The Brute and the Hidden Paradise

Abstract. There is expected to be a hidden paradise of poly-time computable knot polynomials lying just beyond the Alexander polynomial. I will describe my brute attempts to gain entry.

Why "expected"? Gauss diagram formulas [PV, GPV] show that $v_{d,f}(K) = \sum_{Y \subset X(K),\ |Y|=d} f(Y)$ finite-type invariants are all poly-time, and tempt to conjecture that there are no others. But Alexander shows it nonsense:

d	2	3	4	5	6	7	8	⋯
known invts in* $O(n^d)$	1	1	∞	3	4	8	11	⋯

This is an unreasonable picture! **Fresh, numerical, no cheating.*
So there ought to be further poly-time invariants.

Also. • The line above the Alexander line in the Melvin-Morton [MM, Ro] expansion of the coloured Jones polynomial. • The 2-loop contribution to the Kontsevich integral.


Rozansky

Why "paradise"? Foremost answer: *OBVIOUSLY.* Cf. proving (incomputable A)=(incomputable B), or categorifying (incomputable C).

ωεβ/K17: (extend to tangles, perhaps detect non-slice ribbon knots)

Second Answer. The second answer has to do with "Algebraic Knot Theory", so let me start with that. Somewhat informally, a "tangle" is a piece of a knot, or a "knot with endpoints" (an example is on the right). Knots can be assembled by stitching together the strands of several tangles, or the different strands of a single tangle. Some interesting classes of knots can be defined algebraically using tangles and these stitching operations. Here is the most interesting example:

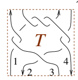


Figure 1. A tangle.

Definition 1. A "ribbon knot" is a knot K that can be presented as the boundary of a disk D which is allowed to have "ribbon singularities" but not "clasp singularities". See Figure 2.

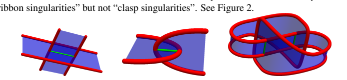
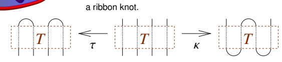


Figure 2. A ribbon singularity, a clasp singularity, and an example of a ribbon knot.

Definition 2. Let $\mathcal{T}_{2n}$ denote the set of all tangles T with $2n$ components that connect $2n$ points along a "top end" with $2n$ points along a "bottom end" inducing the identity permutation of ends (an example is the tangle in Figure 1). Given $T \in \mathcal{T}_{2n}$, let $\tau(T)$ be the result of stitching its components at the top in pairs as on the right — it is an n-component tangle all of whose ends are at the bottom, and we (somewhat loosely) denote the set of all such by $\mathcal{T}_n$, so $\tau\colon \mathcal{T}_{2n} \to \mathcal{T}_n$. Likewise let $\kappa(T)$ be the result of stitching T both at the top and at the bottom, also as on the right. So $\kappa(T)$ is a 1-component tangle, which is the same as a knot, and $\kappa\colon \mathcal{T}_{2n} \to \mathcal{T}_1$.



gives $\mathcal{T}_n \xleftarrow{\tau} \mathcal{T}_{2n} \xrightarrow{\kappa} \mathcal{T}_1$

Theorem 1 (I have not seen this theorem in the literature, yet it is not difficult to prove). The set of ribbon knots is the set of all knots K that can be written as $K = \kappa(T)$ for some tangle T for which $\tau(T)$ is the untangled (crossingless) tangle U:

$$\{\text{ribbon knots}\} = \{\kappa(T)\colon T \in \mathcal{T}_{2n} \text{ and } \tau(T) = U \in \mathcal{T}_n\}.$$

