Some New (v-)Braid-Group Representations

Pensieve header: Some new representations of the (v-)braid groups. Continues pensieve://2016-06/nb/-TurboGassner.pdf.

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```
Kδ /: Kδis__ := KroneckerDelta[1, Length[Union[{is}]]];
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The Burau Representation

Burau

$$B_{i\_,j\_}[\xi\_] := \xi /. v_j \mapsto (1-t) v_i + t v_j$$

BurauR3

```
Column@{lhs = {v₁, v₂, v₃} // B₁,₂ // B₁,₃ // B₂,₃,
   rhs = {v₁, v₂, v₃} // B₂,₃ // B₁,₃ // B₁,₂ ,
   lhs - rhs // Expand}
```

BurauR3

$\{v_1, (1-t) v_1 + t v_2, (1-t) v_1 + t ((1-t) v_2 + t v_3)\}$
$\{v_1, (1-t) v_1 + t v_2, (1-t) ((1-t) v_1 + t v_2) + t ((1-t) v_1 + t v_3)\}$
$\{0, 0, 0\}$

The Gassner Representation

Gassner

$$G_{i\_,j\_}[\xi\_] := \xi /. v_j \mapsto (1-t_i) v_i + t_i v_j$$

```
Column@{lhs = {v₁, v₂, v₃} // G₁,₂ // G₁,₃ // G₂,₃,
   Expand[lhs - ({v₁, v₂, v₃} // G₂,₃ // G₁,₃ // G₁,₂ )]}
```

$\{v_1, (1-t_1) v_1 + t_1 v_2, (1-t_1) v_1 + t_1 ((1-t_2) v_2 + t_2 v_3)\}$
$\{0, 0, 0\}$

GassnerOC

```
Column@{lhs = {v₁, v₂, v₃} // G₁,₂ // G₁,₃,
   Expand[lhs - ({v₁, v₂, v₃} // G₁,₃ // G₁,₂)]}
```

GassnerOC

$\{v_1, (1-t_1) v_1 + t_1 v_2, (1-t_1) v_1 + t_1 v_3\}$
$\{0, 0, 0\}$

GassnerUC

```
Column@{lhs = {v₁, v₂, v₃} // G₁,₃ // G₂,₃,
   rhs = {v₁, v₂, v₃} // G₂,₃ // G₁,₃ ,
   lhs - rhs // Expand}
```

GassnerUC

$\{v_1, v_2, (1-t_1) v_1 + t_1 ((1-t_2) v_2 + t_2 v_3)\}$
$\{v_1, v_2, (1-t_2) v_2 + t_2 ((1-t_1) v_1 + t_1 v_3)\}$
$\{0, 0, v_1 - t_1 v_1 - t_2 v_1 + t_1 t_2 v_1 - v_2 + t_1 v_2 + t_2 v_2 - t_1 t_2 v_2\}$

The Gassner-Plus Representation

GassnerPlus
```
GP_{i_,j_}[𝓈_] := Expand[𝓈 /. {u_j ⟼ (1 - t_i) u_i + t_i u_j,
    f_ . v_j ⟼ f (1 - t_i) v_i + f t_i v_j + (t_i - 1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) u_i + f t_i u_i }];
bas = {f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_2, f[t_1, t_2, t_3] v_3, u_1, u_2, u_3};
```

GPR3l
```
Short[lhs = bas // GP_{1,2} // GP_{1,3} // GP_{2,3}, 2]
```
GPR3l

$$\{f[t_1, t_2, t_3] v_1,$$
$$f[t_1, t_2, t_3] t_1 u_1 + f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + \ll 6\gg + t_1^2 u_1 f^{(1,0,0)}[t_1, t_2, t_3],$$
$$\ll 1\gg + \ll 19\gg + \ll 1\gg, \ll 1\gg, u_1 - t_1 u_1 + t_1 u_2, u_1 - t_1 u_1 + t_1 u_2 - t_1 t_2 u_2 + t_1 t_2 u_3\}$$

GPR3
```
(bas // GP_{2,3} // GP_{1,3} // GP_{1,2}) - lhs
```
GPR3

$$\{0, 0, 0, 0, 0, 0\}$$

GPOC
```
(bas // GP_{1,2} // GP_{1,3}) - (bas // GP_{1,3} // GP_{1,2})
```
GPOC

$$\{0, 0, 0, 0, 0, 0\}$$

Question. Does GP factor through G? How?

