BonnByWebb second talk post-mortem

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Main issue: The talk is "backwards", and hence should have started with a "forwards" preview as in 1,2,3

Dror Bar-Natan: Talks: StonyBrook-1805:

Computation without Representation

Thanks for inviting me to the SCGP! ωεβ:=http://drorbn.net/sb18/

Follows Rozansky [Ro1, Ro2, Ro3] and Overbay [Ov], joint with van der Veen. More at [BV] and at ωεβ/talks.

Abstract. A major part of "quantum topology" is the definition and computation of various knot invariants by carrying out computations in quantum groups. Traditionally these computations are carried out "in a representation", but this is very slow: one has to use tensor powers of these representations, and the dimensions of powers grow exponentially fast. I will describe a direct-participation method for carrying out these computations without having to choose a representation and explain why in many ways the results are better and faster. The two key points we use are a technique for composing infinite-order "perturbed Gaussian" differential operators, and the little-known fact that every semi-simple Lie algebra can be approximated by solvable Lie algebras, where computations are easier.

A Knot Theory Portfolio.
- Has operations $\sqcup$, m, Δ, S.
- All tangles are generated by $R^{\pm 1}$ and $C^{\pm}$ (so "easy" to produce invariants).
- Makes some properties ("genus", "ribbon") be "definable".

(more to say, but not now).

A "Quantum Group" Portfolio consists of an algebra U along with maps (and some axioms...)

PBW Bases. The U's we care about always have "Poincaré-Birkhoff-Witt" bases; there is some finite set B of "generators" and isomorphisms $\mathbb{O}_{y,x,\dots}\colon \hat{\mathcal{S}}(B)\to U$ defined by "ordering monomials" to some fixed $y,x,\dots$ order. The quantum group portfolio now becomes a "symmetric algebra" portfolio, or a "power series" portfolio.

The (Semi-)Category $\mathcal{DO}$. Hence we care about the monoidal (semi-)category $\mathcal{DO}$ whose objects are finite sets B and whose morphisms are $\mathrm{mor}_{\mathcal{DO}}(B,B') := \mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B)\to\mathcal{S}(B')) = \mathcal{S}(B^*,B')$ (by convention, $x^*=\xi$, $y^*=\eta$, etc.).

The Composition Law. If $\mathcal{S}(B_0)\xrightarrow{f}\mathcal{S}(B_1)\xrightarrow{g}\mathcal{S}(B_2)$, then $'(f/\!\!/g) = '(g\circ f) = \left(g|_{\zeta_i\to\partial_{z_i}} f\right)_{z_i=0} = \left(f|_{z_i\to\partial_{\zeta_i}} g\right)_{\zeta_i=0}$.

Note. For $f\in\mathcal{S}(z)$ and $g\in\mathcal{S}(\zeta)$, $\langle f,g\rangle = f(\partial_\zeta)g|_{\zeta=0} = g(\partial_z)f|_{z=0}$.

Proposition. If $F\colon\mathcal{S}(B)\to\mathcal{S}(B')$ is linear and "continuous", then $'F = \exp\left(\sum_{z_i\in B}\zeta_i z_i\right)/\!\!/F$.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

Knotted Candies by Dror Bar-Natan (2008) based on data by Brian Gilbert http://drorbn.net/ap/2008-12/

Examples.
1. The 1-variable identity map $I\colon\mathcal{S}(z)\to\mathcal{S}(z)$ is given by $'I_1 = e^{z\zeta}$ and the n-variable one by $'I_n = e^{z_1\zeta_1+\cdots+z_n\zeta_n}$.
2. The "$z_i\to z_j$ variable rename map $\sigma^i_j\colon\mathcal{S}(z_i)\to\mathcal{S}(z_j)$" becomes $'\sigma^i_j = e^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m^{ij}_k\colon\mathcal{S}(z_i,z_j)\to\mathcal{S}(z_k)$" has $'m = e^{z_k(\zeta_i+\zeta_j)}$.
4. The "archetypal coproduct $\Delta^i_{jk}\colon\mathcal{S}(z_i)\to\mathcal{S}(z_j,z_k)$", given by $z_i\to z_j+z_k$ or $\Delta z = z\otimes 1+1\otimes z$, has $'\Delta = e^{(z_j+z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $e_q^{\hbar y_1x_2}\in\mathcal{U}_q\otimes\mathcal{U}_q$. The "baby R-matrix" is $'R = e^{\hbar yx}\in\mathcal{S}(y,x)$.
6. The "Weyl form of the canonical commutation relations" states that if $[y,x]=t$ is a scalar, then $e^{\xi x}e^{\eta y} = e^{\eta y}e^{\xi x}e^{-\eta\xi t}$. Thus with $\mathcal{S}(y,x)\xrightarrow{\mathbb{O}_{xy}}\mathcal{U}(y,x)$ we have $'SW_{xy} = e^{\eta y+\xi x-\eta\xi t}$.