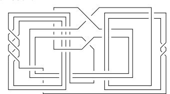
Now suppose we have an invariant $Z\colon \mathcal{T}_k \to A_k$ of tangles, which takes values in some spaces A_k. Suppose also we have operations $\tau_A\colon A_{2n} \to A_n$ and $\kappa_A\colon A_{2n} \to A_1$ such that the diagram on the right is commutative. Then

$$\begin{array}{ccccc} \mathcal{T}_n & \xleftarrow{\tau} & \mathcal{T}_{2n} & \xrightarrow{\kappa} & \mathcal{T}_1 \\ \downarrow Z & & \downarrow Z & & \downarrow Z \\ A_n & \xleftarrow{\tau_A} & A_{2n} & \xrightarrow{\kappa_A} & A_1 \end{array} \qquad (1)$$

$$Z(\{\text{ribbon knots}\}) \subseteq \mathcal{R}_A := \{\kappa_A(\zeta)\colon \zeta \in A_{2n} \text{ and } \tau_A(\zeta) = 1_A \in A_n\} \subset \mathcal{A}_1, \qquad (2)$$

where $1_A := Z(U) \in A_n$. If the target spaces A_k are algebraic (polynomials, matrices, matrices of polynomials, etc.) and the operations τ_A and κ_A are algebraic maps between them (at this stage, meaning just "have simple algebraic formulas"), then $\mathcal{R}_A$ is an algebraically defined set. Hence we potentially have an algebraic way to detect non-ribbon knots: if $Z(K) \notin \mathcal{R}_A$, then K is not ribbon.

As it turns out, it is valuable to detect non-ribbon knots. Indeed the Slice-Ribbon Conjecture (Fox, 1960s) asserts that every slice knot (a knot in S^3 that can be presented as the boundary of a disk embedded in B^4) is ribbon. Gompf, Scharlemann, and Thompson [GST] describe a family of slice knots which they conjecture are not ribbon (the simplest of those is on the right). With the algebraic technology described above it may be possible to show that the [GST] knots are indeed non-ribbon, thus disproving the Slice-Ribbon Conjecture.



What would it take?

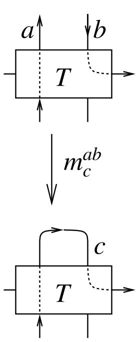
C1. An invariant Z which makes sense on tangles and for which diagram (1) commutes.
C2. Z cannot be a simple extension of the Alexander polynomial to tangles, for by Fox-Milnor [FM] the Alexander polynomial does not detect non-ribbon slice knots.
C3. Z cannot be computable from finitely many finite type invariants, for this would contradict the results of Ng [Ng].[1]
C4. Z must be computable on at least the simplest [GST] knot, which has 48 crossings.
C5. It is better if in some meaningful sense the size of the spaces A_k grows slowly in k. Indeed in (2), if A_{2n} is much bigger than A_n and A_1 then at least generically $\mathcal{R}_A$ will be the full set A_1 and our condition will be empty.

No invariant that I know now meets these criteria. Alexander and Vassiliev fail C2 and C3, respectively. Almost all quantum invariants and knot homologies pass C1-C3, but fail C4. Jones, HOMFLY-PT and Khovanov potentially pass C4, yet fail C5. We must come up with something new.

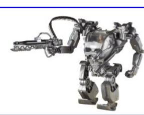
[FM] R. H. Fox and J. W. Milnor, *Singularities of 2-Spheres in 4-Space and Cobordism of Knots,* Osaka J. Math. **3** (1966) 257–267.
[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.
[Ng] K. Y. Ng, *Groups of ribbon knots,* Topology **37** (1998) 441–458, arXiv:q-alg/9502017 (with an addendum at arXiv:math.GT/0310074)

[1] A slight subtlety arises: There is no taking limits here, and C3 does not preclude the possibility that Z is computable from infinitely many finite type invariants. The Fox-Milnor condition on the Alexander polynomial of ribbon knots, for example, is expressible in terms of the full Alexander polynomial, yet not in terms of any finite type reduction thereof. Unfortunately by C2 it cannot be used here.

Moral. Need "stitching":



Why "brute"? Cause it's the only thing I know, for now. There may be better ways in, and it's fair to hope that sooner or later they will be found.

