GWU-1612 handout on 161209

December 9, 2016 7:34 AM

Dror Bar-Natan: Talks: GWU-1612:

On Elves and Invariants

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen.

ωεβ:=http://drorbn.net/GWU-1612/

Work in Progress! Fluid! Help Needed!

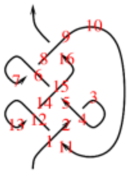
Abstract. Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

Three steps to the computation of ρ_1:

1. **Preparation.** Given K, results $\langle$long *word*$\|$ simple *formulas*$\rangle$.
2. **Rewrite rules.** Make the *word* simpler and the *formulas* more complicated, until the *word* "*elf*" is reached.
3. **Readout.** The invariant ρ_1 is read from the last *formulas*.

$$\text{Knot } K \xrightarrow{\text{preparation}} \langle elf\dots elf\|\omega_0; L_0; Q_0; P_0\rangle \xrightarrow{\text{rewrite rules}} \langle elf\|\omega; -; -; P\rangle \xrightarrow{\text{readout}} \rho_1(K) = \rho_1(\omega, P)$$

Preparation. Draw K using a 0-framed 0-rotation planar diagram D where all crossings are pointing up. Walk along D labeling features by $1,\dots,m$ in order: over-passes, under-passes, and right-heading cups and caps ("±-cuaps"). If x is a xing, let i_x and j_x be the labels on its over/under strands, and let s_x be 0 if it right-handed and -1 otherwise. If c is a cuap, let i_c be its labels and s_c be its sign. Set



$$(L;Q;P) = \sum_{x:\,(i,j,s)} (-)^s \left(l_j;\ t^s e_i f_j;\ (-t)^s e_i l_{(1+s)i-sj} f_j + l_i l_j + \frac{t^{2s} e_i^2 f_j^2}{4} \right) + \sum_{c:\,(i,s)} (0;\ 0;\ s\cdot l_i).$$

This done, output $\langle e_1 l_1 f_1 e_2 l_2 f_2 \cdots e_m l_m f_m \| 1; L; Q; P\rangle$.

In *formulas*, L is always $\mathbb{Z}$-linear in $\{l_i\}$, Q is an R-linear combination of $\{e_i f_j\}$ where $R := \mathbb{Z}[t^{\pm1}]$, and P is an R-linear combination of $\{1,\ l_i,\ l_i l_j,\ e_i f_j,\ e_i l_j f_k,\ e_i e_j f_k f_l\}$.

Rewrite Rules. Manipulate $\langle$*word*$\|$*formulas*$\rangle$ expressions using the rewrite rules below, until you come to the form $\langle e_1 l_1 f_1 \| \omega; -; -; P\rangle$. Output (ω, P).

Rule 1, Deletions. If a letter appears in *word* but not in *formulas*, you can delete it.

Rule 2, Merges. In *word*, you can replace *adjacent* $v_i v_j$ with v_k (for $v \in \{e,l,f\}$) while making the same changes in *formulas* (provided k creates no naming clashes). E.g.,

$$\langle \dots e_i e_j \dots \| Z\rangle \to \langle \dots e_k \dots \| Z|_{e_i,e_j\to e_k}\rangle.$$

Rule 3, *le* Sorts. Provided k introduces no clashes, given $\langle \dots l_j e_i \dots \| \omega; L; Q; P\rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha e_i + Q'$, write $P = P(e_i, l_j)$ (with messy coefficients), set $q = e^\gamma \beta e_k + \gamma l_k$, and output

$$\left\langle \dots e_k l_k \dots \| \omega; L|_{l_j\to l_k}; t^\lambda \alpha e_k + Q'; e^{-q} P(\partial_\beta, \partial_\gamma) e^q|_{\beta\to\alpha/\omega,\, \gamma\to\lambda\log t}\right\rangle.$$

