



Abstract. I will explain how the computation of compositions of maps of a certain natural class, from one polynomial ring into another, naturally leads to a certain composition operation of quadratics and to Feynman diagrams. I will also explain, with very little detail, how this is used in the construction of some very well-behaved poly-time computable knot polynomials.

The PBW Principle Lots of algebras are isomorphic as vector spaces to polynomial algebras. So we want to understand arbitrary linear maps between polynomial algebras.

Gentle Agreement. Everything converges!

Convention. For a finite set A , let $z_A := \{z_i\}_{i \in A}$ and let $\zeta_A := \{\zeta_i^*\}_{i \in A}$. $(y, b, a, x)^* = (\eta, \beta, \alpha, \xi)$

The Generating Series \mathcal{G} : $\text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B]) \rightarrow \mathbb{Q}[\zeta_A, z_B]$.

Claim. $L \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B]) \xrightarrow{\mathcal{G}} \mathbb{Q}[z_B][[\zeta_A]] \ni \mathcal{L}$ via

$$\mathcal{G}(L) := \sum_{n \in \mathbb{N}^A} \frac{\zeta_A^n}{n!} L(z_A^n) = L(\oplus_{a \in A} \zeta_a z_a) = \mathcal{L} = \text{greek } \mathcal{L}_{\text{latin}}$$

$$\mathcal{G}^{-1}(\mathcal{L})(p) = (p|_{z_a \rightarrow \partial_{z_a} \mathcal{L}})_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

Claim. If $L \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B])$, $M \in \text{Hom}(\mathbb{Q}[z_B] \rightarrow \mathbb{Q}[z_C])$, then $\mathcal{G}(L \circ M) = (\mathcal{G}(L)|_{z_b \rightarrow \partial_{z_b} \mathcal{G}(M)})_{\zeta_b=0}$.

Basic Examples. 1. $\mathcal{G}(\text{id}: \mathbb{Q}[y, a, x] \rightarrow \mathbb{Q}[y, a, x]) = e^{\eta y + \alpha a + \xi x}$.

2. The standard commutative product m_k^{ij} of polynomials is given by $z_i, z_j \rightarrow z_k$. Hence $\mathcal{G}(m_k^{ij}) = m_k^{ij}(\oplus_{\zeta_i, \zeta_j} z_i, z_j) = e^{(\zeta_i + \zeta_j) z_k}$.

$$\begin{array}{ccc} \mathbb{Q}[z_i] \otimes \mathbb{Q}[z_j] & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z_k] \\ \parallel & & \parallel \\ \mathbb{Q}[z_i, z_j] & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z_k] \end{array}$$

3. The standard co-commutative co-product Δ_{jk}^i of polynomials is given by $z_i \rightarrow z_j + z_k$. Hence $\mathcal{G}(\Delta_{jk}^i) = \Delta_{jk}^i(\oplus_{\zeta_i} z_i) = e^{\zeta_i(z_j + z_k)}$.

$$\begin{array}{ccc} \mathbb{Q}[z_i] & \xrightarrow{\Delta_{jk}^i} & \mathbb{Q}[z_j] \otimes \mathbb{Q}[z_k] \\ \parallel & & \parallel \\ \mathbb{Q}[z_i] & \xrightarrow{\Delta_{jk}^i} & \mathbb{Q}[z_j, z_k] \end{array}$$

Heisenberg Algebras. Let $\mathbb{H} = \langle x, y \rangle / [x, y] = \hbar$ (with \hbar a scalar), let $\mathbb{O}_i: \mathbb{Q}[x_i, y_i] \rightarrow \mathbb{H}_i$ is the “ x before y ” PBW ordering map and let hm_k^{ij} be the composition

$$\mathbb{Q}[x_i, y_i, x_j, y_j] \xrightarrow{\mathbb{O}_i \otimes \mathbb{O}_j} \mathbb{H}_i \otimes \mathbb{H}_j \xrightarrow{m_k^{ij}} \mathbb{H}_k \xrightarrow{\mathbb{O}_k^{-1}} \mathbb{Q}[x_k, y_k].$$

Then $\mathcal{G}(hm_k^{ij}) = e^{\Lambda_{\hbar}}$, where $\Lambda_{\hbar} = -\hbar \eta_i \xi_j + (\xi_i + \xi_j) x_k + (\eta_i + \eta_j) y_k$.