The End(G) Representation

```
EG_{i_,j_}[𝓈_] := Expand[𝓈 /. {u_j → (1 - t_i) u_i + t_i u_j, w_i → w_i + (1 - t_i^{-1}) w_j, w_j → t_i^{-1} w_j}];
EGchecks = Flatten@Table[u_i w_j, {i, 3}, {j, 3}]
```

$$\{u_1 w_1, u_1 w_2, u_1 w_3, u_2 w_1, u_2 w_2, u_2 w_3, u_3 w_1, u_3 w_2, u_3 w_3\}$$

```
Short[R3l = EGchecks // EG_{1,2} // EG_{1,3} // EG_{2,3}, 10]
```

$$\left\{u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1}, \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2}, \frac{u_1 w_3}{t_1 t_2},\right.$$

$$u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 -$$

$$u_2 w_3 + t_1 u_2 w_3, -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$

$$-\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2}, u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$

$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$

$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$

$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$

$$\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3, -\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\right\}$$

```
R3r = EGchecks // EG₂,₃ // EG₁,₃ // EG₁,₂; R3l - R3r
```

$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$

```
(# → Collect[EGᵢ,ⱼ[#], u_ w_, Simplify]) & /@ {uₖ wⱼ, uₖ wᵢ, uⱼ wₖ, uⱼ wᵢ}
```

$\left\{u_k w_j \rightarrow \dfrac{u_k w_j}{t_i}, \; u_k w_i \rightarrow u_k w_i + \left(1 - \dfrac{1}{t_i}\right) u_k w_j, \; u_j w_k \rightarrow (1 - t_i) u_i w_k + t_i u_j w_k,\right.$

$\left. u_j w_i \rightarrow (1 - t_i) u_i w_i + t_i u_j w_i - \dfrac{(-1 + t_i)^2 u_i w_j}{t_i} + (-1 + t_i) u_j w_j\right\}$

The End(G)+c Representation

Is there topology behind this representation?

```
EGcᵢ_,ⱼ_[ξ_] := Expand[ξ /. {
        cᵢ → cᵢ - (1 - tᵢ⁻¹) uᵢ wⱼ, cⱼ → cⱼ + (1 - tᵢ⁻¹) uᵢ wⱼ,
        uⱼ → (1 - tᵢ) uᵢ + tᵢ uⱼ,
        wᵢ → wᵢ + (1 - tᵢ⁻¹) wⱼ, wⱼ → tᵢ⁻¹ wⱼ}];
EGcchecks = {c₁, c₂, c₃, u₁ w₁, u₁ w₂, u₁ w₃, u₂ w₁, u₂ w₂, u₂ w₃, u₃ w₁, u₃ w₂, u₃ w₃};
```

```
(# → Collect[EGc₁,₂[#], u_ w_, Simplify]) & /@ EGcchecks
```

$\left\{c_1 \rightarrow c_1 + \left(-1 + \dfrac{1}{t_1}\right) u_1 w_2, \; c_2 \rightarrow c_2 + \left(1 - \dfrac{1}{t_1}\right) u_1 w_2,\right.$

$c_3 \rightarrow c_3, \; u_1 w_1 \rightarrow u_1 w_1 + \left(1 - \dfrac{1}{t_1}\right) u_1 w_2, \; u_1 w_2 \rightarrow \dfrac{u_1 w_2}{t_1}, \; u_1 w_3 \rightarrow u_1 w_3,$

$u_2 w_1 \rightarrow (1 - t_1) u_1 w_1 + t_1 u_2 w_1 - \dfrac{(-1 + t_1)^2 u_1 w_2}{t_1} + (-1 + t_1) u_2 w_2, \; u_2 w_2 \rightarrow \left(-1 + \dfrac{1}{t_1}\right) u_1 w_2 + u_2 w_2,$

