Mods to StonyBrook-1805

May 16, 2018 11:31 AM

Dror Bar-Natan: Talks: StonyBrook-1805:

Computation without Representation

Thanks for inviting me to the SCGP! ωεβ:=http://drorbn.net/sb18/

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at [BV] and at ωεβ/talks.

Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast. I will describe a direct-participation method for carrying out these computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

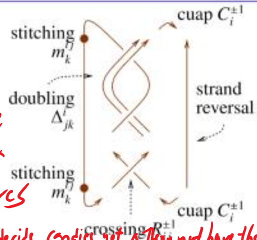
Zoom in on this?

A Knot Theory Portfolio.

- Has operations $\sqcup$, m, Δ, S.
- All tangles are generated by $R^{\pm1}$ and $C^{\pm}$ (so "easy" to produce invariants).
- Makes some properties ("genus", "ribbon") be "definable".

Include genus & ribbon pictures

(more to say, but not now).



(I decide candies get, they must have them, and we remain close!)

A "Quantum Group" Portfolio consists of an algebra U along with maps (and some axioms...)

$$\sqcup \to \otimes$$

$$\begin{tikzcd}
\mathbb{Q} = \hat{U}^{\otimes\emptyset} \arrow[r, "C_i"] & \hat{U}^{\otimes\{i\}} \arrow[loop above, "S_i"] \arrow[r, bend left, "\Delta^i_{jk}"] & \hat{U}^{\otimes\{j,k\}} \arrow[l, bend left, "m^{jk}_i"] & \mathbb{Q} = \hat{U}^{\otimes\emptyset} \arrow[l, "R_{jk}"'] \\
\hat{\mathcal{S}}(\emptyset) \arrow[u, "\mathbb{O}_{()}"] \arrow[r, "C_i"] & \hat{\mathcal{S}}(B_i) \arrow[u, "\mathbb{O}_{y_ix_i\ldots}"] \arrow[loop below, "S_i"] \arrow[r, bend left, "\Delta^i_{jk}"] & \hat{\mathcal{S}}(B_j, B_k) \arrow[u, "\mathbb{O}_{y_jx_j\ldots\otimes y_kx_k\ldots}"] \arrow[l, bend left, "m^{jk}_i"] & \hat{\mathcal{S}}(\emptyset) \arrow[u, "\mathbb{O}_{()}"] \arrow[l, "R_{jk}"']
\end{tikzcd}$$

$$\sqcup \to \cdot$$

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set B of "generators" and isomorphisms $\mathbb{O}_{y,x,\ldots}: \hat{\mathcal{S}}(B) \to U$ defined by "ordering monomials" to some fixed $y, x, \ldots$ order. The quantum group portfolio now becomes a "symmetric algebra" potfolio, or a "power series" portfolio.

The Category $\mathcal{DO}$. Hence we care about the monoidal category $\mathcal{DO}$ whose objects are finite sets B and whose morphisms are $\mathrm{mor}_{\mathcal{DO}}(B, B') := \mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B) \to \mathcal{S}(B')) = \mathcal{S}(B^*, B')$ (by convention, $x^* = \xi$, $y^* = \eta$, etc.).

The Composition Law. If

$$\mathcal{S}(B_0) \xrightarrow[{}^t f \in \mathbb{Q}[\![\zeta_{0i}, z_{1j}]\!]]{f} \mathcal{S}(B_1) \xrightarrow[{}^t g \in \mathbb{Q}[\![\zeta_{1j}, z_{2k}]\!]]{g} \mathcal{S}(B_2)$$

then ${}^t(f /\!\!/ g) = {}^t(g \circ f) = \left(g|_{\zeta_{1j} \to \partial_{z_{1j}}} f \right)_{z_{1j}=0} = \left(f|_{z_{1j} \to \partial_{\zeta_{1j}}} g \right)_{\zeta_{1j}=0}$.

Note. For $f \in \mathcal{S}(z)$ and $g \in \mathcal{S}(\zeta)$, $\langle f, g \rangle = f(\partial_\zeta) g|_{\zeta=0} = g(\partial_z) f|_{z=0}$.

Proposition. If $F: \mathcal{S}(B) \to \mathcal{S}(B')$ is linear and "continuous", then ${}^t F = \exp\left(\sum_{z_i \in B} \zeta_i z_i \right) /\!\!/ F$.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

Examples.