Proof 1. Recall the “Weyl form of the CCR” $e^{\eta y} e^{\xi x} = e^{-\hbar \eta \xi} e^{\xi x} e^{\eta y}$, and compute

$$\begin{aligned} \mathcal{G}(hm_k^{ij}) &= e^{\xi_i x_i + \eta_i y_i + \xi_j x_j + \eta_j y_j} \parallel \mathbb{O}_i \otimes \mathbb{O}_j \parallel m_k^{ij} \parallel \mathbb{O}_k^{-1} \\ &= e^{\xi_i x_i} e^{\eta_i y_i} e^{\xi_j x_j} e^{\eta_j y_j} \parallel m_k^{ij} \parallel \mathbb{O}_k^{-1} = e^{\xi_i x_k} e^{\eta_i y_k} e^{\xi_j x_k} e^{\eta_j y_k} \parallel \mathbb{O}_k^{-1} \\ &= e^{-\hbar \eta_i \xi_j} e^{(\xi_i + \xi_j) x_k} e^{(\eta_i + \eta_j) y_k} \parallel \mathbb{O}_k^{-1} = e^{\Lambda_{\hbar}}. \end{aligned}$$

Proof 2. We compute in a faithful 3D representation ρ of \mathbb{H} :

$$\left\{ \hat{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \hat{y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hbar \\ 0 & 0 & 0 \end{pmatrix}, \hat{c} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}; \quad (\omega \epsilon \beta / \text{hm})$$

$$\{\hat{x} \cdot \hat{y} - \hat{y} \cdot \hat{x} = \hbar \hat{c}, \hat{x} \cdot \hat{c} = \hat{c} \cdot \hat{x}, \hat{y} \cdot \hat{c} = \hat{c} \cdot \hat{y}\}$$

{True, True, True}

$$\Lambda = -\hbar \eta_i \xi_j c_k + (\xi_i + \xi_j) x_k + (\eta_i + \eta_j) y_k;$$

Simplify@With [{ $\mathbb{E} = \text{MatrixExp}$ },

$$\begin{aligned} &\mathbb{E}[\hat{x} \xi_i] \cdot \mathbb{E}[\hat{y} \eta_i] \cdot \mathbb{E}[\hat{x} \xi_j] \cdot \mathbb{E}[\hat{y} \eta_j] = \\ &\mathbb{E}[\hat{x} \partial_{x_k} \Lambda] \cdot \mathbb{E}[\hat{y} \partial_{y_k} \Lambda] \cdot \mathbb{E}[\hat{c} \partial_{c_k} \Lambda] \end{aligned}$$

True

A Real DoPeGDO Example (DoPeGDO:=Docile Perturbed Gaussian Differential Operators). Let $sl_{2+}^{\epsilon} := L\langle y, b, a, x \rangle$ subject to $[a, x] = x$, $[b, y] = -\epsilon y$, $[a, b] = 0$, $[a, y] = -y$, $[b, x] = \epsilon x$, and $[x, y] = \epsilon a + b$. So $t := \epsilon a - b$ is central and if $\exists \epsilon^{-1}$, $sl_{2+}^{\epsilon} \cong sl_2 \oplus \langle t \rangle$. Let $CU := \mathcal{U}(sl_{2+}^{\epsilon})$, and let cm_k^{ij} be the composition below, where $\mathbb{O}_i: \mathbb{Q}[y_i, b_i, a_i, x_i] \rightarrow CU_i$ be the PBW ordering map in the order ybx :

$$\begin{array}{ccc} CU_i \otimes CU_j & \xrightarrow{m_k^{ij}} & CU_k \\ \uparrow \mathbb{O}_{i,j} & & \uparrow \mathbb{O}_k \\ \mathbb{Q}[y_i, b_i, a_i, x_i, y_j, b_j, a_j, x_j] & \xrightarrow{cm_k^{ij}} & \mathbb{Q}[y_k, b_k, a_k, x_k] \end{array}$$

Claim. Let (all drawn and no brains)

$$\Lambda = \left(\eta_i + \frac{e^{-\alpha_i - \epsilon \beta_i} \eta_j}{1 + \epsilon \eta_j \xi_i} \right) y_k + \left(\beta_i + \beta_j + \frac{\log(1 + \epsilon \eta_j \xi_i)}{\epsilon} \right) b_k + \left(\alpha_i + \alpha_j + \log(1 + \epsilon \eta_j \xi_i) \right) a_k + \left(\frac{e^{-\alpha_j - \epsilon \beta_j} \xi_i}{1 + \epsilon \eta_j \xi_i} + \xi_j \right) x_k$$

Then $e^{\eta_i y_i + \beta_i b_i + \alpha_i a_i + \xi_i x_i + \eta_j y_j + \beta_j b_j + \alpha_j a_j + \xi_j x_j} \parallel \mathbb{O}_{i,j} \parallel cm_k^{ij} = e^{\Lambda} \parallel \mathbb{O}_k$, and hence $\mathcal{G}(cm_k^{ij}) = e^{\Lambda}$.