$\left. u_2 w_3 \rightarrow (1 - t_1) u_1 w_3 + t_1 u_2 w_3, \; u_3 w_1 \rightarrow u_3 w_1 + \left(1 - \dfrac{1}{t_1}\right) u_3 w_2, \; u_3 w_2 \rightarrow \dfrac{u_3 w_2}{t_1}, \; u_3 w_3 \rightarrow u_3 w_3\right\}$

```
u₁ w₂ + c₂ // EGc₁,₂
```

$c_2 + u_1 w_2$

```
(# → Simplify[EGc₁,₂[#] /. {uᵢ_ wᵢ_ → 1, uᵢ_ wⱼ_ /; i ≠ j ⇸ 0}]) & /@ EGcchecks
```

$\{c_1 \rightarrow c_1, \; c_2 \rightarrow c_2, \; c_3 \rightarrow c_3, \; u_1 w_1 \rightarrow 1, \; u_1 w_2 \rightarrow 0, \; u_1 w_3 \rightarrow 0,$

$u_2 w_1 \rightarrow 0, \; u_2 w_2 \rightarrow 1, \; u_2 w_3 \rightarrow 0, \; u_3 w_1 \rightarrow 0, \; u_3 w_2 \rightarrow 0, \; u_3 w_3 \rightarrow 1\}$

Short[R3l = EGcchecks // EGc$_{1,2}$ // EGc$_{1,3}$ // EGc$_{2,3}$, 10]

$$\Big\{c_1 - u_1\,w_2 + \frac{u_1\,w_2}{t_1} - u_1\,w_3 + \frac{u_1\,w_3}{t_1},\ c_2 + u_1\,w_2 - \frac{u_1\,w_2}{t_1} + u_1\,w_3 - \frac{u_1\,w_3}{t_1} - \frac{u_1\,w_3}{t_2} + \frac{u_1\,w_3}{t_1\,t_2} - u_2\,w_3 + \frac{u_2\,w_3}{t_2},$$

$$c_3 + \frac{u_1\,w_3}{t_2} - \frac{u_1\,w_3}{t_1\,t_2} + u_2\,w_3 - \frac{u_2\,w_3}{t_2},\ u_1\,w_1 + u_1\,w_2 - \frac{u_1\,w_2}{t_1} + u_1\,w_3 - \frac{u_1\,w_3}{t_1},$$

$$\frac{u_1\,w_2}{t_1} + \frac{u_1\,w_3}{t_1} - \frac{u_1\,w_3}{t_1\,t_2},\ \frac{u_1\,w_3}{t_1\,t_2},\ u_1\,w_1 - t_1\,u_1\,w_1 + t_1\,u_2\,w_1 + 2\,u_1\,w_2 -$$

$$\frac{u_1\,w_2}{t_1} - t_1\,u_1\,w_2 - u_2\,w_2 + t_1\,u_2\,w_2 + 2\,u_1\,w_3 - \frac{u_1\,w_3}{t_1} - t_1\,u_1\,w_3 - u_2\,w_3 + t_1\,u_2\,w_3,$$

$$-u_1\,w_2 + \frac{u_1\,w_2}{t_1} + u_2\,w_2 - u_1\,w_3 + \frac{u_1\,w_3}{t_1} + \frac{u_1\,w_3}{t_2} - \frac{u_1\,w_3}{t_1\,t_2} + u_2\,w_3 - \frac{u_2\,w_3}{t_2},\ -\frac{u_1\,w_3}{t_2} + \frac{u_1\,w_3}{t_1\,t_2} + \frac{u_2\,w_3}{t_2},$$

$$u_1\,w_1 - t_1\,u_1\,w_1 + t_1\,u_2\,w_1 - t_1\,t_2\,u_2\,w_1 + t_1\,t_2\,u_3\,w_1 + 2\,u_1\,w_2 - \frac{u_1\,w_2}{t_1} - t_1\,u_1\,w_2 -$$

$$u_2\,w_2 + t_1\,u_2\,w_2 + t_2\,u_2\,w_2 - t_1\,t_2\,u_2\,w_2 - t_2\,u_3\,w_2 + t_1\,t_2\,u_3\,w_2 + 2\,u_1\,w_3 - \frac{u_1\,w_3}{t_1} -$$

