

Dror Bar-Natan: Talks: Greece-1607: <http://drorbn.net/Greece-1607/>  
 Work in Progress! **The Brute and the Hidden Paradise**

**Abstract.** There is expected to be a hidden paradise of poly-time computable knot polynomials lying just beyond the Alexander polynomial. I will describe my brute attempts to gain entry.

**Why "expected"?** Gauss diagram  $v_{d,f}(K) = \sum_{Y \subset X(K), |Y|=d} f(Y)$  formulas [PV, GPV] show that finite-type invariants are all poly-time, and tempt to conjecture that there are no others. But Alexander shows it nonsense:

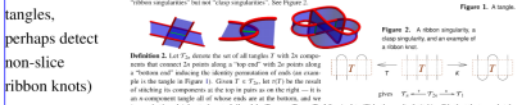
$d$	2	3	4	5	6	7	8	...
known invts* in $O(n^d)$	1	1	$\infty$	3	4	8	11	...

This is an unreasonable picture! \*Fresh, numerical, no cheating. So there ought to be further poly-time invariants.

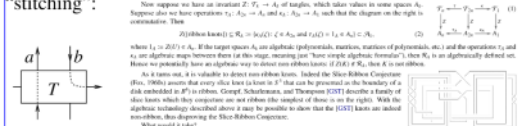
**Also.** • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.

**Why "paradise"?** Foremost answer: **OBVIOUSLY.** Cf. proving (incomputable A)-(incomputable B), or categorifying (incomputable C).