1. The 1-variable identity map $I: \mathcal{S}(z) \to \mathcal{S}(z)$ is given by ${}^t I_1 = \mathbb{e}^{z\zeta}$ and the n-variable one by ${}^t I_n = \mathbb{e}^{z_1\zeta_1 + \cdots + z_n\zeta_n}$.
2. The "$z_i \to z_j$ variable rename map $\sigma^i_j: \mathcal{S}(z_i) \to \mathcal{S}(z_j)$" becomes ${}^t\sigma^i_j = \mathbb{e}^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m^{ij}_k: \mathcal{S}(z_i, z_j) \to \mathcal{S}(z_k)$" has ${}^t m = \mathbb{e}^{z_k(\zeta_i + \zeta_j)}$.
4. The "archetypal coproduct $\Delta^i_{jk}: \mathcal{S}(z_i) \to \mathcal{S}(z_j, z_k)$", given by $z_i \to z_j + z_k$ or $\Delta z = z \otimes 1 + 1 \otimes z$, has ${}^t\Delta = \mathbb{e}^{(z_j + z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1 x_2} \in \mathcal{U}_q \otimes \mathcal{U}_q$. The "baby R-matrix" is ${}^t R = \mathbb{e}^{\hbar yx} \in \mathcal{S}(y, x)$.
6. The "Weyl form of the canonical commutation relations" states that if $[y, x] = t$ is a scalar, then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. Thus with

$$\begin{tikzcd}
\mathcal{S}(y, x) \arrow[loop left, "SW_{xy}"] \arrow[r, shift left, "\mathbb{O}_{xy}"] \arrow[r, shift right, "\mathbb{O}_{yx}"'] & \mathcal{U}(y, x)
\end{tikzcd}$$

we have ${}^t SW_{xy} = \mathbb{e}^{\eta y + \xi x - \eta\xi t}$.

The Zipping Issue. (between unbound and bound lies half-zipped).

Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set

$$\left\langle P(\zeta^j, z_i) \right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i=0}.$$

(E.g., if $P = \sum a_{nm}\zeta^n z^m$ then $\langle P \rangle_\zeta = \sum n! a_{nn}$).

Implementation. ωεβ/Zip

```
z* = ζ; ζ* = z; Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] :=
 (Expand[P // Zip{ζs}] /. f_. ζ^d_. :> ∂{ζ*,d} f) /. ζ* → 0
{Zip{ζ}[ζ^2 e^(δ z^2)], Zip{ζ}[ζ^4 e^(δ z^2)]}
```

{2 δ, 12 δ^2}

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{\eta^i z_i+y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i+y_i,\zeta^j)\mathbb{e}^{\eta^i(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j\to\hbar y_j$ and test at $\hbar=0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde q)\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i}\Big|_{z_i\to\tilde q^k_i(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde q$ is the inverse matrix of $1-q$: $(\delta^i_j-q^i_j)\tilde q^j_k=\delta^i_k$ (proof: replace $q^i_j\to\hbar q^i_j$ and test at $\hbar=0$ and at $\partial_\hbar$).

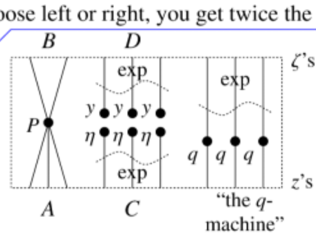
reduce / improve 2 6

Exponential Reservoirs.. The true Hilbert hotel is exp! Remove one x from an "exponential reservoir" of x's and you are left with the same exponential reservoir:

$$\mathbb{e}^x=\left[\dots+\frac{xxxxx}{120}+\dots\right]\xrightarrow{\partial_x}\left[\dots+\frac{\cancel{x}xxxx}{120}+\dots\right]=(\mathbb{e}^x)'=\mathbb{e}^x,$$

and if you let each element choose left or right, you get twice the same reservoir:

$$\mathbb{e}^x\xrightarrow{x\to x_l+x_r}\mathbb{e}^{x_l+x_r}=\mathbb{e}^{x_l}\mathbb{e}^{x_r}.$$



A Graphical Proof. Glue top to bottom on the right, in all possible ways. Several scenarios occur:

1. Start at A, go through the q-machine $k\geq 0$ times, stop at B. Get $\left\langle P\left(\sum_{k\geq 0}q^k z,\zeta\right)\right\rangle=\langle P(\tilde q z,\zeta)\rangle$.
2. Loop through the q-machine and swallow your own tail. Get $\exp\left(\sum q^k/k\right)=\exp(-\log(1-q))=\tilde q$.
3. …