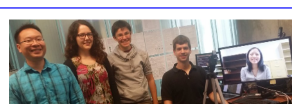


The Gold Standard is set by the formulas [BNS, BN] for Alexander. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A \end{array}\right\}$ with $R_S := \mathbb{Z}(\{t_a\colon a \in S\})$:

$$\left({}_a{\nearrow\!\!\!\!\nwarrow}_b,\ {}_b{\nearrow\!\!\!\!\nwarrow}_a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm1} \\ b & 0 & t_a^{\pm1} \end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}\right)$$

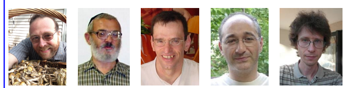
Help Needed! Disorganized videos of talks in a private seminar are at ωεβ/PP.

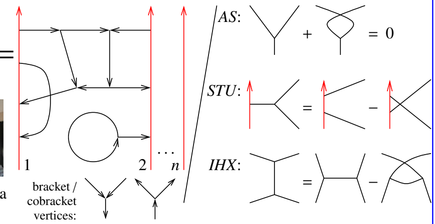

Vo, Halacheva, Dalvit, Ens, Lee (van der Veen, Schaveling)

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know!
Dunfield: 1000-crossing fast.

Theorem [EK, Ha, En, Se]. There is a "homomorphic expansion"

$$Z\colon \left\{\begin{array}{l} S\text{-component} \\ (v/b\text{-})\text{tangles} \end{array}\right\} \to \mathcal{A}^v_S :=$$

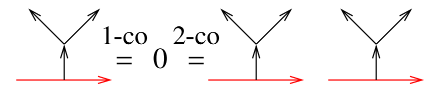

Etingof Kazhdan Haviv Enriquez Ševera

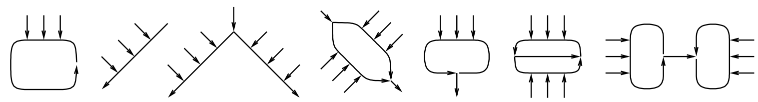


(it is enough to know Z on ⤫ and have disjoint union and stitching formulas) *. . . exponential and too hard!*

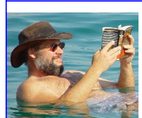
Idea. Look for "ideal" quotients of $\mathcal{A}^v_S$ that have poly-sized descriptions; . . . specifically, limit the co-brackets.

1-co and 2-co, aka TC and TC^2, on the right. The primitives that remain are:

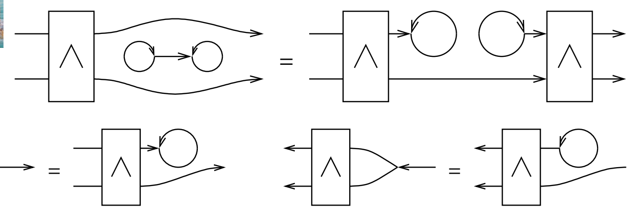




. . . manageable but still exponential!


Jones

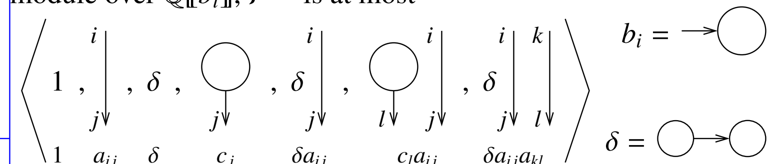
The 2D relations come from the relation with 2D Lie bialgebras:



We let $\mathcal{A}^{2,2}$ be $\mathcal{A}^v$ modulo 2-co and $2D$, and $z^{2,2}$ be the projection of $\log Z$ to $\mathcal{P}^{2,2} := \pi\mathcal{P}^v$, where $\mathcal{P}^v$ are the primitives of $\mathcal{A}^v$.

Main Claim. $z^{2,2}$ is poly-time computable.