**oeß/K17:** (extend to tangles, perhaps detect non-slice ribbon knots)



**Moral. Need "stitching":**



Now suppose we have an invariant  $Z: \mathcal{T} \rightarrow A$ , of tangles, which takes values in some space  $A$ . Suppose also we have operators  $T_1, T_2, \dots, T_n: A \rightarrow A$ , such that the diagram on the right is commutative. This is the "stitching" operation. The operators  $T_i$  are defined as follows:  $T_1$  is the identity,  $T_2$  is the crossing,  $T_3$  is the crossing with a resolution,  $T_4$  is the crossing with a resolution and a crossing,  $T_5$  is the crossing with a resolution and two crossings,  $T_6$  is the crossing with a resolution and three crossings,  $T_7$  is the crossing with a resolution and four crossings,  $T_8$  is the crossing with a resolution and five crossings,  $T_9$  is the crossing with a resolution and six crossings,  $T_{10}$  is the crossing with a resolution and seven crossings,  $T_{11}$  is the crossing with a resolution and eight crossings,  $T_{12}$  is the crossing with a resolution and nine crossings,  $T_{13}$  is the crossing with a resolution and ten crossings,  $T_{14}$  is the crossing with a resolution and eleven crossings,  $T_{15}$  is the crossing with a resolution and twelve crossings,  $T_{16}$  is the crossing with a resolution and thirteen crossings,  $T_{17}$  is the crossing with a resolution and fourteen crossings,  $T_{18}$  is the crossing with a resolution and fifteen crossings,  $T_{19}$  is the crossing with a resolution and sixteen crossings,  $T_{20}$  is the crossing with a resolution and seventeen crossings,  $T_{21}$  is the crossing with a resolution and eighteen crossings,  $T_{22}$  is the crossing with a resolution and nineteen crossings,  $T_{23}$  is the crossing with a resolution and twenty crossings,  $T_{24}$  is the crossing with a resolution and twenty-one crossings,  $T_{25}$  is the crossing with a resolution and twenty-two crossings,  $T_{26}$  is the crossing with a resolution and twenty-three crossings,  $T_{27}$  is the crossing with a resolution and twenty-four crossings,  $T_{28}$  is the crossing with a resolution and twenty-five crossings,  $T_{29}$  is the crossing with a resolution and twenty-six crossings,  $T_{30}$  is the crossing with a resolution and twenty-seven crossings,  $T_{31}$  is the crossing with a resolution and twenty-eight crossings,  $T_{32}$  is the crossing with a resolution and twenty-nine crossings,  $T_{33}$  is the crossing with a resolution and thirty crossings,  $T_{34}$  is the crossing with a resolution and thirty-one crossings,  $T_{35}$  is the crossing with a resolution and thirty-two crossings,  $T_{36}$  is the crossing with a resolution and thirty-three crossings,  $T_{37}$  is the crossing with a resolution and thirty-four crossings,  $T_{38}$  is the crossing with a resolution and thirty-five crossings,  $T_{39}$  is the crossing with a resolution and thirty-six crossings,  $T_{40}$  is the crossing with a resolution and thirty-seven crossings,  $T_{41}$  is the crossing with a resolution and thirty-eight crossings,  $T_{42}$  is the crossing with a resolution and thirty-nine crossings,  $T_{43}$  is the crossing with a resolution and forty crossings,  $T_{44}$  is the crossing with a resolution and forty-one crossings,  $T_{45}$  is the crossing with a resolution and forty-two crossings,  $T_{46}$  is the crossing with a resolution and forty-three crossings,  $T_{47}$  is the crossing with a resolution and forty-four crossings,  $T_{48}$  is the crossing with a resolution and forty-five crossings,  $T_{49}$  is the crossing with a resolution and forty-six crossings,  $T_{50}$  is the crossing with a resolution and forty-seven crossings,  $T_{51}$  is the crossing with a resolution and forty-eight crossings,  $T_{52}$  is the crossing with a resolution and forty-nine crossings,  $T_{53}$  is the crossing with a resolution and fifty crossings,  $T_{54}$  is the crossing with a resolution and fifty-one crossings,  $T_{55}$  is the crossing with a resolution and fifty-two crossings,  $T_{56}$  is the crossing with a resolution and fifty-three crossings,  $T_{57}$  is the crossing with a resolution and fifty-four crossings,  $T_{58}$  is the crossing with a resolution and fifty-five crossings,  $T_{59}$  is the crossing with a resolution and fifty-six crossings,  $T_{60}$  is the crossing with a resolution and fifty-seven crossings,  $T_{61}$  is the crossing with a resolution and fifty-eight crossings,  $T_{62}$  is the crossing with a resolution and fifty-nine crossings,  $T_{63}$  is the crossing with a resolution and sixty crossings,  $T_{64}$  is the crossing with a resolution and sixty-one crossings,  $T_{65}$  is the crossing with a resolution and sixty-two crossings,  $T_{66}$  is the crossing with a resolution and sixty-three crossings,  $T_{67}$  is the crossing with a resolution and sixty-four crossings,  $T_{68}$  is the crossing with a resolution and sixty-five crossings,  $T_{69}$  is the crossing with a resolution and sixty-six crossings,  $T_{70}$  is the crossing with a resolution and sixty-seven crossings,  $T_{71}$  is the crossing with a resolution and sixty-eight crossings,  $T_{72}$  is the crossing with a resolution and sixty-nine crossings,  $T_{73}$  is the crossing with a resolution and seventy crossings,  $T_{74}$  is the crossing with a resolution and seventy-one crossings,  $T_{75}$  is the crossing with a resolution and seventy-two crossings,  $T_{76}$  is the crossing with a resolution and seventy-three crossings,  $T_{77}$  is the crossing with a resolution and seventy-four crossings,  $T_{78}$  is the crossing with a resolution and seventy-five crossings,  $T_{79}$  is the crossing with a resolution and seventy-six crossings,  $T_{80}$  is the crossing with a resolution and seventy-seven crossings,  $T_{81}$  is the crossing with a resolution and seventy-eight crossings,  $T_{82}$  is the crossing with a resolution and seventy-nine crossings,  $T_{83}$  is the crossing with a resolution and eighty crossings,  $T_{84}$  is the crossing with a resolution and eighty-one crossings,  $T_{85}$  is the crossing with a resolution and eighty-two crossings,  $T_{86}$  is the crossing with a resolution and eighty-three crossings,  $T_{87}$  is the crossing with a resolution and eighty-four crossings,  $T_{88}$  is the crossing with a resolution and eighty-five crossings,  $T_{89}$  is the crossing with a resolution and eighty-six crossings,  $T_{90}$  is the crossing with a resolution and eighty-seven crossings,  $T_{91}$  is the crossing with a resolution and eighty-eight crossings,  $T_{92}$  is the crossing with a resolution and eighty-nine crossings,  $T_{93}$  is the crossing with a resolution and ninety crossings,  $T_{94}$  is the crossing with a resolution and ninety-one crossings,  $T_{95}$  is the crossing with a resolution and ninety-two crossings,  $T_{96}$  is the crossing with a resolution and ninety-three crossings,  $T_{97}$  is the crossing with a resolution and ninety-four crossings,  $T_{98}$  is the crossing with a resolution and ninety-five crossings,  $T_{99}$  is the crossing with a resolution and ninety-six crossings,  $T_{100}$  is the crossing with a resolution and ninety-seven crossings,  $T_{101}$  is the crossing with a resolution and ninety-eight crossings,  $T_{102}$  is the crossing with a resolution and ninety-nine crossings,  $T_{103}$  is the crossing with a resolution and one hundred crossings.