Proof. We compute in a faithful 2D representation ρ of CU :

$$\left\{ \hat{y} = \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}, \hat{b} = \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon \end{pmatrix}, \hat{a} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \hat{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}; \quad (\omega \epsilon \beta / sl_2)$$

$$\begin{aligned} \{\hat{a} \cdot \hat{x} - \hat{x} \cdot \hat{a} = \hat{x}, \hat{a} \cdot \hat{y} - \hat{y} \cdot \hat{a} = -\hat{y}, \hat{b} \cdot \hat{y} - \hat{y} \cdot \hat{b} = -\epsilon \hat{y}, \\ \hat{b} \cdot \hat{x} - \hat{x} \cdot \hat{b} = \epsilon \hat{x}, \hat{x} \cdot \hat{y} - \hat{y} \cdot \hat{x} = \hat{b} + \epsilon \hat{a}\} \end{aligned}$$

{True, True, True, True, True}

Simplify@With [{ $\mathbb{E} = \text{MatrixExp}$ },

$$\begin{aligned} &\mathbb{E}[\eta_i \hat{y}] \cdot \mathbb{E}[\beta_i \hat{b}] \cdot \mathbb{E}[\alpha_i \hat{a}] \cdot \mathbb{E}[\xi_i \hat{x}] \cdot \mathbb{E}[\eta_j \hat{y}] \cdot \mathbb{E}[\beta_j \hat{b}] \cdot \\ &\mathbb{E}[\alpha_j \hat{a}] \cdot \mathbb{E}[\xi_j \hat{x}] = \mathbb{E}[\hat{y} \partial_{y_k} \Lambda] \cdot \mathbb{E}[\hat{b} \partial_{b_k} \Lambda] \cdot \mathbb{E}[\hat{a} \partial_{a_k} \Lambda] \cdot \\ &\mathbb{E}[\hat{x} \partial_{x_k} \Lambda] \end{aligned}$$

True

Series [Λ , { $\epsilon, 0, 2$ }]

$$\begin{aligned} &(a_k (\alpha_i + \alpha_j) + y_k (\eta_i + e^{-\alpha_i} \eta_j) + \\ &b_k (\beta_i + \beta_j + \eta_j \xi_i) + x_k (e^{-\alpha_j} \xi_i + \xi_j)) + \\ &\left(a_k \eta_j \xi_i - \frac{1}{2} b_k \eta_j^2 \xi_i^2 - e^{-\alpha_i} y_k \eta_j (\beta_i + \eta_j \xi_i) - \right. \\ &\left. e^{-\alpha_j} x_k \xi_i (\beta_j + \eta_j \xi_i) \right) \epsilon + \\ &\left(-\frac{1}{2} a_k \eta_j^2 \xi_i^2 + \frac{1}{3} b_k \eta_j^3 \xi_i^3 + \frac{1}{2} e^{-\alpha_i} y_k \eta_j (\beta_i^2 + 2 \beta_i \eta_j \xi_i + 2 \eta_j^2 \xi_i^2) + \right. \\ &\left. \frac{1}{2} e^{-\alpha_j} x_k \xi_i (\beta_j^2 + 2 \beta_j \eta_j \xi_i + 2 \eta_j^2 \xi_i^2) \right) \epsilon^2 + 0[\epsilon]^3 \end{aligned}$$

Note 1. If the lower half of the alphabet (a, b, α, β) is regarded as constants, then $\Lambda = C + Q + \sum_{k \geq 1} \epsilon^k P^{(k)}$ is a docile perturbed Gaussian relative to the upper half of the alphabet (x, y, ξ, η) : C is a scalar, Q is a quadratic, and $\deg P^{(k)} \leq 2k + 2$.

Note 2. $\text{wt}(x, y, \xi, \eta; a, b, \alpha, \beta; \epsilon) = (1, 1, 1, 1; 2, 0, 0, 2; -2)$.

Quadratic Casimirs. If $t \in \mathfrak{g} \otimes \mathfrak{g}$ is the quadratic Casimir of a semi-simple Lie algebra \mathfrak{g} , then e^t , regarded by PBW as an element of $\mathcal{S}^{\otimes 2} = \text{Hom}(\mathcal{S}(\mathfrak{g})^{\otimes 0} \rightarrow \mathcal{S}(\mathfrak{g})^{\otimes 2})$, has a latin-latin dominant Gaussian factor. Likewise for R -matrices.

(Baby) DoPeGDO := The category with objects finite sets^{†1} and

$$\text{mor}(A \rightarrow B) = \{ \mathcal{L} = \omega \exp(Q + P) \} \subset \mathbb{Q}[\zeta_A, z_B, \epsilon],$$

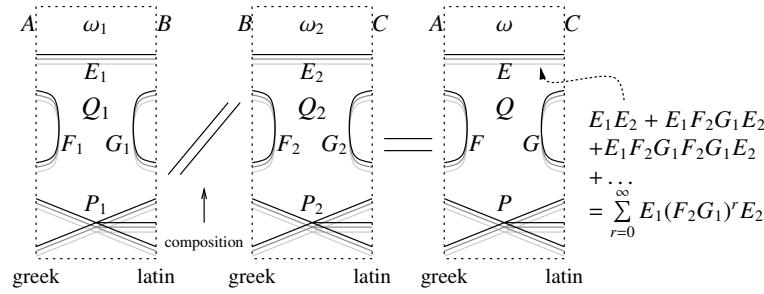
where: • ω is a scalar.^{†2} • Q is a “small” ϵ -free quadratic in $\zeta_A \cup z_B$.^{†3} • P is a “docile perturbation”: $P = \sum_{k \geq 1} \epsilon^k P^{(k)}$, where $\deg P^{(k)} \leq 2k + 2$.^{†4} • Compositions:^{†6} $\mathcal{L} \circ \mathcal{M} := (\mathcal{L}|_{z_i \rightarrow \partial_{z_i} \mathcal{M}})_{\zeta_i=0}$.