Rule 4, *fl* Sorts. Provided k introduces no clashes, given $\langle \dots f_i l_j \dots \| \omega; L; Q; P\rangle$, decompose $L = \lambda l_j + L'$, $Q = \alpha f_i + Q'$, write $P = P(f_i, l_j)$ (with messy coefficients), set $q = e^\gamma \beta f_k + \gamma l_k$, and output

$$\left\langle \dots l_k f_k \dots \| \omega; L|_{l_j\to l_k}; t^\lambda \alpha f_k + Q'; e^{-q} P(\partial_\beta, \partial_\gamma) e^q|_{\beta\to\alpha/\omega,\, \gamma\to\lambda\log t}\right\rangle.$$

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

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more elves here.

Rule 5, *fe* Sorts. Provided k introduces no clashes, given $\langle \dots f_i e_j \dots \| \omega; L; Q; P\rangle$, decompose $Q = Q_{fe} f_i e_j + Q_f f_i + Q_e e_j + Q'$ write $P = P(f_i, e_j)$ (with messy coefficients), set $\mu = 1 + (t-1)\delta$ and $q = ((1-t)\alpha\beta + \beta e_k + \alpha f_k + \delta e_k f_k)/\mu$, and output

$$\left\langle \dots e_k f_k \dots \left\| \begin{matrix} \mu\omega; L; \mu\omega q + \mu Q'; \\ \omega^4 \Lambda_k + e^{-q} P(\partial_\alpha, \partial_\beta)(e^q) \end{matrix} \right.\right\rangle\Bigg|_{\substack{\alpha\to Q^f/\omega,\, \beta\to Q^e/\omega, \\ \delta\to Q^{fe}/\omega}},$$

where Λ_k is the Λόγος, "a principle of order and knowledge":

$$\begin{aligned}\Lambda_k = \frac{t+1}{4}\Big(&-\delta(\mu+1)\left(\beta^2 e_k^2 + \alpha^2 f_k^2\right) - \delta^3(3\mu+1) e_k^2 f_k^2 \\ &- 2(\beta e_k + \alpha f_k)\left(\alpha\beta + 2\delta\mu + \delta^2(2\mu+1) e_k f_k + 2\delta\mu^2 l_k\right) \\ &- 4(\alpha\beta + \delta\mu)\left(\delta(\mu+1) e_k f_k + \mu^2 l_k\right) - 4\delta^2\mu^2 e_k f_k l_k \\ &+ (t-1)\left(2(\alpha\beta+\delta\mu)^2 - \alpha^2\beta^2\right)\Big).\end{aligned}$$

***elf* merges,** m_k^{ij}, are defined as compositions

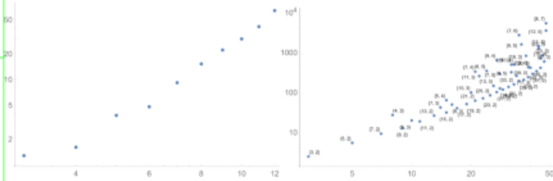
$$e_i l_i \overline{f_i e_j} l_j f_j \xrightarrow{S_x^{f_i e_j}} e_i \overline{l_i e_x}\, \overline{f_x l_j} f_j \xrightarrow{S_x^{l_i e_x} /\!/ S_x^{f_x l_j}} \overline{e_i e_x}\, \overline{l_x l_x}\, \overline{f_x f_j} \xrightarrow{i,j,x\to k} e_k l_k f_k$$

Readout. Given $\langle elf \| \omega; -; -; P\rangle$, output

$$\rho_1(K) := \frac{t(t\omega'\omega^3 - P|_{e,l,f\to 0})}{(t-1)^2\omega^2}.$$

(ω is the Alexander polynomial, L and Q are not interesting).

Experimental Analysis (ωεβ/Exp). Log-log plot of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1, always $\deg \rho_1^+ \le 2g - 1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Why Works? The Lie algebra $\mathfrak{g}_1$ (below) is a "solvable approximation of sl_2".