**Why "brute"?** Cause it's the only thing I know, for now. There may be better ways in, and it's fair to hope that sooner or later they will be found.

**The Gold Standard** is set by the formulas [BNS, BN] for Alexander. An  $S$ -component tangle  $T$  has  $\Gamma(T) \in R_S \times M_{S \times S}(R_S) = \left\{ \begin{matrix} \omega & S \\ S & A \end{matrix} \right\}$  with  $R_S := \mathbb{Z}\langle t_a : a \in S \rangle$ :

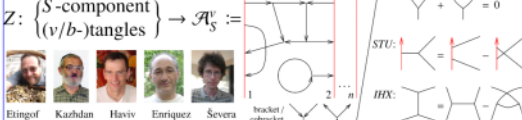
$$\left( \begin{matrix} a & b \\ a' & b' \end{matrix} \right) \rightarrow \begin{matrix} a & b \\ a & 1 - t_a^{-1} \\ b & 0 \\ b & t_a^{-1} \end{matrix} \quad T_1 \sqcup T_2 \rightarrow \begin{matrix} \omega_1 \omega_2 & S_1 & S_2 \\ S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{matrix}$$

$$\begin{matrix} \omega & a & b & S \\ a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{matrix} \xrightarrow{m_c^{ab}} \begin{matrix} (1-\beta)\omega & c & S \\ c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{matrix}$$

**Help Needed!** Disorganized videos of talks in a private seminar are at [oeß/PP](http://oeß/PP).  
 Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

For long knots,  $\omega$  is Alexander, and that's the fastest Alexander algorithm I know!  
 Dunfield: 1000-crossing fast.

**Theorem [EK, Ha, En, Se].** There is a "homomorphic expansion"  $Z: \left\{ \begin{matrix} S\text{-component} \\ (v/b)\text{-tangles} \end{matrix} \right\} \rightarrow \mathcal{A}_S^v :=$



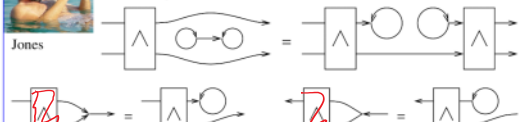
(it is enough to know  $Z$  on  $\mathcal{Z}$  and have disjoint union and stitching formulas) ... exponential and too hard!

**Idea.** Look for "ideal" quotients of  $\mathcal{A}_S^v$  that have poly-sized descriptions; ... specifically, limit the co-brackets.

**1-co and 2-co, aka TC and TC<sup>2</sup>,** on the right. The primitives that remain are:



**The 2D relations** come from the relation with 2D Lie bialgebras:



We let  $\mathcal{A}^{2,2}$  be  $\mathcal{A}^v$  modulo 2-co and 2D, and  $z^{2,2}$  be the projection of  $\log Z$  to  $\mathcal{P}^{2,2} := \pi \mathcal{P}^v$ , where  $\mathcal{P}^v$  are the primitives of  $\mathcal{A}^v$ .

**Main Claim.**  $z^{2,2}$  is poly-time computable.

**Main Point.**  $\mathcal{P}^{2,2}$  is poly-size, so how hard can it be? Indeed, as a module over  $\mathbb{Q}\langle b_i \rangle$ ,  $\mathcal{P}^{2,2}$  is at most

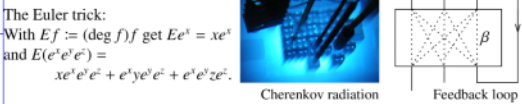
$$\left\langle \begin{matrix} i \\ 1 \\ j \end{matrix} \middle| \delta, \begin{matrix} i \\ j \end{matrix} \middle| \delta, \begin{matrix} i \\ j \\ i \end{matrix} \middle| \delta, \begin{matrix} i \\ j \\ i \\ j \end{matrix} \middle| \delta, \begin{matrix} i \\ j \\ i \\ j \\ i \end{matrix} \middle| \delta, \begin{matrix} i \\ j \\ i \\ j \\ i \\ j \end{matrix} \middle| \delta \right\rangle \quad b_i = \begin{matrix} \circ \\ \circ \end{matrix} \quad \delta = \begin{matrix} \circ \\ \circ \end{matrix}$$

**Claim.**  $R_{jk} = e^{a_j} e^{\rho_{jk}}$  is a solution of the Yang-Baxter / R3 equation  $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$  in  $\exp \mathcal{P}^{2,2}$ , with  $\rho_{jk} :=$

$$\psi(b_j) \left( -c_k + \frac{c_k a_{jk}}{b_j} - \frac{\delta a_{jk} a_{jk}}{b_j^2} \right) + \frac{\phi(b_j) \psi(b_k)}{b_k \phi(b_k)} \left( c_k a_{kk} - \frac{\delta a_{jk} a_{kk}}{b_j} \right),$$

and with  $\phi(x) := e^{-x} - 1 = -x + x^2/2 - \dots$ , and  $\psi(x) := (x+2)e^{-x} - 2 + x / (2x) = x^2/12 - x^3/24 + \dots$ . (This already gives some new (v-)braid group representations, as below).