$$t_1\,u_1\,w_3 - u_2\,w_3 + t_1\,u_2\,w_3 + t_2\,u_2\,w_3 - t_1\,t_2\,u_2\,w_3 - t_2\,u_3\,w_3 + t_1\,t_2\,u_3\,w_3,$$

$$-u_1\,w_2 + \frac{u_1\,w_2}{t_1} + u_2\,w_2 - t_2\,u_2\,w_2 + t_2\,u_3\,w_2 - u_1\,w_3 + \frac{u_1\,w_3}{t_1} + \frac{u_1\,w_3}{t_2} - \frac{u_1\,w_3}{t_1\,t_2} + 2\,u_2\,w_3 -$$

$$\frac{u_2\,w_3}{t_2} - t_2\,u_2\,w_3 - u_3\,w_3 + t_2\,u_3\,w_3,\ -\frac{u_1\,w_3}{t_2} + \frac{u_1\,w_3}{t_1\,t_2} - u_2\,w_3 + \frac{u_2\,w_3}{t_2} + u_3\,w_3\Big\}$$

R3r = EGcchecks // EGc$_{2,3}$ // EGc$_{1,3}$ // EGc$_{1,2}$; R3l - R3r

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

The Turbo-Gassner Representation

TG

K$\delta_{i\_,j\_}$:= KroneckerDelta[i, j];

TG$_{i\_,j\_}$[$\xi\_$] := Expand$\Big[\xi$ /. $\{$

    $f\_$. v$_{k\_}$:> Plus$\big[f$ v$_k$ /. v$_j$ → $(1 - t_i)$ v$_i$ + t$_i$ v$_j$,

      $(1 - t_i^{-1})$ (t$_i$ ∂$_{t_i}$$f$ - t$_j$ ∂$_{t_j}$$f$) $*$ (u$_k$ /. u$_j$ → $(1 - t_i)$ u$_i$ + t$_i$ u$_j$) $*$ u$_i$ w$_j$,

      K$\delta_{k,i}$ f (u$_j$ - u$_i$) u$_i$ w$_j\big]$,

    u$_j$ → $(1 - t_i)$ u$_i$ + t$_i$ u$_j$,

    w$_i$ → w$_i$ + $(1 - t_i^{-1})$ w$_j$, w$_j$ → t$_i^{-1}$ w$_j\}\Big]$;

bas = {f[t$_1$, t$_2$, t$_3$] v$_1$, f[t$_1$, t$_2$, t$_3$] v$_2$, f[t$_1$, t$_2$, t$_3$] v$_3$, u$_1$, u$_2$, u$_3$, w$_1$, w$_2$, w$_3$};

Short[R31 = bas // TG$_{1,2}$ // TG$_{1,3}$ // TG$_{2,3}$, 10]

$\big\{ f[t_1, t_2, t_3] \, v_1 - f[t_1, t_2, t_3] \, u_1^2 \, w_2 + f[t_1, t_2, t_3] \, u_1 \, u_2 \, w_2 - f[t_1, t_2, t_3] \, u_1^2 \, w_3 +$

$\quad f[t_1, t_2, t_3] \, u_1 \, u_3 \, w_3 - \dfrac{t_3 \, u_1^2 \, w_3 \, f^{(0,0,1)}[t_1, t_2, t_3]}{t_2} + \dfrac{t_3 \, u_1^2 \, w_3 \, f^{(0,0,1)}[t_1, t_2, t_3]}{t_1 \, t_2} -$

$\quad t_3 \, u_1 \, u_2 \, w_3 \, f^{(0,0,1)}[t_1, t_2, t_3] + \dfrac{t_3 \, u_1 \, u_2 \, w_3 \, f^{(0,0,1)}[t_1, t_2, t_3]}{t_2} -$

$\quad t_2 \, u_1^2 \, w_2 \, f^{(0,1,0)}[t_1, t_2, t_3] + \dfrac{t_2 \, u_1^2 \, w_2 \, f^{(0,1,0)}[t_1, t_2, t_3]}{t_1} + u_1^2 \, w_3 \, f^{(0,1,0)}[t_1, t_2, t_3] -$