So What? If V is a representation, then $V^{\otimes n}$ explodes as a function of n , while in **DoPeGDO** up to a fixed power of ϵ , the ranks of $\text{mor}(A \rightarrow B)$ grow polynomially as a function of $|A|$ and $|B|$.

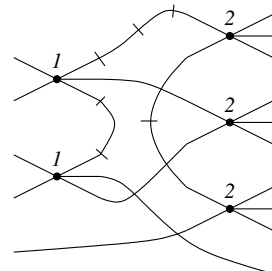
Compositions. In $\text{mor}(A \rightarrow B)$,

$$Q = \sum_{i \in A, j \in B} E_{ij} \zeta_i \zeta_j + \frac{1}{2} \sum_{i, j \in A} F_{ij} \zeta_i \zeta_j + \frac{1}{2} \sum_{i, j \in B} G_{ij} \zeta_i \zeta_j,$$

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \dots$)



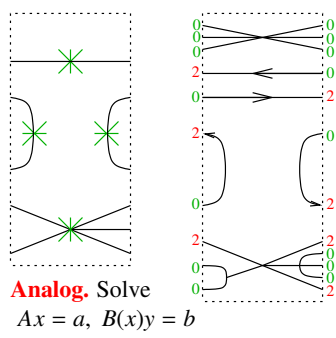
- $E = E_1(I - F_2G_1)^{-1}E_2$.
- $F = F_1 + E_1F_2(I - G_1F_2)^{-1}E_1^T$.
- $G = G_2 + E_2^T G_1(I - F_2G_1)^{-1}E_2$.
- $\omega = \omega_1 \omega_2 \det(I - F_2G_1)^{-1}$.
- P is computed as the solution of a messy PDE or using “connected Feynman diagrams” (yet we’re still in pure algebra!). Docility is preserved.



DoPeGDO Footnotes. Each variable has a “weight” $\in \{0, 1, 2\}$, and always $\text{wt } z_i + \text{wt } \zeta_i = 2$.

- †1. Really, “weight-graded finite sets” $A = A_0 \sqcup A_1 \sqcup A_2$.
- †2. Really, a power series in the weight-0 variables†5.
- †3. The weight of Q must be 2, so it decomposes as $Q = Q_{20} + Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series†5.
- †4. Setting $\text{wt } \epsilon = -2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained)†5.
- †5. In the knot-theoretic case, all weight-0 power series are rational functions of bounded degree in the exponentials of the weight-0 variables.
- †6. There’s also an obvious product $\text{mor}(A_1 \rightarrow B_1) \times \text{mor}(A_2 \rightarrow B_2) \rightarrow \text{mor}(A_1 \sqcup A_2 \rightarrow B_1 \sqcup B_2)$.

Full DoPeGDO. Compute compositions in two phases:
 • A 1-1 phase over the ring of power series in the weight-0 variables, in which the weight-2 variables are spectators.
 • A (slightly modified) 2-0 phase over \mathbb{Q} , in which the weight-1 variables are spectators.

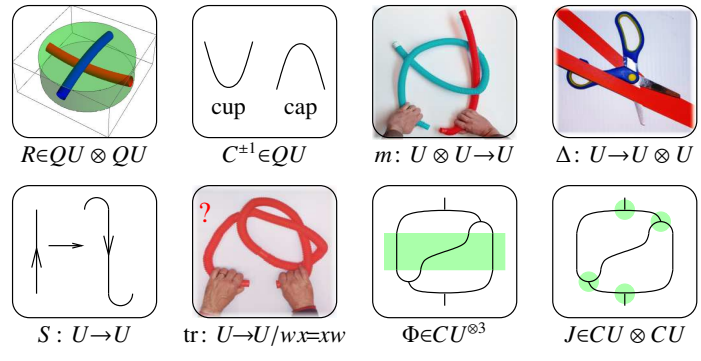


Questions. • Are there QFT precedents for “two-step Gaussian integration”?

- In QFT, one saves even more by considering “one-particle-irreducible” diagrams and “effective actions”. Does this mean anything here?
- Understanding $\text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B])$ seems like a good cause. Can you find other applications for the technology here?