By the reservoir splitting principle, these scenarios contribute multiplicatively. □

Implementation. ωεβ/Zip

```
𝔼 /: Zip_{ξs_List}@𝔼[Q_, P_] :=      (* 𝔼[Q,P] means e^Q P *)
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ξs}];
   c = Q /. Alternatives @@ (ξs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ξs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ξs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule; Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[Q2, Det[qt] e^{-Q2} Zip_{ξs}[e^{Q1} (P /. zrule)]] ];
```

Fuller program & testing suite: ωεβ/mm, ωεβ/port.

The Real Thing. In the algebra QU_ϵ (explained later), over $\mathbb{Q}[\![\hbar]\!]$ using the *yaxt* order, $T=\mathbb{e}^{\hbar t}$, $\bar T=T^{-1}$, $\mathcal{A}=\mathbb{e}^{\alpha}$, and $\bar{\mathcal{A}}=\mathcal{A}^{-1}$, we have

$$^t R_{ij}=\mathbb{e}^{\hbar(y_ix_j-t_ia_j)}\left(1+\epsilon\hbar\left(a_ia_j-\hbar^2y_i^2x_j^2/4\right)+O(\epsilon^2)\right)$$

in $\mathcal{S}(B_i,B_j)$, and in $\mathcal{S}(B_1^*,B_2^*,B)$ we have

$$^t m=\mathbb{e}^{(\alpha_1+\alpha_2)a+\eta_2\xi_1(1-T)/\hbar+(\xi_1\bar{\mathcal{A}}_2+\xi_2)x+(\eta_1+\eta_2\bar{\mathcal{A}}_1)y}\left(1+\epsilon\lambda+O(\epsilon^2)\right),$$

where $\lambda=2a\eta_2\xi_1T+\eta_2^2\xi_1^2\left(3T^2-4T+1\right)/4\hbar-\eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2/2$
$-\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1/2+\eta_2\xi_1xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$.

Finally,

$$^t\Delta=\mathbb{e}^{\tau(t_1+t_1)+\eta(y_1+T_1y_2)+\alpha(a_1+a_2)+\xi(x_1+x_2)}\left(1+O(\epsilon)\right)\in\mathcal{S}(B^*,B_1,B_2),$$

and $^tS=\mathbb{e}^{-\tau t-\alpha a-\eta\xi(1-\bar T)\mathcal{A}/\hbar-\bar T\eta y\mathcal{A}-\xi x\mathcal{A}}\left(1+O(\epsilon)\right)\in\mathcal{S}(B^*,B)$.

Real Zipping is a minor mess, and is done in two phases:

	τa-phase		~~ξx~~ $\eta\xi$-phase	
ζ-like variables	τ	a	ξ	y
z-like variables	t	α	x	η

Already at $\epsilon=0$ we get the best known formulas for the Alexander polynomial!

Generic Docility. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form

$$\mathbb{e}^{q^{ij}z_iz_j}P=\mathbb{e}^{q^{ij}z_iz_j}\left(\sum_{k\geq 0}\epsilon^kP_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k\leq 4k$.

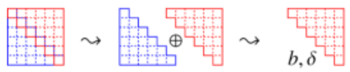
Our Docility. In the case of QU_ϵ, all invariants and operations are of the form $\mathbb{e}^{L+Q}P$, where

- L is a quadratic of the form $\sum l_{z\zeta}z\zeta$, where z runs over $\{t_i,\alpha_i\}_{i\in S}$ and ζ over $\{\tau_i,a_i\}_{i\in S}$, with integer coefficients $l_{z\zeta}$.
- Q is a quadratic for the form $\sum q_{z\zeta}z\zeta$, where z runs over $\{x_i,\eta_i\}_{i\in S}$ and ζ over $\{\xi_i,y_i\}_{i\in S}$, with coefficients $q_{z\zeta}$ in the ring R_S of rational functions in $\{T_i,\mathcal{A}_i\}_{i\in S}$.
- P is a docile power series in $\{y_i,a_i,x_i,\eta_i,\xi_i\}_{i\in S}$ with coefficients in R_S, and where $\deg(y_i,a_i,x_i,\eta_i,\xi_i)=(1,2,1,1,1)$.