$\quad \dfrac{u_1^2 \, w_3 \, f^{(0,1,0)}[t_1, t_2, t_3]}{t_1} - t_2 \, u_1^2 \, w_3 \, f^{(0,1,0)}[t_1, t_2, t_3] + \dfrac{t_2 \, u_1^2 \, w_3 \, f^{(0,1,0)}[t_1, t_2, t_3]}{t_1} -$

$\quad u_1 \, u_2 \, w_3 \, f^{(0,1,0)}[t_1, t_2, t_3] + t_2 \, u_1 \, u_2 \, w_3 \, f^{(0,1,0)}[t_1, t_2, t_3] - u_1^2 \, w_2 \, f^{(1,0,0)}[t_1, t_2, t_3] +$

$\quad t_1 \, u_1^2 \, w_2 \, f^{(1,0,0)}[t_1, t_2, t_3] - u_1^2 \, w_3 \, f^{(1,0,0)}[t_1, t_2, t_3] + t_1 \, u_1^2 \, w_3 \, f^{(1,0,0)}[t_1, t_2, t_3],$

$\quad f[t_1, t_2, t_3] \, v_1 - f[t_1, t_2, t_3] \, t_1 \, v_1 + f[t_1, t_2, t_3] \, t_1 \, v_2 - f[t_1, t_2, t_3] \, u_1 \, u_2 \, w_3 +$

$\quad \ll 55 \gg + 2 \, t_1 \, u_1^2 \, w_3 \, f^{(1,0,0)}[t_1, t_2, t_3] - t_1^2 \, u_1^2 \, w_3 \, f^{(1,0,0)}[t_1, t_2, t_3] -$

$\quad t_1 \, u_1 \, u_2 \, w_3 \, f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 \, u_1 \, u_2 \, w_3 \, f^{(1,0,0)}[t_1, t_2, t_3],$

$\quad \ll 99 \gg + \ll 1 \gg, \; u_1, \; \ll 1 \gg, \; \ll 1 \gg, \; w_1 + \ll 6 \gg, \; \dfrac{w_2}{t_1} + \dfrac{w_3}{t_1} - \dfrac{w_3}{t_1 \, t_2}, \; \dfrac{w_3}{t_1 \, t_2} \big\}$

(bas // TG$_{1,2}$ // TG$_{1,3}$ // TG$_{2,3}$) − (bas // TG$_{2,3}$ // TG$_{1,3}$ // TG$_{1,2}$)

$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$

Short[OC1 = bas // TG$_{1,2}$ // TG$_{1,3}$]

$\big\{ f[t_1, t_2, t_3] \, v_1 - f[t_1, t_2, t_3] \, u_1^2 \, w_2 + \ll 15 \gg + t_1 \, u_1^2 \, w_3 \, f^{(1,0,0)}[t_1, t_2, t_3],$

$\quad \ll 39 \gg + t_1^2 \ll 3 \gg \; \ll 1 \gg, \; \ll 1 \gg, \; \ll 4 \gg, \; \dfrac{\ll 1 \gg}{\ll 1 \gg}, \; \dfrac{w_3}{t_1} \big\}$

TGOC

(bas // TG$_{1,2}$ // TG$_{1,3}$) − (bas // TG$_{1,3}$ // TG$_{1,2}$)

TGOC

$\{0, \, -f[t_1, t_2, t_3] \, u_1 \, u_2 \, w_3 + f[t_1, t_2, t_3] \, t_1 \, u_1 \, u_2 \, w_3 + f[t_1, t_2, t_3] \, u_1 \, u_3 \, w_3 -$

$\quad f[t_1, t_2, t_3] \, t_1 \, u_1 \, u_3 \, w_3, \, -f[t_1, t_2, t_3] \, u_1 \, u_2 \, w_2 + f[t_1, t_2, t_3] \, t_1 \, u_1 \, u_2 \, w_2 +$

$\quad f[t_1, t_2, t_3] \, u_1 \, u_3 \, w_2 - f[t_1, t_2, t_3] \, t_1 \, u_1 \, u_3 \, w_2, \, 0, \, 0, \, 0, \, 0, \, 0, \, 0\}$