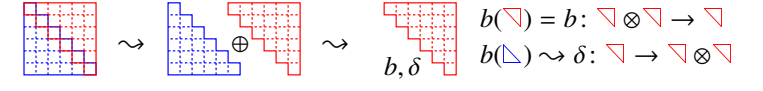
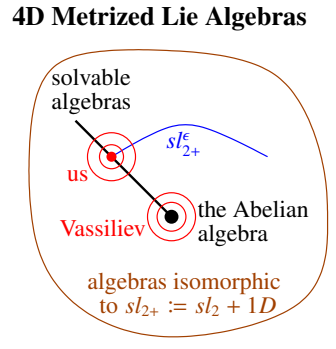
$$\left(\begin{aligned} QU &= \mathcal{U}_h(sl_{2+}^{\epsilon}) = A(y, b, a, x) / \hbar \text{ with } [a, x] = x, [b, y] = -\epsilon y, [a, b] = 0, \\ [a, y] &= -y, [b, x] = \epsilon x, \text{ and } xy - qyx = (1 - AB) / \hbar, \text{ where } q = e^{\hbar \epsilon}, A = e^{-\hbar \epsilon a}, \\ \text{and } B &= e^{-\hbar b}. \text{ Also } \Delta(y, b, a, x) = (y_1 + B_1 y_2, b_1 + b_2, a_1 + a_2, x_1 + A_1 x_2), \\ S(y, b, a, x) &= (-B^{-1}y, -b, -a, -A^{-1}x), \text{ and } R = \sum \hbar^{j+k} y^j b^k \otimes a^j x^k / j! k! q^j. \end{aligned} \right)$$

Theorem. Everything of value regarding $U = CU$ and/or its quantization $U = QU$ is **DoPeGDO**:



also Cartan’s θ , the Dequantizator, and more, and all of their compositions.

Solvable Approximation. In sl_n , half is enough! Indeed $sl_n \oplus \mathfrak{a}_{n-1} = \mathcal{D}(\nabla, b, \delta)$. Now define $sl_{n+}^{\epsilon} := \mathcal{D}(\nabla, b, \epsilon \delta)$. Schematically, this is $[\nabla, \nabla] = \nabla$, $[\Delta, \Delta] = \epsilon \Delta$, and $[\nabla, \Delta] = \Delta + \epsilon \nabla$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1} = 0$ always yields a solvable Lie algebra.











Conclusion. There are lots of poly-time-computable well-behaved near-Alexander knot invariants:
 • They extend to tangles with appropriate multiplicative behaviour.
 • They have cabling and strand reversal formulas. $\omega \epsilon \beta / \text{aft}$
 The invariant for $sl_{2+}^{\epsilon} / (\epsilon^2 = 0)$ (prior art: $\omega \epsilon \beta / \text{Ov}$) attains 2,883 distinct values on the 2,978 prime knots with ≤ 12 crossings. HOMFLY-PT and Khovanov homology together attain only 2,786 distinct values.

knot diag	n_k^t $(\rho_1^t)^+$	Alexander's ω^+ $(\rho_2^t)^+$	genus / ribbon unknotting # / amphi?	knot diag	n_k^t $(\rho_1^t)^+$	Alexander's ω^+ $(\rho_2^t)^+$	genus / ribbon unknotting # / amphi?	knot diag	n_k^t $(\rho_1^t)^+$	Alexander's ω^+ $(\rho_2^t)^+$	genus / ribbon unknotting # / amphi?
	0_1^a 0	1	0 / ✓ 0 / ✓		3_1^a T	T-1	1 / ✗ 1 / ✗		4_1^a 0	3-T	1 / ✗ 1 / ✓
	5_1^a $2T^3 + 3T$	$T^2 - T + 1$	2 / ✗ 2 / ✗		5_2^a $5T - 4$	$2T - 3$	1 / ✗ 1 / ✗		6_1^a T-4	$5 - 2T$	1 / ✓ 1 / ✗
	6_2^a $T^3 - 4T^2 + 4T - 4$	$-T^2 + 3T - 3$	2 / ✗ 1 / ✗		6_3^a 0	$T^2 - 3T + 5$	2 / ✗ 1 / ✓		7_1^a $3T^5 + 5T^3 + 6T$	$T^3 - T^2 + T - 1$	3 / ✗ 3 / ✗
		$3T^8 - 217T^7 + 497T^6 + 157T^5 - 433T^4 + 1543T^3 - 3431T^2 + 5482T - 6410$				$4T^8 - 337T^7 + 1217T^6 - 2037T^5 - 1117T^4 + 1499T^3 - 4210T^2 + 7186T - 8510$				$7T^{11} - 287T^{10} + 777T^9 - 168T^8 + 3227T^7 - 5607T^6 + 8917T^5 - 13107T^4 + 17777T^3 - 22387T^2 + 2604T - 2772$	