Docililty Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$. !!!!!

- At $\epsilon^2=0$ we get the Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] invariant, which is stronger than HOMFLY-PT polynomial and Khovanov homology taken together!
- In general, get "higher diagonals in the Melvin-Morton-Rozansky expansion of the coloured Jones polynomial" [MM, BNG], but why spoil something good?

Solvable Approximation. In gl_n, half is enough! Indeed $gl_n\oplus\mathfrak{a}_n=\mathcal{D}(◹,b,\delta)$:



$b(◹)=b\colon ◹\otimes◹\to◹$
$b(◺)\rightsquigarrow\delta\colon ◹\to◹\otimes◹$

Now define $gl_n^\epsilon:=\mathcal{D}(◹,b,\epsilon\delta)$. Schematically, this is $[◹,◹]=◹$, $[◺,◺]=\epsilon◺$, and $[◹,◺]=◺+\epsilon◹$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1}=0$ always yields a solvable Lie algebra.

CU and *QU*. Starting from sl_2, get $CU_\epsilon=\langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ [x,y]=2\epsilon a-t)$. Quantize using standard tools (I'm sorry) and get $QU_\epsilon=\langle y,a,x,t\rangle/([t,-]=0,\ [a,y]=-y,\ [a,x]=x,\ xy-\mathbb{e}^{\hbar\epsilon}yx=(1-T\mathbb{e}^{-2\hbar\epsilon a})/\hbar)$.

References.

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[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function*, Commun. Math. Phys. **169** (1995) 501–520.

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[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.

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[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, arXiv:math/0201139.

Computation without Representation

The Simons Center for Geometry and Physics, May 2018

http://drorbn.net/sb18
modified 11 Ma'"iou 2018, 08:50

The Algebras H and H^*. Let $q = e^\epsilon$ and set $H = \langle a, x\rangle/([a,x] = x)$ with

$$A = e^{-\epsilon a}, \quad xA = qAx, \quad S_H(a, A, x) = (-a, A^{-1}, -A^{-1}x),$$
$$\Delta_H(a, A, x) = (a_1 + a_2, A_1A_2, x_1 + A_1x_2)$$

and dual $H^* = \langle b, y\rangle/([b,y] = -\epsilon y)$ with

$$B = e^{-b}, \quad By = qyB, \quad S_{H^*}(b, B, y) = (-b, B^{-1}, -yB^{-1}),$$
$$\Delta_{H^*}(b, B, y) = (b_1 + b_2, B_1B_2, y_1B_2 + y_2).$$

Pairing by $(a,x)^* = (b,y)$ ($\Rightarrow \langle B, A\rangle = q$) making $\langle y^lb^i, a^jx^k\rangle = \delta_{ij}\delta_{kl}j![k]_q!$ so $R = \sum \frac{y^kb^j\otimes a^jx^k}{j![k]_q!}$.

The Algebra QU. By the Drinfel'd double procedure, $QU = H^{*cop}\otimes H$ with $(\phi f)(\psi g) = \langle\psi_1 S^{-1}f_3\rangle\langle\psi_3, f_1\rangle(\phi\psi_2)(f_2g)$ and

$$S(y, b, a, x) = (-B^{-1}y, -b, -a, -A^{-1}x),$$
$$\Delta(y, b, a, x) = (y_1 + y_2B_1, b_1 + b_2, a_1 + a_2, x_1 + A_1x_2).$$

The 2D Lie Algebra. Clever people know* that if $[a,x] = \gamma x$ then $e^{\xi x}e^{\alpha a} = e^{\alpha a}e^{e^{-\gamma\alpha}\xi x}$. Ergo with

$$SW_{ax} \circlearrowright \mathcal{S}(a,x) \underset{\mathbb{O}_{xa}}{\overset{\mathbb{O}_{ax}}{\rightleftarrows}} \mathcal{U}(a,x)$$

we have ${}^tSW_{ax} = e^{\alpha a + e^{-\gamma\alpha}\xi x}$.

* Indeed $xa = (a-\gamma)x$ thus $xa^n = (a-\gamma)^n x$ thus $xe^{\alpha a} = e^{\alpha(a-\gamma)}x = e^{-\gamma\alpha}e^{\alpha a}x$ thus $x^ne^{\alpha a} = e^{\alpha a}(e^{-\gamma\alpha})^nx^n$ thus $e^{\xi x}e^{\alpha a} = e^{\alpha a}e^{e^{-\gamma\alpha}\xi x}$.