Theorem. The map (as defined below)

$$\langle w \| \omega; L; Q; P\rangle \mapsto \mathbb{O}\left(\omega^{-1} e^{L\log t + \omega^{-1} Q}(1 + \epsilon\omega^{-4} P):\ w\right) \in \hat{\mathcal{U}}(\mathfrak{g}_1)$$

is well defined modulo the sorting rules. It maps the initial preparation to a product of "R-matrices" and "cuap values" satisfying the usual moves for Morse knots (R3, etc.). (And hence the result is a "quantum invariant", except computed very differently; no rep theory!).

orientation

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle h, l, e', f\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with h central and with $[f,l] = f$, $[l,e'] = e'$, and $[e',f] = h - 2\epsilon l$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a **solvable approximation of sl_2**: $\mathfrak{g}_1 \supset \langle h, e', f, \epsilon h, \epsilon l, \epsilon e', \epsilon f\rangle \supset \langle h, \epsilon h, \epsilon l, \epsilon e', \epsilon f\rangle \supset 0$. Pragmatics: declare $\deg(h, l, e', f, \epsilon) = (1, 0, 1, 0, 1)$ and set $t := e^h$ and $e := (t-1)e'/h$.

How did it arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle l, f\rangle/[f,l] = f$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (l,f) \mapsto (0, l\wedge f)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle h', e', l, f\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 1$, get $[f,l] = f$, $[f,h'] = -\epsilon f$, $[l,e'] = e'$, $[h',e'] = -\epsilon e'$,

Differential Polynomials (Implementing $P(\partial_\alpha, \partial_\beta)(f)$)

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] :=
  Total[CoefficientRules[P, {x, y}] /.
    ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

***le* and *fl* Sorts**

```
S_{1_j_ (x:e|f)_i_ → k_}[E[ω_, L_, Q_, P_]] :=
  With[{λ = ∂_{1_j} L, α = ∂_{x_i} Q, q = e^γ β x_k + γ 1_k}, CF[
    E[ω, L /. 1_j → 1_k, t^λ α x_k + (Q /. x_i → 0),
      e^-q DP_{1_j→D_γ, x_i→D_β}[P][e^q] /. {β → α / ω, γ → λ Log[t]}] ]];
```

The Λόγος

```
Λ[k_] := ((t - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 e_k 1_k f_k δ^2 μ^2 -
    δ (1 + μ) (f_k^2 α^2 + e_k^2 β^2) - e_k^2 f_k^2 δ^3 (1 + 3 μ) -
    2 (α β + 2 δ μ + e_k f_k δ^2 (1 + 2 μ) + 2 1_k δ μ^2) (f_k α + e_k β) -
```

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle h, l, e', f\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with h central and with $[f, l] = f$, $[l, e'] = e'$, and $[e', f] = h - 2\epsilon l$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle h, e', f, \epsilon h, \epsilon l, \epsilon e', \epsilon f\rangle \supset \langle h, \epsilon h, \epsilon l, \epsilon e', \epsilon f\rangle \supset 0$. Pragmatics: declare $\deg(h, l, e', f, \epsilon) = (1, 0, 1, 0, 1)$ and set $t := \mathbb{e}^h$ and $e := (t-1)e'/h$.

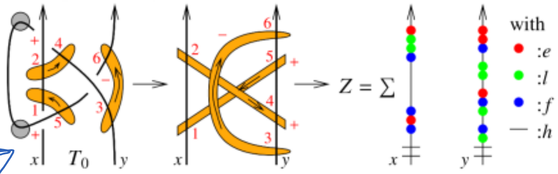
How did it arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^- / \mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle l, f\rangle/[f, l] = f$ is a Lie bialgebra with $\delta\colon \mathfrak{b}^+ \to \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta\colon (l, f) \mapsto (0, l \wedge f)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle h', e', l, f\rangle/\cdots$. **Idea.** Replace $\delta \to \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 1$, get $[f, l] = f$, $[f, h'] = -\epsilon f$, $[l, e'] = e'$, $[h', e'] = -\epsilon e'$, $[h', l] = 0$, and $[e', f] = h' - \epsilon l$. Now note that $h' + \epsilon l$ is central, so switch to $h := h' + \epsilon l$. This is $\mathfrak{g}_1$.