knot diag	n_k^r $(\rho_1^r)^+$	Alexander's ω^+ $(\rho_2^r)^+$	genus / ribbon unknotting # / amphi?	knot diag	n_k^r $(\rho_1^r)^+$	Alexander's ω^+ $(\rho_2^r)^+$	genus / ribbon unknotting # / amphi?	knot diag	n_k^r $(\rho_1^r)^+$	Alexander's ω^+ $(\rho_2^r)^+$	genus / ribbon unknotting # / amphi?
	7_2^a	$3T-5$ $14T-16$ $-129T^4+1177T^3-4421T^2+9226T-11718$	1 / ✗ 1 / ✗		7_3^a	$2T^2-3T+3$ $-9T^3+8T^2-16T+12$ $-18T^8+208T^7-917T^6+2666T^5-6049T^4+11283T^3-17671T^2+23356T-25736$	2 / ✗ 2 / ✗		7_4^a	$4T-7$ $32-24T$ $-352T^4+3616T^3-14378T^2+30700T-39188$	1 / ✗ 2 / ✗
	7_5^a	$2T^2-4T+5$ $9T^3-16T^2+29T-28$ $-18T^8+264T^7-1548T^6+5680T^5-15107T^4+31152T^3-51476T^2+69252T-76414$	2 / ✗ 2 / ✗		7_6^a	$-T^2+5T-7$ $T^3-8T^2+19T-20$ $3T^8-35T^7+128T^6+105T^5-2610T^4+11225T^3-28031T^2+47186T-55946$	2 / ✗ 1 / ✗		7_7^a	T^2-5T+9 $8-3T$ $4T^8-55T^7+3107T^6-805T^5+86T^4+6349T^3-22686T^2+43610T-53622$	2 / ✗ 1 / ✗
	8_1^a	$7-3T$ $5T-16$ $42T^4+215T^3-2542T^2+7562T-10542$	1 / ✗ 1 / ✗		8_2^a	$-T^3+3T^2-3T+3$ $2T^5-8T^4+10T^3-12T^2+13T-12$ $5T^{12}-39T^{11}+119T^{10}-139T^9-249T^8+1660T^7-4959T^6+11131T^5-20813T^4+33595T^3-47521T^2+58988T-63556$	3 / ✗ 2 / ✗		8_3^a	$9-4T$ 0 $224T^4-224T^3-3910T^2+14100T-20364$	1 / ✗ 2 / ✓
	8_4^a	$-2T^2+5T-5$ $3T^3-8T^2+6T-4$ $54T^8-344T^7+865T^6-650T^5-2723T^4+12243T^3-28461T^2+45792T-53540$	2 / ✗ 2 / ✗		8_5^a	$-T^3+3T^2-4T+5$ $-2T^5+8T^4-13T^3+20T^2-22T+24$ $5T^{12}-39T^{11}+128T^{10}-182T^9-274T^8+2476T^7-8642T^6+21517T^5-42924T^4+71719T^3-102448T^2+126480T-135628$	3 / ✗ 2 / ✗		8_6^a	$-2T^2+6T-7$ $5T^3-20T^2+28T-32$ $38T^8-216T^7+112T^6+2880T^5-14787T^4+42444T^3-85415T^2+128406T-146916$	2 / ✗ 2 / ✗
	8_7^a	T^3-3T^2+5T-5 $-T^5+4T^4-10T^3+12T^2-13T+12$ $8T^{12}-75T^{11}+343T^{10}-979T^9+1821T^8-1782T^7-1623T^6+12083T^5-33001T^4+64599T^3-101194T^2+131404T-143216$	3 / ✗ 1 / ✗		8_8^a	$2T^2-6T+9$ $-T^3+4T^2-12T+16$ $62T^8-504T^7+1736T^6-2408T^5-3717T^4+26492T^3-68493T^2+113418T-133180$	2 / ✓ 2 / ✗		8_9^a	$-T^3+3T^2-5T+7$ 0 $9T^{12}-87T^{11}+417T^{10}-1305T^9+2858T^8-4134T^7+2114T^6+8285T^5-31925T^4+69235T^3-112773T^2+148508T-162396$	3 / ✓ 1 / ✓
	8_{10}^a	T^3-3T^2+6T-7 $-T^5+4T^4-11T^3+16T^2-21T+20$ $8T^{12}-75T^{11}+362T^{10}-1122T^9+2306T^8-2540T^7-2198T^6+18817T^5-54380T^4+110103T^3-175694T^2+230080T-251346$	3 / ✗ 2 / ✗		8_{11}^a	$-2T^2+7T-9$ $5T^3-24T^2+39T-44$ $38T^8-264T^7+301T^6+3514T^5-21716T^4+68785T^3-146898T^2+227828T-263172$	2 / ✗ 1 / ✗		8_{12}^a	$T^2-7T+13$ 0 $4T^8-77T^7+583T^6-1991T^5+987T^4+17311T^3-71802T^2+147914T-185846$	2 / ✗ 2 / ✓
	8_{13}^a	$2T^2-7T+11$ $-T^3+4T^2-14T+20$ $62T^8-592T^7+2351T^6-3918T^5-4235T^4+40079T^3-111533T^2+191500T-227432$	2 / ✗ 1 / ✗		8_{14}^a	$-2T^2+8T-11$ $5T^3-28T^2+57T-68$ $38T^8-312T^7+444T^6+5096T^5-3477T^4+116368T^3-255750T^2+401632T-465478$	2 / ✗ 1 / ✗		8_{15}^a	$3T^2-8T+11$ $21T^3-64T^2+120T-140$ $-123T^8+2128T^7-15241T^6+66120T^5-19999T^4+451912T^3-79241T^2+1101720T-1228222$	2 / ✗ 2 / ✗
	8_{16}^a	T^3-4T^2+8T-9 $T^5-6T^4+17T^3-28T^2+35T-36$ $8T^{12}-100T^{11}+598T^{10}-2205T^9+5292T^8-7164T^7-23807T^6+43100T^5-137314T^4+291750T^3-478742T^2+636488T-698666$	3 / ✗ 2 / ✗		8_{17}^a	$-T^3+4T^2-8T+11$ 0 $9T^{12}-116T^{11}+722T^{10}-2843T^9+7656T^8-13668T^7+11117T^6+21968T^5-113086T^4+273778T^3-475622T^2+649064T-717954$	3 / ✗ 1 / ✓		8_{18}^a	$-T^3+5T^2-10T+13$ 0 $9T^{12}-145T^{11}+1075T^{10}-4842T^9+14504T^8-28560T^7+27957T^6+35195T^5-225204T^4+573797T^3-1021641T^2+1411484T-1567262$	3 / ✗ 2 / ✓
	8_{19}^a	T^3-T^2+1 $-3T^5-4T^2-3T$ $7T^{11}-19T^{10}+67T^9+48T^8-52T^7-91T^6+211T^5+167T^4-431T^3+289T^2+536T-1060$	3 / ✗ 3 / ✗		8_{20}^a	T^2-2T+3 $4T-4$ $4T^8-22T^7+66T^6-124T^5+52T^4+478T^3-1652T^2+3014T-3640$	2 / ✓ 1 / ✗		8_{21}^a	$-T^2+4T-5$ $T^3-8T^2+16T-20$ $3T^8-28T^7+49T^6+352T^5-2489T^4+8164T^3-17530T^2+27092T-31226$	2 / ✗ 1 / ✗