A Full Implementation at $\epsilon^2 = 0$.

```
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := ExpandDenominator@
   ExpandNumerator@
    Together[Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CF[x]];
```

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] :=
  𝔼[L1 + L2, Q1 + Q2, P1 * P2];
```

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};
(u_i_)* := (u*)_i;
```

```
expand[sd_SeriesData] := MapAt[expand, sd, 3];
expand[ℰ_] := Expand[ℰ];
Zip_{}[P_] := P;
Zip_{ζ_,ζs___}[P_] :=
 (expand[P // Zip_{ζs}] /. f_. ζ^d_. :→ ∂_{ζ*,d} f) /. ζ* → 0
```

```
QZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q2 = (Q1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   simp /@ 𝔼[L, Q2, Det[qt] e^(-Q2) Zip_ζs[e^Q1 (P /. zrule)]] ];
QZip_{ζs_List} := QZip_{ζs,CF};
```

```
U2l = {B_i_^p_. → e^(-p b_i), B^p_. → e^(-p b), T_i_^p_. → e^(p t_i), T^p_. → e^(p t),
   𝒜_i_^p_. → e^(p α_i), 𝒜^p_. → e^(p α)};
l2U = {e^(c_. b_i_ + d_.) :→ B_i^-c e^d, e^(c_. b + d_.) :→ B^-c e^d,
   e^(c_. t_i_ + d_.) :→ T_i^c e^d, e^(c_. t + d_.) :→ T^c e^d,
   e^(c_. α_i_ + d_.) :→ 𝒜_i^c e^d, e^(c_. α + d_.) :→ 𝒜^c e^d,
   e^ℰ_ :→ e^Expand@ℰ};
```

```
LZip_{ζs_List,simp_}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives @@ zs → 0;
   Q2 = (Q1 = Q /. U2l /. zrule) /. Alternatives @@ zs → 0;
   simp /@
    𝔼[L2, Q2, Det[lt] e^(-L2-Q2)
      Zip_ζs[e^(L1+Q1) (P /. U2l /. zrule)]] //. l2U ];
LZip_{ζs_List} := LZip_{ζs,CF};
```

```
Bind_{}[L_, R_] := L R;
Bind_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i},
       {i, {is}}],
     R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Flatten@Table[{β_{n@i}, τ_{n@i}, a_{n@i}}, {i, {is}}]} //
   QZip_{Flatten@Table[{ξ_{n@i}, y_{n@i}}, {i, {is}}]} ];
B_{l_List}[L_, R_] := Bind_l[L, R];
B_{is___}[L_, R_] := Bind_{is}[L, R];
```

```
am_{i_,j_->k_} := 𝔼[(α_i + α_j) a_k, (e^(-α_j) ξ_i + ξ_j) x_k, 1 + O[ϵ]^2]
aΔ_{i_->j_,k_} := 𝔼[α_i (a_j + a_k), ξ_i (x_j + x_k),
  1 + ϵ ξ_i x_k (-a_j + ξ_i x_j / 2) + O[ϵ]^2]
aS_i_ := 𝔼[-α_i a_i, - e^(α_i) ξ_i x_i,
  1 - ϵ e^(α_i) ξ_i x_i (a_i + e^(α_i) ξ_i x_i / 2) + O[ϵ]^2]
aSi_i_ := 𝔼[-α_i a_i, -e^(α_i) ξ_i x_i,
  1 - ϵ e^(α_i) ξ_i x_i (a_i - 1 + e^(α_i) ξ_i x_i / 2) + O[ϵ]^2]
```

```
bm_{i_,j_→k_} := 𝔼[(β_i + β_j) b_k, (η_i + η_j) y_k, 1 - ϵ η_j y_k β_i + O[ϵ]^2]
bΔ_{i_->j_,k_} := 𝔼[β_i (b_j + b_k), η_i (e^(-b_k) y_j + y_k),
  1 + ϵ η_i^2 y_j y_k e^(-b_k) / 2 + O[ϵ]^2]
bS_i_ := 𝔼[-β_i b_i, -e^(b_i) η_i y_i,
  1 - ϵ e^(b_i) η_i y_i (β_i + e^(b_i) η_i y_i / 2) + O[ϵ]^2]
bSi_i_ := 𝔼[-β_i b_i, -e^(b_i) η_i y_i,
  1 - ϵ e^(b_i) η_i y_i (β_i - 1 + e^(b_i) η_i y_i / 2) + O[ϵ]^2]
```

```
tP_{i_,j_} := 𝔼[β_i α_j, η_i ξ_j, 1 + ϵ η_i^2 ξ_j^2 / 4]
```

```
Block[{i, j, k},
 dm_{i_,j_→k_} =
  (𝔼[β_i b_i + α_j a_j, η_i y_i + ξ_j x_j, 1] (aΔ_{i→h1,h2} ~B_h2~ aΔ_{h2→h2,h3})
     (bΔ_{j→t1,t2} ~B_t2~ bΔ_{t2→t2,t3})) ~B_h3~ aSi_h3 ~B_{t1,h3}~
   (tP_{t1,h3}) ~B_{t3,h1}~ (tP_{t3,h1}) ~B_{h2,j,i,t2}~ (am_{h2,j→k} bm_{i,t2→k})]
```