**Problem.** How do we multiply in  $\exp(\mathcal{P}^{2,2})$ ? How do we stitch? BCH is a theoretical dream. Instead, use "scatter and glow" and "feedback loops":



The Euler trick: With  $E f := (\deg f) f$  get  $E e^x = x e^x$  and  $E(e^x e^y) = x e^x e^y + e^x e^y z$ .

No need for these antisymmetric 2x2 boxes.

### The Brute and the Hidden Paradise

**Local Algebra** (with van der Veen) Much can be reformulated as (non-standard) “quantum algebra” for the 4D Lie algebra  $\mathfrak{g} = \langle b, c, u, w \rangle$  over  $\mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$ , with  $b$  central and  $[w, c] = w, [c, u] = u$ , and  $[u, w] = b - 2\epsilon c$ . The key:  $a_{ij} = (b_j - \epsilon c_j) c_j + u_i w_j$  in  $\mathcal{U}(\mathfrak{g})^{\otimes(i,j)}$ .



van der Veen

Some (new) representations of the (v-)braid groups.

```
B_{i,j}[\epsilon] := \epsilon / . v_j \mapsto (1-t) v_i + t v_j
Column@{lhs = {v1, v2, v3} // B_{1,2} // B_{1,3} // B_{2,3},
rhs = {v1, v2, v3} // B_{2,3} // B_{1,3} // B_{1,2},
lhs - rhs // Expand}
```

```
{v1, (1-t) v1 + t v2, (1-t) v1 + t ((1-t) v2 + t v3)}
{v1, (1-t) v1 + t v2,
(1-t) ((1-t) v1 + t v2) + t ((1-t) v1 + t v3)}
{0, 0, 0}
```

```
G_{i,j}[\epsilon] := \epsilon / . v_j \mapsto (1-t_i) v_i + t_i v_j
```

```
Column@{lhs = {v1, v2, v3} // G_{1,2} // G_{1,3},
Expand[lhs - ({v1, v2, v3} // G_{1,3} // G_{1,2})]}
{v1, (1-t1) v1 + t1 v2, (1-t1) v1 + t1 v3}
{0, 0, 0}
```

... Undercrossings Commute (UC):

```
Column@{lhs = {v1, v2, v3} // G_{1,3} // G_{2,3},
rhs = {v1, v2, v3} // G_{2,3} // G_{1,3},
lhs - rhs // Expand}
{v1, v2, (1-t1) v1 + t1 ((1-t2) v2 + t2 v3)}
{v1, v2, (1-t2) v2 + t2 ((1-t1) v1 + t1 v3)}
{0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}
```

Gassner Plus (new?)

```
GP_{i,j}[\epsilon] := Expand[\epsilon / . {u_j \mapsto (1-t_i) u_i + t_i u_j,
\epsilon_i . v_j \mapsto \epsilon (1-t_i) v_i + \epsilon t_i v_j + (t_i - 1) (t_i \partial_{\epsilon_i} \epsilon - t_j \partial_{\epsilon_j} \epsilon) u_i +
\epsilon t_i u_i}];
bas = {\epsilon[t1, t2, t3] v1, \epsilon[t1, t2, t3] v2, \epsilon[t1, t2, t3] v3,
u1, u2, u3};
```

Short[lhs = bas // GP\_{1,2} // GP\_{1,3} // GP\_{2,3}, 2] ... R3 (left)

```
{\epsilon[t1, t2, t3] v1, \epsilon[t1, t2, t3] t1 u1 + \epsilon[t1, t2, t3] v1 -
\epsilon[t1, t2, t3] t1 v1 + \langle\langle 6 \rangle\rangle + t1^2 u1 f^{(1,0,0)}[t1, t2, t3],
\langle\langle 1 \rangle\rangle + \langle\langle 19 \rangle\rangle + \langle\langle 1 \rangle\rangle, \langle\langle 1 \rangle\rangle, u1 - t1 u1 + t1 u2,
u1 - t1 u1 + t1 u2 - t1 t2 u2 + t1 t2 u3}
```

(bas // GP\_{2,3} // GP\_{1,3} // GP\_{1,2}) - lhs ... R3 (rest)

```
{0, 0, 0, 0, 0, 0}
(bas // GP_{1,2} // GP_{1,3}) - (bas // GP_{1,3} // GP_{1,2})
{0, 0, 0, 0, 0, 0} ... OC
```

Question. Does Gassner Plus factor through Gassner?