knot diag	n_k^r $(\rho_1^r)^+$	Alexander's ω^+ $(\rho_2^r)^+$	genus / ribbon unknotting # / amphi?	knot diag	n_k^r $(\rho_1^r)^+$	Alexander's ω^+ $(\rho_2^r)^+$	genus / ribbon unknotting # / amphi?
	9_1^a	$T^4-T^3+T^2-T+1$ $4T^7+7T^5+9T^3+10T$ $9T^{15}-36T^{14}+99T^{13}-216T^{12}+414T^{11}-720T^{10}+1170T^9-1800T^8+2630T^7-3662T^6+4853T^5-6142T^4+7423T^3-852T^2+9420T-9780$	4 / ✗ 4 / ✗		9_2^a	$4T-7$ $30T-40$ $-728T^4+6088T^3-21946T^2+44788T-56420$	1 / ✗ 1 / ✗
	9_3^a	$2T^3-3T^2+3T-3$ $-13T^5+12T^4-25T^3+20T^2-32T+24$ $-26T^{12}+296T^{11}-1311T^{10}+3838T^9-8867T^8+17613T^7-31407T^6+51061T^5-76085T^4+104297T^3-131779T^2+152840T-160976$	3 / ✗ 3 / ✗		9_4^a	$3T^2-5T+5$ $23T^3-28T^2+46T-44$ $-219T^8+1999T^7-8389T^6+23799T^5-52835T^4+96723T^3-149121T^2+194698T-213338$	2 / ✗ 2 / ✗
	9_5^a	$6T-11$ $100-65T$ $-3234T^4+29792T^3-113241T^2+236818T-300294$	1 / ✗ 2 / ✗		9_6^a	$2T^3-4T^2+5T-5$ $13T^5-24T^4+45T^3-52T^2+68T-64$ $-26T^{12}+376T^{11}-2212T^{10}+8280T^9-23249T^8+53488T^7-106013T^6+185990T^5-292853T^4+416673T^3-537062T^2+626488T-659788$	3 / ✗ 3 / ✗
	9_7^a	$3T^2-7T+9$ $23T^3-56T^2+99T-108$ $-219T^8+2717T^7-15720T^6+58389T^5-157698T^4+329265T^3-548657T^2+741610T-819394$	2 / ✗ 2 / ✗		9_8^a	$-2T^2+8T-11$ $3T^3-16T^2+29T-28$ $54T^8-552T^7+2124T^6-2216T^5-12641T^4+67112T^3-172118T^2+289304T-342134$	2 / ✗ 2 / ✗
	9_9^a	$2T^3-4T^2+6T-7$ $13T^5-24T^4+55T^3-72T^2+98T-96$ $-26T^{12}+376T^{11}-2296T^{10}+9328T^9-28988T^8+73584T^7-158399T^6+295928T^5-486916T^4+712094T^3-930993T^2+1092074T-1151564$	3 / ✗ 3 / ✗		9_{10}^a	$4T^2-8T+9$ $-40T^3+72T^2-114T+120$ $-608T^8+6720T^7-33776T^6+110928T^5-273462T^4+537040T^3-862768T^2+1145784T-1259748$	2 / ✗ 2, 3 / ✗
	9_{11}^a	$-T^3+5T^2-7T+7$ $-2T^5+16T^4-41T^3+52T^2-66T+64$ $5T^{12}-65T^{11}+312T^{10}-4637T^9-2042T^8+14588T^7-50444T^6+126967T^5-258750T^4+444545T^3-654213T^2+827220T-895336$	3 / ✗ 2 / ✗		9_{12}^a	$-2T^2+9T-13$ $5T^3-36T^2+84T-100$ $38T^8-312T^7+45T^6+9790T^5-60473T^4+202775T^3-453255T^2+722176T-841572$	2 / ✗ 1 / ✗
	9_{13}^a	$4T^2-9T+11$ $-40T^3+92T^2-154T+168$ $-608T^8+7680T^7-43650T^6+158004T^5-417129T^4+856533T^3-1412461T^2+1899222T-2095210$	2 / ✗ 2, 3 / ✗		9_{14}^a	$2T^2-9T+15$ $-T^3+8T^2-35T+60$ $62T^8-752T^7+3655T^6-7178T^5-9502T^4+97737T^3-294656T^2+531720T-642168$	2 / ✗ 1 / ✗
	9_{15}^a	$-2T^2+10T-15$ $-5T^3+40T^2-108T+136$ $38T^8-360T^7+208T^6+12328T^5-84103T^4+298764T^3-691161T^2+1121034T-1313504$	2 / ✗ 2 / ✗		9_{16}^a	$2T^3-5T^2+8T-9$ $-13T^5+36T^4-80T^3+120T^2-161T+168$ $-26T^{12}+456T^{11}-3331T^{10}+15554T^9-53941T^8+149494T^7-345106T^6+680900T^5-1167591T^4+1759576T^3-234749T^2+2786466T-2949428$	3 / ✗ 3 / ✗