Main Point. $\mathcal{P}^{2,2}$ is poly-size, so how hard can it be? Indeed, as a module over $\mathbb{Q}[\![b_i]\!]$, $\mathcal{P}^{2,2}$ is at most



Claim. $R_{jk} = e^{a_{jk}}e^{\rho_{jk}}$ is a solution of the Yang-Baxter / R3 equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in $\exp\mathcal{P}^{2,2}$, with $\rho_{jk} :=$

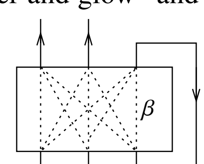
$$\psi(b_j)\left(-c_k + \frac{c_k a_{jk}}{b_j} - \frac{\delta a_{jk}a_{jk}}{b_j^2}\right) + \frac{\phi(b_j)\psi(b_k)}{b_k\phi(b_k)}\left(c_k a_{kk} - \frac{\delta a_{jk}a_{kk}}{b_j}\right),$$

and with $\phi(x) := e^{-x} - 1 = -x + x^2/2 - \ldots$, and $\psi(x) := ((x+2)e^{-x} - 2 + x)/(2x) = x^2/12 - x^3/24 + \ldots$. (This already gives some new (v-)braid group representations, as below).

Problem. How do we multiply in $\exp(\mathcal{P}^{2,2})$? How do we stitch? BCH is a theoretical dream. Instead, use "scatter and glow" and "feedback loops":

The Euler trick:
With $Ef := (\deg f)f$ get $Ee^x = xe^x$ and $E(e^xe^ye^z) = xe^xe^ye^z + e^xye^ye^z + e^xe^yze^z$.


Cherenkov radiation



Feedback loop

Work in Progress!

The Brute and the Hidden Paradise

van der Veen

Local Algebra (with van der Veen) Much can be reformulated as (non-standard) "quantum algebra" for the 4D Lie algebra $\mathfrak{g} = \langle b, c, u, w \rangle$ over $\mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$. The key: $a_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g})^{\otimes\{i,j\}}$.

Some (new) representationss of the (v-)braid groups. ωεβ/Reps

Burau (old)

```
B_{i_,j_}[ξ_] := ξ /. v_j :→ (1 - t) v_i + t v_j
```

…testing R3

```
Column@{lhs = {v1, v2, v3} // B1,2 // B1,3 // B2,3,
  rhs = {v1, v2, v3} // B2,3 // B1,3 // B1,2 ,
  lhs - rhs // Expand}

{v1, (1 - t) v1 + t v2, (1 - t) v1 + t ((1 - t) v2 + t v3)}
{v1, (1 - t) v1 + t v2,
  (1 - t) ((1 - t) v1 + t v2) + t ((1 - t) v1 + t v3)}
{0, 0, 0}
```

Gassner (old)

```
G_{i_,j_}[ξ_] := ξ /. v_j :→ (1 - t_i) v_i + t_i v_j
```

…Overcrossings Commute (OC):

```
Column@{lhs = {v1, v2, v3} // G1,2 // G1,3,
  Expand[lhs - ({v1, v2, v3} // G1,3 // G1,2)]}

{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}
{0, 0, 0}
```

…Undercrossings Commute (UC):

```
Column@{lhs = {v1, v2, v3} // G1,3 // G2,3,
  rhs = {v1, v2, v3} // G2,3 // G1,3 ,
  lhs - rhs // Expand}

{v1, v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3)}
{v1, v2, (1 - t2) v2 + t2 ((1 - t1) v1 + t1 v3)}
{0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}
```

Gassner Plus (new?)

```
GP_{i_,j_}[ξ_] := Expand[ξ /. {u_j :→ (1 - t_i) u_i + t_i u_j,
    f_. v_j :→ f (1 - t_i) v_i + f t_i v_j + (t_i - 1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) u_i +
      f t_i u_i }];
bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3,
   u1, u2, u3};
```

…R3 (left)

```
Short[lhs = bas // GP1,2 // GP1,3 // GP2,3, 2]

{f[t1, t2, t3] v1, f[t1, t2, t3] t1 u1 + f[t1, t2, t3] v1 -
   f[t1, t2, t3] t1 v1 + <<6>> + t1^2 u1 f^(1,0,0)[t1, t2, t3],
  <<1>> + <<19>> + <<1>>, <<1>>, u1 - t1 u1 + t1 u2,
  u1 - t1 u1 + t1 u2 - t1 t2 u2 + t1 t2 u3}
```