$$\mathbb{E}\Big[a_k\alpha_i+a_k\alpha_j+b_k\beta_i+b_k\beta_j,\ \frac{1}{\mathcal{A}_i\mathcal{A}_j}(y_k\mathcal{A}_i\mathcal{A}_j\eta_i+y_k\mathcal{A}_j\eta_j+x_k\mathcal{A}_i\xi_i+\mathcal{A}_i\mathcal{A}_j\eta_j\xi_i-B_k\mathcal{A}_i\mathcal{A}_j\eta_j\xi_i+x_k\mathcal{A}_i\mathcal{A}_j\xi_j),\ 1+\frac{1}{4\mathcal{A}_i\mathcal{A}_j}\big(-4y_k\mathcal{A}_j\beta_i\eta_j-4x_k\mathcal{A}_i\beta_j\xi_i+4x_ky_k\eta_j\xi_i+4a_kB_k\mathcal{A}_i\mathcal{A}_j\eta_j\xi_i+2y_k\mathcal{A}_j\eta_j^2\xi_i-6B_ky_k\mathcal{A}_j\eta_j^2\xi_i+2x_k\mathcal{A}_i\eta_j\xi_i^2-6B_kx_k\mathcal{A}_i\eta_j\xi_i^2+\mathcal{A}_i\mathcal{A}_j\eta_j^2\xi_i^2-4B_k\mathcal{A}_i\mathcal{A}_j\eta_j^2\xi_i^2+3B_k^2\mathcal{A}_i\mathcal{A}_j\eta_j^2\xi_i^2\big)\epsilon+O[\epsilon]^2\Big]$$

$$\texttt{Block}[\{i\},\ dS_{i\_}=\mathbb{E}[\beta_i b_1+\alpha_i a_2,\ \eta_i y_1+\xi_i x_2,\ 1]\sim B_{1,2}\sim(bSi_1\,aS_2)\sim B_{1,2}\sim dm_{2,1\to i}]$$

$$\mathbb{E}\Big[-a_i\alpha_i-b_i\beta_i,\ \frac{-y_i\mathcal{A}_i\eta_i-B_ix_i\mathcal{A}_i\xi_i+\mathcal{A}_i\eta_i\xi_i-B_i\mathcal{A}_i\eta_i\xi_i}{B_i},\ 1+\frac{1}{4B_i^2}\big(4B_iy_i\mathcal{A}_i\eta_i-4B_iy_i\mathcal{A}_i\beta_i\eta_i-2y_i^2\mathcal{A}_i^2\eta_i^2-4a_iB_i^2x_i\mathcal{A}_i\xi_i-4B_i^2x_i\mathcal{A}_i\beta_i\xi_i-4B_i\mathcal{A}_i\eta_i\xi_i+4a_iB_i\mathcal{A}_i\eta_i\xi_i+4B_i^2\mathcal{A}_i\eta_i\xi_i-4B_ix_iy_i\mathcal{A}_i^2\eta_i\xi_i+4B_i\mathcal{A}_i\beta_i\eta_i\xi_i-4B_i^2\mathcal{A}_i\beta_i\eta_i\xi_i+6y_i\mathcal{A}_i^2\eta_i^2\xi_i-2B_iy_i\mathcal{A}_i^2\eta_i^2\xi_i-2B_i^2x_i^2\mathcal{A}_i^2\xi_i^2+6B_ix_i\mathcal{A}_i^2\eta_i\xi_i^2-2B_i^2x_i\mathcal{A}_i^2\eta_i\xi_i^2-3\mathcal{A}_i^2\eta_i^2\xi_i^2+4B_i\mathcal{A}_i^2\eta_i^2\xi_i^2-B_i^2\mathcal{A}_i^2\eta_i^2\xi_i^2\big)\epsilon+O[\epsilon]^2\Big]$$