knot diag	n'_k Alexander's ω^+ $(\rho'_1)^+$	genus / ribbon unknotting # / amphi?	knot diag	n'_k Alexander's ω^+ $(\rho'_1)^+$	genus / ribbon unknotting # / amphi?
	10_{158}^n $2T^2 - 7T + 12$	$-T^3 + 4T^2 - 10T + 15$ 3 / ✗ 2 / ✗		10_{159}^n $T^5 - 6T^4 + 26T^3 - 60T^2 + 98T - 112$	$T^3 - 4T^2 + 9T - 11$ 3 / ✗ 1 / ✗
	$9T^{12} - 116T^{11} + 764T^{10} - 3275T^9 + 9743T^8 - 19422T^7 + 18439T^6 + 32898T^5 - 196271T^4 + 513374T^3 - 940025T^2 + 1323614T - 1479452$			$8T^{12} - 100T^{11} + 609T^{10} - 2267T^9 + 5047T^8 - 3237T^7 - 23513T^6 + 115362T^5 - 318739T^4 + 648093T^3 - 1045247T^2 + 1379659T - 1511358$	
	10_{160}^n $-2T^5 + 12T^4 - 20T^3 + 14T^2 - 16T + 12$	$-T^3 + 4T^2 - 4T + 3$ 3 / ✗ 2 / ✗		10_{161}^n $3T^5 + 6T^4 - 3T^3 + 4T^2 + 14T - 12$	$T^3 - 2T + 3$ 3 / ✗ 3 / ✗
	$5T^{12} - 52T^{11} + 198T^{10} - 255T^9 - 522T^8 + 3092T^7 - 8443T^6 + 18756T^5 - 37588T^4 + 67858T^3 - 108568T^2 + 148444T - 165862$			$30T^{10} - 53T^9 - 145T^8 + 630T^7 - 674T^6 - 870T^5 + 3591T^4 - 4450T^3 + 581T^2 + 6166T - 9640$	
	10_{162}^n $10T^3 - 38T^2 + 58T - 68$	$-3T^2 + 9T - 11$ 2 / ✗ 2 / ✗		10_{163}^n $-T^5 + 8T^4 - 30T^3 + 62T^2 - 89T + 96$	$T^3 - 5T^2 + 12T - 15$ 3 / ✗ 1, 2 / ✗
	$222T^8 - 1473T^7 + 2609T^6 + 8829T^5 - 65543T^4 + 206079T^3 - 427536T^2 + 647498T - 741358$			$8T^{12} - 125T^{11} + 923T^{10} - 4154T^9 + 12040T^8 - 19732T^7 - 4345T^6 + 140575T^5 - 506052T^4 + 1171653T^3 - 2040193T^2 + 2809224T - 3119648$	
	10_{164}^n $T^3 - 10T^2 + 29T - 40$	$3T^2 - 11T + 17$ 2 / ✗ 1 / ✗		10_{165}^n $-5T^3 + 50T^2 - 146T + 196$	$-2T^2 + 10T - 15$ 2 / ✗ 2 / ✗
	$321T^8 - 3179T^7 + 12782T^6 - 20103T^5 - 32876T^4 + 254013T^3 - 688337T^2 + 1170838T - 1386922$			$38T^8 - 344T^7 - 848T^6 + 23020T^5 - 137555T^4 + 465256T^3 - 1047705T^2 + 1673914T - 1951560$	