The Zipping Issue. (between unbound and bound lies half-zipped).

Zipping. If $P(\zeta^j,z_i)$ is a polynomial, or whenever otherwise convergent, set
$$\left\langle P(\zeta^j,z_i)\right\rangle_{(\zeta^j)} = P\left(\partial_{z_j},z_i\right)\Big|_{z_i=0}.$$
(E.g., if $P=\sum a_{nm}\zeta^n z^m$ then $\langle P\rangle_\zeta = \sum n!a_{nn}$).

Implementation. ωεβ/Zip

```
z* = ζ; ζ* = z; Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] :=
 (Expand[P // Zip{ζs}] /. f_. ζ^d_. :> ∂{ζ*,d}f) /. ζ* → 0
{Zip{ζ}[ζ² e^{δ z²}], Zip{ζ}[ζ⁴ e^{δ z²}]}
{2 δ, 12 δ²}
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must make this part more understandable.

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i,\zeta^j)e^{\eta^i z_i + y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i+y_i,\zeta^j)e^{\eta^i(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i + y_j\zeta^j + q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde{q})\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i}\Big|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde{q}$ is the inverse matrix of $1-q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar = 0$ and at $\partial_\hbar$).

Exponential Reservoirs.. The true Hilbert hotel is exp! Remove one x from an "exponential reservoir" of x's and you are left with the same exponential reservoir:

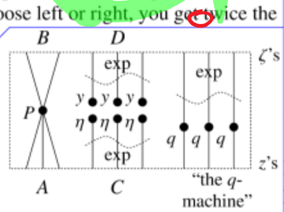
$$e^x = \left[\ldots + \frac{xxxxx}{120} + \ldots\right] \xrightarrow{\partial_x} \left[\ldots + \frac{xxxxx}{120} + \ldots\right] = (e^x)' = e^x,$$

Remove?

and if you let each element choose left or right, you get twice the same reservoir:

$$e^x \xrightarrow{x\to x_l+x_r} e^{x_l+x_r} = e^{x_l}e^{x_r}.$$

A Graphical Proof. Glue top to bottom on the right, in all possible ways. Several scenarios occur:



1. Start at A, go through the q-machine $k \ge 0$ times, stop at B. Get $\left\langle P\left(\sum_{k\ge 0} q^k z, \zeta\right)\right\rangle = \langle P(\tilde{q}z,\zeta)\rangle$.
2. Loop through the q-machine and swallow your own tail. Get $\exp\left(\sum q^k/k\right) = \exp(-\log(1-q)) = \tilde{q}$.
3. …

By the reservoir splitting principle, these scenarios contribute multiplicatively. □

Implementation. ωεβ/Zip