$$\texttt{Block}[\{i,j,k\},\ d\Delta_{i\_\to j\_,k\_}=(b\Delta_{i\to3,1}\,a\Delta_{i\to2,4})\sim B_{1,2,3,4}\sim(dm_{3,4\to k}\,dm_{1,2\to j})]$$

$$\mathbb{E}\Big[a_j\alpha_i+a_k\alpha_i+b_j\beta_i+b_k\beta_i,\ y_j\eta_i+B_jy_k\eta_i+x_j\xi_i+x_k\xi_i,\ 1+\frac{1}{2}\big(B_jy_jy_k\eta_i^2-2a_jx_k\xi_i+x_jx_k\xi_i^2\big)\epsilon+O[\epsilon]^2\Big]$$

$$R_{i\_,j\_}:=\mathbb{E}\big[b_i a_j,\ y_i x_j,\ 1-\epsilon y_i^2x_j^2/4+O[\epsilon]^2\big]$$

$$\texttt{Block}\big[\{i,j\},\ \overline{R}_{i\_,j\_}=\texttt{Expand}\,/@\,R_{i,j}\sim B_j\sim dS_j\big]$$

$$\mathbb{E}\Big[-a_jb_i,\ -\frac{x_jy_i}{B_i},\ 1+\frac{(-4a_jB_ix_jy_i-3x_j^2y_i^2)\epsilon}{4B_i^2}+O[\epsilon]^2\Big]$$

$$\texttt{Block}[\{i\},\ \{u_{i\_}=R_{1,2}\sim B_1\sim dS_1\sim B_{1,2}\sim dm_{2,1\to i},\ ui_{i\_}:=R_{1,2}\sim B_2\sim dS_2\sim B_2\sim dS_2\sim B_{1,2}\sim dm_{2,1\to i}\}]$$

$$\Big\{\mathbb{E}\Big[-a_ib_i,\ -\frac{x_iy_i}{B_i},\ B_i+\frac{(-4a_iB_i^2-4B_ix_iy_i-4a_iB_ix_iy_i-3x_i^2y_i^2)\epsilon}{4B_i}+O[\epsilon]^2\Big],\ \texttt{Null}\Big\}$$

$$\texttt{Block}\big[\{i\},\ \{CC_{i\_}=\mathbb{E}[0,0,B_i^{1/2}e^{-\epsilon a_i/2}+O[\epsilon]^2],\ \overline{CC}_{i\_}=\mathbb{E}[0,0,B_i^{-1/2}e^{\epsilon a_i/2}+O[\epsilon]^2]\}\big]$$

$$\Big\{\mathbb{E}\Big[0,0,\sqrt{B_i}-\frac{1}{2}(a_i\sqrt{B_i})\epsilon+O[\epsilon]^2\Big],\ \mathbb{E}\Big[0,0,\frac{1}{\sqrt{B_i}}+\frac{a_i\epsilon}{2\sqrt{B_i}}+O[\epsilon]^2\Big]\Big\}$$

$$\texttt{Block}\big[\{i,j\},\ \{\texttt{Kink}_{i\_}=(R_{1,3}\,\overline{CC}_2)\sim B_{1,2}\sim dm_{1,2\to1}\sim B_{1,3}\sim dm_{1,3\to i},\ \overline{\texttt{Kink}}_{j\_}=(\overline{R}_{1,3}\,CC_2)\sim B_{1,2}\sim dm_{1,2\to1}\sim B_{1,3}\sim dm_{1,3\to j}\}\big]$$

$$\Big\{\mathbb{E}\Big[a_ib_i,\ x_iy_i,\ \frac{1}{\sqrt{B_i}}+\frac{(2a_i-x_i^2y_i^2)\epsilon}{4\sqrt{B_i}}+O[\epsilon]^2\Big],\ \mathbb{E}\Big[-a_jb_j,\ -\frac{x_jy_j}{B_j},\ \sqrt{B_j}+\frac{(-2a_jB_j^2-4a_jB_jx_jy_j-3x_j^2y_j^2)\epsilon}{4B_j^{3/2}}+O[\epsilon]^2\Big]\Big\}$$