A Finite-Rank Turbo-Gassner Representation

$\eta \, /: \, \eta[i\_]^2 = 0; \; \eta \, /: \, \eta[i\_] \, \eta[j\_] = 0;$

$\text{FTG}_{i\_,j\_}[\xi\_] \; := \; \text{Expand}\big[\xi \, /. \, \big\{$

$\quad\quad f\_ . \, v_{k\_} \, :\to \, \text{Plus}\big[f \, v_k \, /. \, v_j \to \big(1 - t_i - \eta[i]\big) \, v_i + (t_i + \eta[i]) \, v_j,$

$\quad\quad\quad (t_i \, \text{Coefficient}[f, \eta[i]] - t_j \, \text{Coefficient}[f, \eta[j]])$

$\quad\quad\quad\quad \big(1 - t_i^{-1}\big) \, \big(u_k \, /. \, u_j \to \big(1 - t_i\big) \, u_i + t_i \, u_j\big) \, u_i \, w_j,$

$\quad\quad\quad K\delta_{k,i} \, \big(f \, /. \, \_\eta \to 0\big) \, (u_j - u_i) \, u_i \, w_j\big],$

$\quad\quad u_j \to \big(1 - t_i\big) \, u_i + t_i \, u_j,$

$\quad\quad w_i \to w_i + \big(1 - t_i^{-1}\big) \, w_j, \; w_j \to t_i^{-1} \, w_j \big\}\big];$

```
ff = f₀ + f₁ η[1] + f₂ η[2] + f₃ η[3];
FTGchecks = {ff v₁, ff v₂, ff v₃, u₁, u₂, u₃, w₁, w₂, w₃};

{v₁, v₂} // FTG₁,₂ // Column
```

$v_1 - u_1^2 w_2 + u_1 u_2 w_2$
$v_1 - t_1 v_1 + t_1 v_2 - v_1 \eta[1] + v_2 \eta[1]$

```
Short[R3l = FTGchecks // FTG₁,₂ // FTG₁,₃ // FTG₂,₃, 10]
```

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```
R3r = FTGchecks // FTG₂,₃ // FTG₁,₃ // FTG₁,₂; R3l - R3r
```

{0, 0, 0, 0, 0, 0, 0, 0, 0}

```
Short[OCl = FTGchecks // FTG₁,₂ // FTG₁,₃]
```

≪1≫

```
Short[OCr = FTGchecks // FTG₁,₃ // FTG₁,₂]
```

≪1≫

```
OCl - OCr
```

$\{0, -f_0 u_1 u_2 w_3 + f_0 t_1 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 - f_0 t_1 u_1 u_3 w_3,$
$-f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_3 w_2 - f_0 t_1 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0\}$

The Turbo-Burau Representation

TB

```
η /: η[i_]² = 0; η /: η[i_] η[j_] = 0;
TB_{i_,j_}[ξ_] :=
  Expand[ξ /. {
    f_ . v_{k_} :> Plus[f v_k /. v_j → (1 - t - η[i]) v_i + (t + η[i]) v_j,
      (t - 1) (Coefficient[f, η[i]] - Coefficient[f, η[j]]) *
        (u_k /. u_j → (1 - t) u_i + t u_j) * u_i w_j,
      Kδ_{k,i} (f /. _η → 0) (u_j - u_i) u_i w_j],
    u_j → (1 - t) u_i + t u_j,
    w_i → w_i + (1 - t⁻¹) w_j, w_j → t⁻¹ w_j}];
ff = f₀ + f₁ η[1] + f₂ η[2] + f₃ η[3];
bas = {ff v₁, ff v₂, ff v₃, u₁² w₁, u₁² w₂, u₁, u₂, u₃, w₁, w₂, w₃};
```

```
Short[R3l = bas // TB₁,₂ // TB₁,₃ // TB₂,₃, 2]
```

$\{f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + t f_1 u_1^2 w_2 + f_2 u_1^2 w_2 - t f_2 u_1^2 w_2 + \ll17\gg + f_3 u_1 u_2 w_3 - t f_3 u_1 u_2 w_3 +$
$\quad f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \ll75\gg + t \ll3\gg, \ll7\gg, \ll1\gg, \frac{w_3}{t^2}\}$