```
𝔼 /: Zip_{ζs_List}@𝔼[Q_, P_] :=      (* 𝔼[Q,P] means e^Q P *)
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule; Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[Q2, Det[qt] e^{-Q2} Zip_{ζs}[e^{Q1} (P /. zrule)]] ];
```

Fuller program & testing suite: ωεβ/mm, ωεβ/port.

The Real Thing. In the algebra QU_ϵ (explained later), over $\mathbb{Q}[[\hbar]]$ using the *yaxt* order, $T = e^{\hbar t}$, $\bar{T} = T^{-1}$, $\mathcal{A} = e^{\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

$${}^tR_{ij} = e^{\hbar(y_ix_j - t_ia_j)}\left(1 + \epsilon\hbar\left(a_ia_j - \hbar^2y_i^2x_j^2/4\right) + O(\epsilon^2)\right)$$

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$${}^tm = e^{(\alpha_1+\alpha_2)a + \eta_2\xi_1(1-T)/\hbar + (\xi_1\bar{\mathcal{A}}_2+\xi_2)x + (\eta_1+\eta_2\bar{\mathcal{A}}_1)y}\left(1 + \epsilon\lambda + O(\epsilon^2)\right),$$

where $\lambda = 2a\eta_2\xi_1T + \eta_2^2\xi_1^2\left(3T^2 - 4T + 1\right)/4\hbar - \eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2/2$
$-\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1/2 + \eta_2\xi_1xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$.

Finally,

$${}^t\Delta = e^{\tau(t_1+t_1)+\eta(y_1+T_1y_2)+\alpha(a_1+a_2)+\xi(x_1+x_2)}(1 + O(\epsilon)) \in \mathcal{S}(B^*, B_1, B_2),$$

and $${}^tS = e^{-\tau t - \alpha a - \eta\xi(1-\bar{T})\mathcal{A}/\hbar - \bar{T}\eta y\mathcal{A} - \xi x\mathcal{A}}(1 + O(\epsilon)) \in \mathcal{S}(B^*, B).$$

Real Zipping is a minor mess, and is done in two phases:

	τa-phase		ξy-phase	
ζ-like variables	τ	a	ξ	y
z-like variables	t	α	x	η

Already at $\epsilon = 0$ we get the best known formulas for the Alexander polynomial!

Generic Docility. A "docile perturbed Gaussian" in the variables $(z_i)_{i\in S}$ over the ring R is an expression of the form

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$$e^{q^{ij}z_iz_j}P = e^{q^{ij}z_iz_j}\left(\sum_{k\ge 0}\epsilon^kP_k\right),$$

where all coefficients are in R and where P is a "docile series": $\deg P_k \le 4k$.

Our Docility. In the case of QU_ϵ, all invariants and operations are of the form $e^{L+Q}P$, where

- L is a quadratic of the form $\sum l_{z\zeta}z\zeta$, where z runs over $\{t_i, \alpha_i\}_{i\in S}$ and ζ over $\{\tau_i, a_i\}_{i\in S}$, with integer coefficients $l_{z\zeta}$.
- Q is a quadratic for the form $\sum q_{z\zeta}z\zeta$, where z runs over $\{x_i, \eta_i\}_{i\in S}$ and ζ over $\{\xi_i, y_i\}_{i\in S}$, with coefficients $q_{z\zeta}$ in the ring R_S of rational functions in $\{T_i, \mathcal{A}_i\}_{i\in S}$.
- P is a docile power series in $\{y_i, a_i, x_i, \eta_i, \xi_i\}_{i\in S}$ with coefficients in R_S, and where $\deg(y_i, a_i, x_i, \eta_i, \xi_i) = (1, 2, 1, 1, 1)$.

Docility Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$. !!!!!

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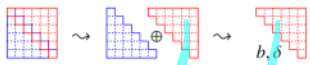
- At $\epsilon^2 = 0$ we get the Rozansky-Overbay [Ro1, Ro2, Ro3, Ov] invariant, which is stronger than HOMFLY-PT polynomial and Khovanov homology taken together!
- In general, get "higher diagonals in the Melvin-Morton-Rozansky expansion of the coloured Jones polynomial" [MM, BNG], but why spoil something good?

Solvable Approximation. In gl_n, half is enough! Indeed $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:



$b(◹) = b: ◹\otimes◹ \to ◹$
$b(◺) \rightsquigarrow \delta: ◹ \to ◹\otimes◹$

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Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon◺$, and $[◹, ◺] = ◺ + \epsilon◹$. The same process works for all semi-simple Lie algebras, and at $\epsilon^{k+1} = 0$ always yields a solvable Lie algebra.

CU and *QU*. Starting from sl_2, get $CU_\epsilon = \langle y, a, x, t\rangle/([t,-] = 0, [a,y] = -y, [a,x] = x, [x,y] = 2\epsilon a - t)$. Quantize using standard tools (I'm sorry) and get $QU_\epsilon = \langle y, a, x, t\rangle/([t,-] = 0, [a,y] = -y, [a,x] = x, xy - e^{\hbar\epsilon}yx = (1 - Te^{-2\hbar\epsilon a})/\hbar)$.

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