$$Z=R_{1,5}\,R_{6,2}\,R_{3,7}\,\overline{CC}_4\,\overline{\texttt{Kink}}_8\,\overline{\texttt{Kink}}_9\,\overline{\texttt{Kink}}_{10};$$
$$\texttt{Do}[Z=Z\sim B_{1,r}\sim dm_{1,r\to1},\ \{r,2,10\}];$$
$$\texttt{Simplify}\,/@\,Z$$

$$\mathbb{E}\Big[0,0,\ \frac{B_1}{1-B_1+B_1^2}+\frac{1}{(1-B_1+B_1^2)^3}B_1\big(-B_1+2B_1^2+2B_1^4+a_1(-1+B_1-B_1^3+B_1^4)-2x_1y_1-B_1^3(3+2x_1y_1)\big)\epsilon+O[\epsilon]^2\Big]$$

$$b2t_{i\_}:=\mathbb{E}\big[\alpha_i a_i-\beta_i t_i,\ \xi_i x_i+\eta_i y_i,\ 1+\epsilon\beta_i a_i+O[\epsilon]^2\big]$$
$$t2b_{i\_}:=\mathbb{E}\big[\alpha_i a_i-\tau_i b_i,\ \xi_i x_i+\eta_i y_i,\ 1+\epsilon\tau_i a_i+O[\epsilon]^2\big]$$

Faddeev's Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n-1}{q-1}$, with $[n]_q! := [1]_q[2]_q\cdots[n]_q$ and with $\mathbb{e}_q^x := \sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we have

$$\log\mathbb{e}_q^x=\sum_{k\geq1}\frac{(1-q)^kx^k}{k(1-q^k)}=x+\frac{(1-q)^2x^2}{2(1-q^2)}+\dots.$$

Proof. We have that $\mathbb{e}_q^x=\frac{\mathbb{e}_q^{qx}-\mathbb{e}_q^x}{qx-x}$ ("the q-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx}=(1+(1-q)x)\mathbb{e}_q^x$, and

$$\log\mathbb{e}_q^{qx}=\log(1+(1-q)x)+\log\mathbb{e}_q^x.$$

Writing $\log\mathbb{e}_q^x=\sum_{k\geq1}a_kx^k$ and comparing powers of x, we get $q^ka_k=-(1-q)^k/k+a_k$, or $a_k=\frac{(1-q)^k}{k(1-q^k)}$. □

A Partial To Do List.

- Complete all "docility" arguments by identifying a "contained" docile substructure.
- Understand denominators and get rid of them.
- See if much can be gained by including P in the exponential: $\mathbb{e}^{L+Q}P\rightsquigarrow\mathbb{e}^{L+Q+P}$?
- Clean the program and make it efficient.
- Run it for all small knots and links, at $k=2,3$.
- Understand the centre and figure out how to read the output.
- Extend to sl_3 and beyond.
- Prove a genus bound and a Seifert formula.
- Obtain "Gauss-Gassner formulas" (ωεβ/NCSU).
- Relate with Melvin-Morton-Rozansky and with Rozansky-Overbay.
- Understand the braid group representations that arise.
- Find a topological interpretation. The Garoufalidis-Rozansky "loop expansion" [GR]?
- Figure out the action of the Cartan automorphism.
- Disprove the ribbon-slice conjecture!
- Figure out the action of the Weyl group.
- Do everything at the "arrow diagram" level of finite-type invariants of (rotational) virtual tangles.
- What else can you do with the "solvable approximations"?
- And with the "Gaussian zip and bind" technology?

Further References.

[GR] S. Garoufalidis and L. Rozansky, *The Loop Exapnsion of the Kontsevich Integral, the Null-Move, and S-Equivalence*, arXiv:math.GT/0003187.

[Fa] L. Faddeev, *Modular Double of a Quantum Group*, arXiv:math/9912078.

[Qu] C. Quesne, *Jackson's q-Exponential as the Exponential of a Series*, arXiv:math-ph/0305003.

[Za] D. Zagier, *The Dilogarithm Function*, in Cartier, Moussa, Julia, and Vanhove (eds) *Frontiers in Number Theory, Physics, and Geometry II*. Springer, Berlin, Heidelberg, and ωεβ/Za.