K\delta\_{i,j} := KroneckerDelta[i, j]; Turbo-Gassner (new!)

```
TG_{i,j}[\epsilon] := Expand[\epsilon / . {
\epsilon_i . v_k \mapsto Plus[\epsilon v_k / . v_j \mapsto (1-t_i) v_i + t_i v_j,
(1-t_i^{-1}) (t_i \partial_{\epsilon_i} \epsilon - t_j \partial_{\epsilon_j} \epsilon) *
(u_k / . u_j \mapsto (1-t_i) u_i + t_i u_j) * u_i w_j,
K\delta_{k,i} \epsilon (u_j - u_i) u_i w_j],
u_j \mapsto (1-t_i) u_i + t_i u_j,
w_i \mapsto w_i + (1-t_i^{-1}) w_j, w_j \mapsto t_i^{-1} w_j}];
bas = {\epsilon[t1, t2, t3] v1, \epsilon[t1, t2, t3] v2, \epsilon[t1, t2, t3] v3,
u1, u2, u3, w1, w2, w3};
```

Satisfies R3...

(bas // TG\_{1,2} // TG\_{1,3}) - (bas // TG\_{1,3} // TG\_{1,2}) ... OC

```
{0, -\epsilon[t1, t2, t3] u1 u2 w3 + \epsilon[t1, t2, t3] t1 u1 u2 w3 +
\epsilon[t1, t2, t3] u1 u3 w3 - \epsilon[t1, t2, t3] t1 u1 u3 w3,
-\epsilon[t1, t2, t3] u1 u2 w2 + \epsilon[t1, t2, t3] t1 u1 u2 w2 +
\epsilon[t1, t2, t3] u1 u3 w2 -
\epsilon[t1, t2, t3] t1 u1 u3 w2, 0, 0, 0, 0, 0, 0}
```

\eta / : \eta[\epsilon\_i] = 0; \eta / : \eta[\epsilon\_i] \eta[\epsilon\_j] = 0; Turbo-Bureau (new!)

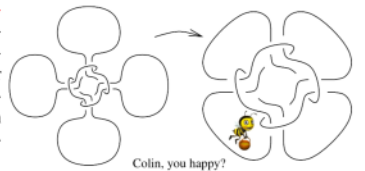
```
TB_{i,j}[\epsilon] :=
Expand[\epsilon / . {
\epsilon_i . v_k \mapsto Plus[\epsilon v_k / . v_j \mapsto (1-t-\eta[\epsilon_i]) v_i + (t+\eta[\epsilon_i]) v_j,
(t-1) (Coefficient[\epsilon, \eta[\epsilon_i]] - Coefficient[\epsilon, \eta[\epsilon_j]]) *
(u_k / . u_j \mapsto (1-t) u_i + t u_j) * u_i w_j,
K\delta_{k,i} (\epsilon / . \eta \rightarrow 0) (u_j - u_i) u_i w_j],
u_j \mapsto (1-t) u_i + t u_j,
w_i \mapsto w_i + (1-t^{-1}) w_j, w_j \mapsto t^{-1} w_j}];
f\epsilon = {f\epsilon_0 + f\epsilon_1 \eta[1] + f\epsilon_2 \eta[2] + f\epsilon_3 \eta[3]};
bas = {f\epsilon v1, f\epsilon v2, f\epsilon v3, u1^2 w1, u2^2 w2, u1, u2, u3, w1, w2, w3};
```

(bas // TB\_{1,2} // TB\_{1,3}) - (bas // TB\_{1,3} // TB\_{1,2}) ... OC

```
{0, -\epsilon_0 u1 u2 w3 + t \epsilon_0 u1 u2 w2 + \epsilon_0 u1 u3 w3 - t \epsilon_0 u1 u3 w3,
-\epsilon_0 u1 u2 w2 + t \epsilon_0 u1 u2 w2 + \epsilon_0 u1 u3 w2 - t \epsilon_0 u1 u3 w2,
0, 0, 0, 0, 0, 0, 0, 0}
```

Flower Surgery Theorem.

A knot is ribbon iff it is the result of  $n$ -petal flower surgery (from thin petals to wide petals) on an  $n$ -component unlink, for some  $n$ .



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“God created the knots, all else in topology is the work of mortals.” Leopold Kronecker (modified)