```
R3r = bas // TB₂,₃ // TB₁,₃ // TB₁,₂; R3l - R3r
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

TBOC

$$(\text{bas} \mathbin{//} \text{TB}_{1,2} \mathbin{//} \text{TB}_{1,3}) - (\text{bas} \mathbin{//} \text{TB}_{1,3} \mathbin{//} \text{TB}_{1,2})$$

TBOC

$$\{0,\ -f_0\,u_1\,u_2\,w_3 + t\,f_0\,u_1\,u_2\,w_3 + f_0\,u_1\,u_3\,w_3 - t\,f_0\,u_1\,u_3\,w_3,$$
$$-f_0\,u_1\,u_2\,w_2 + t\,f_0\,u_1\,u_2\,w_2 + f_0\,u_1\,u_3\,w_2 - t\,f_0\,u_1\,u_3\,w_2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

Exporting the above as PDF files

The below is adapted from pensieve://2016-04/GaussGassner/GaussGassnerDemo.nb.

```mathematica
ConditionalExport[fname_String, rest___] := Module[{temp, exists},
  temp = "ConditionalExportTemporary" <> "." <> FileExtension[fname];
  exists = FileExistsQ[fname];
  Export[temp, rest];
  If[exists && FileByteCount[fname] === FileByteCount[temp],
   DeleteFile[temp],
   (* else *) Print["Exporting " <> fname <> "..."];
   If[exists, DeleteFile[fname]];
   RenameFile[temp, fname]
  ];
  fname
 ]


ExportButton = Button["Export",
  SetOptions[$FrontEndSession, PrintingStyleEnvironment → "Working"];
  TagProperties[_] := {};
  TagProperties["ct-def"] = {PageWidth → 6/0.65};
  Options[CellExport] = {
    PageWidth → 4/0.65, CellFilter → Identity,
    ExportDirectory → "Snips", ExportBaseFilename → Automatic,
    ExportFormat → ".pdf", ExportOptions → {}, Split → False
   };
  CellExport[tag_String, opts___Rule] := CellExport[
    NotebookGet[EvaluationNotebook[]],
    tag, opts
   ];
  CellExport[nb_Notebook, tag_String] :=
   CellExport[nb, tag, TagProperties[tag]];
  CellExport[nb_Notebook, tag_String, OptionsPattern[]] := Module[
    {cells, cell, filename, format},
    filename = FileNameJoin[{
       OptionValue[ExportDirectory] /. Automatic → Directory[],
       OptionValue[ExportBaseFilename] /. Automatic → tag
```

```mathematica
      }];
      format = OptionValue[ExportFormat];
      cells = OptionValue[CellFilter][Cases[
         nb, c_Cell /; FreeQ[List @@ c, Cell] && ! FreeQ[c, CellTags → tag],
         Infinity
        ]];
      If[! OptionValue[Split],
       If[Length[cells] ≥ 1,
        If[Length[cells] == 1,
          cells = Append[First[cells], PageWidth → 1.2 × 72 OptionValue[PageWidth]],
          cells = Cell[CellGroup[cells], PageWidth → 72 OptionValue[PageWidth]]
         ];
        ConditionalExport[
          filename <> format, cells,
          ImageResolution → 300,
          OptionValue[ExportOptions]
         ]
        ],
        k = 0;
        Table[
         ++k;
         ConditionalExport[
          filename <> "-" <> ToString[k] <> format,
          Append[cell, PageWidth → 72 OptionValue[PageWidth]],
          ImageResolution → 300,
          OptionValue[ExportOptions]
         ],
         {cell, cells}
        ]
       ]
      ];
    nb = NotebookGet[EvaluationNotebook[]];
    tags = Cases[nb, (CellTags → tag_) ⧴ tag, Infinity] // Union;
    CellExport /@ tags;
    Print["Done."]
   ]
```

Export

```
Exporting Snips\TBOC.pdf...
Done.
```