…R3 (rest)

```
(bas // GP2,3 // GP1,3 // GP1,2) - lhs
{0, 0, 0, 0, 0, 0}
```

…OC

```
(bas // GP1,2 // GP1,3) - (bas // GP1,3 // GP1,2)
{0, 0, 0, 0, 0, 0}
```

Question. Does Gassner Plus factor through Gassner?

Turbo-Gassner (new!)

```
Kδ_{i_,j_} := KroneckerDelta[i, j];
TG_{i_,j_}[ξ_] := Expand[ξ /. {
     f_. v_k_ :→ Plus[f v_k /. v_j → (1 - t_i) v_i + t_i v_j,
       (1 - t_i^-1) (t_i ∂_{t_i} f - t_j ∂_{t_j} f) *
        (u_k /. u_j → (1 - t_i) u_i + t_i u_j) * u_i w_j,
       Kδ_{k,i} f (u_j - u_i) u_i w_j],
     u_j → (1 - t_i) u_i + t_i u_j,
     w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3,
   u1, u2, u3, w1, w2, w3};
```

Satisfies R3…

…OC

```
(bas // TG1,2 // TG1,3) - (bas // TG1,3 // TG1,2)

{0, -f[t1, t2, t3] u1 u2 w3 + f[t1, t2, t3] t1 u1 u2 w3 +
  f[t1, t2, t3] u1 u3 w3 - f[t1, t2, t3] t1 u1 u3 w3,
 -f[t1, t2, t3] u1 u2 w2 + f[t1, t2, t3] t1 u1 u2 w2 +
  f[t1, t2, t3] u1 u3 w2 -
  f[t1, t2, t3] t1 u1 u3 w2, 0, 0, 0, 0, 0, 0}
```

Turbo-Burau (new!)

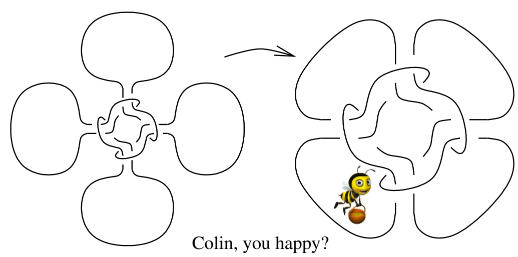
```
η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
TB_{i_,j_}[ξ_] :=
 Expand[ξ /. {
    f_. v_k_ :→ Plus[f v_k /. v_j → (1 - t - η[i]) v_i + (t + η[i]) v_j,
      (t - 1) (Coefficient[f, η[i]] - Coefficient[f, η[j]]) *
       (u_k /. u_j → (1 - t) u_i + t u_j) * u_i w_j,
      Kδ_{k,i} (f /. _η → 0) (u_j - u_i) u_i w_j],
    u_j → (1 - t) u_i + t u_j,
    w_i → w_i + (1 - t^-1) w_j, w_j → t^-1 w_j}];
ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
bas = {ff v1, ff v2, ff v3, u1^2 w1, u1^2 w2, u1, u2, u3, w1, w2, w3};
```

…OC

```
(bas // TB1,2 // TB1,3) - (bas // TB1,3 // TB1,2)

{0, -f0 u1 u2 w3 + t f0 u1 u2 w3 + f0 u1 u3 w3 - t f0 u1 u3 w3,
 -f0 u1 u2 w2 + t f0 u1 u2 w2 + f0 u1 u3 w2 - t f0 u1 u3 w2,
 0, 0, 0, 0, 0, 0, 0}
```

Flower Surgery Theorem. A knot is ribbon iff it is the result of n-petal flower surgery (from thin petals to wide petals) on an n-componenet unlink, for some n.



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"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)