Ordering Symbols. $\mathbb{O}(poly \mid specs)$ plants the variables of $poly$ in $\hat{\mathcal{S}}(\oplus_i \mathfrak{g})$ along $\hat{\mathcal{U}}(\mathfrak{g})$ according to $specs$. E.g.,

$$\mathbb{O}\left(e_1 \mathbb{e}^{e_3} l_1^3 l_2 f_3^9 \mid f_3 l_1 e_1 e_3 l_2\right) = f^9 l^3 e \mathbb{e}^e l \in \hat{\mathcal{U}}(\mathfrak{g}).$$

This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})$ using commutative polynomials / power series. In $\mathfrak{g}_1$, no need to specify h / t.

Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A \sim \hat{\mathcal{U}}(\mathfrak{g})$), appropriate orange $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$-valued invariant of pure S-component tangles:



Demo Programs. ωεβ/Demo **Formatting**

```
CF[ℰ_] := Module[{vars = Union@Cases[ℰ, e_ | l_ | f_, ∞]},
   If[vars === {}, Factor[ℰ],
    Total[CoefficientRules[ℰ, vars] /.
      (p_ → c_) ↦ Factor[c] Times @@ (vars^p)] ]];
CF[ℰ_𝔼] := CF /@ ℰ;
```

Preparation

```
𝔼[i_, j_, s_] := 𝔼[1, (-1)^s l_j, (-t)^s e_i f_j,
   t^s e_i l_(1+s) i-s j f_j + (-1)^s l_i l_j + (-t^2)^s e_i^2 f_j^2/4];
𝔼[i_, s_] := 𝔼[1, 0, 0, s l_i];
𝔼 /: 𝔼[1, L1_, Q1_, P1_] 𝔼[1, L2_, Q2_, P2_] :=
  𝔼[1, L1 + L2, Q1 + Q2, P1 + P2];
```

Preparing the Trefoil

```
z1 = (𝔼[1, 11, 0] 𝔼[4, 2, -1] 𝔼[15, 5, 0]
   𝔼[6, 8, -1] 𝔼[9, 16, 0] 𝔼[12, 14, -1] 𝔼[3, -1] 𝔼[7, +1]
   𝔼[10, -1] 𝔼[13, +1])
```

$$\mathbb{E}\Big[1,\ -l_2 + l_5 - l_8 + l_{11} - l_{14} + l_{16},$$
$$-\frac{e_4 f_2}{t} + e_{15} f_5 - \frac{e_6 f_8}{t} + e_1 f_{11} - \frac{e_{12} f_{14}}{t} + e_9 f_{16},$$
$$-\frac{e_4^2 f_2^2}{4t^2} + \frac{1}{4} e_{15}^2 f_5^2 - \frac{e_6^2 f_8^2}{4t^2} + \frac{1}{4} e_1^2 f_{11}^2 - \frac{e_{12}^2 f_{14}^2}{4t^2} + \frac{1}{4} e_9^2 f_{16}^2 + e_1 f_{11} l_1 +$$
$$\frac{e_4 f_2 l_2}{t} - l_3 - l_2 l_4 + l_7 + \frac{e_6 f_8 l_8}{t} - l_6 l_8 + e_9 f_{16} l_9 - l_{10} +$$
$$l_1 l_{11} + l_{13} + \frac{e_{12} f_{14} l_{14}}{t} - l_{12} l_{14} + e_{15} f_5 l_{15} + l_5 l_{15} + l_9 l_{16}\Big]$$

Differential Polynomials

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] :=
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) ↦ c D[f, {α, m}, {β, n}]]
```

(Implementing $P(\partial_\alpha, \partial_\beta)(f)$)

***le* and *fl* Sorts**

