{i_,j_}[ℰ_] :=
  Expand[ℰ /. {
     f_. v_{k_} ↦ Plus[f v_k /. v_j → (1 - t - η[i]) v_i + (t + η[i]) v_j,
       (t - 1) (Coefficient[f, η[i]] - Coefficient[f, η[j]]) *
        (u_k /. u_j → (1 - t) u_i + t u_j) * u_i w_j,
       Kδ_{k,i} (f /. _η → 0) (u_j - u_i) u_i w_j],
     u_j → (1 - t) u_i + t u_j,
     w_i → w_i + (1 - t^-1) w_j, w_j → t^-1 w_j}];
ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
bas = {ff v1, ff v2, ff v3, u1^2 w1, u1^2 w2, u1, u2, u3, w1, w2, w3};
```

…OC

```
(bas // TB1,2 // TB1,3) - (bas // TB1,3 // TB1,2)

{0, -f0 u1 u2 w3 + t f0 u1 u2 w3 + f0 u1 u3 w3 - t f0 u1 u3 w3,
 -f0 u1 u2 w2 + t f0 u1 u2 w2 + f0 u1 u3 w2 - t f0 u1 u3 w2,
 0, 0, 0, 0, 0, 0, 0, 0}
```

Flower Surgery Theorem. A knot is ribbon iff it is the result of n-petal flower surgery (from thin petals to wide petals) on an n-componenet unlink, for some n.



Colin, you happy?

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"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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