```
S_{l_j_ (x:e|f)_i_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{λ = ∂_{l_j} L, α = ∂_{x_i} Q, q = e^γ β x_k + γ l_k}, CF[
    𝔼[ω, L /. l_j → l_k, t^λ α x_k + (Q /. x_i → 0),
     e^-q DP_{l_j→D_γ, x_i→D_β}[P][e^q] /. {β → α/ω, γ → λ Log[t]}] ]];
```

The Λόγος

```
Λ[k_] := ((t - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 e_k l_k f_k δ^2 μ^2 -
    δ (1 + μ) (f_k^2 α^2 + e_k^2 β^2) - e_k^2 f_k^2 δ^3 (1 + 3 μ) -
    2 (α β + 2 δ μ + e_k f_k δ^2 (1 + 2 μ) + 2 l_k δ μ^2) (f_k α + e_k β) -
    4 (l_k μ^2 + e_k f_k δ (1 + μ)) (α β + δ μ)) (1 + t) / 4;
```

***fe* Sorts**

```
S_{f_i_ e_j_ → k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{q = ((1 - t) α β + β e_k + α f_k + δ e_k f_k) / μ}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. f_i | e_j → 0),
       μ^4 e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + ω^4 Λ[k]] /. μ → 1 + (t - 1) δ /.
     {α → ω^-1 (∂_{f_i} Q /. e_j → 0), β → ω^-1 (∂_{e_j} Q /. f_i → 0),
      δ → ω^-1 ∂_{f_i,e_j} Q}]];
```

Elf Merges

```
m_{i_,j_→k_}[Z_𝔼] := Module[{x, z},
  CF[(Z // S_{f_i e_j→x} // S_{l_i e_x→x} // S_{f_x l_j→x}) /. z_{i|j|x} → z_k]]
```

Rewriting the Trefoil (by merging 16 elves)

```
(Do[z1 = z1 // m_{1,k→1}, {k, 2, 16}]; z1)
```

$$\mathbb{E}\Big[\frac{1-t+t^2}{t}, 0, 0, \frac{(-1+t)(1-t+t^2)^2(1-t+2t^2)}{t^3} - \frac{2(1+t)(1-t+t^2)^3 e_1 f_1}{t^4} - \frac{2(-1+t)(1+t)(1-t+t^2)^3 l_1}{t^4}\Big]$$

Readout

```
ρ1[𝔼[ω_, _, _, P_]] := CF[ t ((P /. e_ | l_ | f_ → 0) - t ω^3 (∂_t ω)) / ((t - 1)^2 ω^2) ]
```

$\rho_1(3_1)$

```
ρ1[z1] // Expand
```

$\frac{1}{t} + t$

What we didn't say (more, including videos, in ωεβ/Talks).

- ρ_1 is "line" in the coloured Jones polynomial; related to Melvin-Morton-Rozansky.
- ρ_1 extends to "rotational virtual tangles" and is a projection of the universal finite type invariant of such.
- ρ_1 seems to have a better chance than anything else we know to detect a counterexample to slice=ribbon.
- ρ_1 leads to a very long to-do list. Have fun!

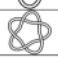
References.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, University of North Carolina PhD thesis, ωεβ/Ov.

[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representaion and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.

[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, arXiv:math/0201139.

diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's A_+	genus / ribbon / unknotting number / amphicheiral	diagram	n_k^t / Today's / Rozansky's ρ_1^+	Alexander's A_+	genus / ribbon / unknotting number / amphicheiral
	0_1^a / 0	1	0 / ✔ / 0 / ✔		3_1^a / t	$t - 1$	1 / ✘ / 1 / ✘
	4_1^a / 0	$3 - t$	1 / ✘ / 1 / ✔		5_1^a / $2t^3 + 3t$	$t^2 - t + 1$	2 / ✘ / 2 / ✘