

Implementation (see IType.nb of $\omega \epsilon \beta/ap$).

© Once[<< KnotTheory`; << Rot.m];</pre>

- Loading KnotTheory` version of February 2, 2020, 10:53:45.2097. Read more at http://katlas.org/wiki/KnotTheory.
- Loading Rot.m from http://drorbn.net/AP/Talks/Beijing-2407 to compute rotation numbers.

 $\widehat{\odot} \operatorname{CF}[\omega_{-} \mathcal{S}_{-}\mathbb{E}] := \operatorname{CF}[\omega] \times \operatorname{CF}/\mathbb{Q}\mathcal{S}; \\ \operatorname{CF}[\mathcal{S}_{-}List] := \operatorname{CF}/\mathbb{Q}\mathcal{S}; \\ \operatorname{CF}[\mathcal{S}_{-}] := \operatorname{Module}[\{\operatorname{vs}, \operatorname{ps}, \operatorname{c}\}, \\ \operatorname{vs} = \operatorname{Cases}[\mathcal{S}, (\operatorname{x} \mid \operatorname{p} \mid \mathcal{S} \mid \pi \mid \operatorname{g})_{-}, \infty] \cup \{\operatorname{e}\}; \\ \operatorname{Total}[\operatorname{CoefficientRules}[\operatorname{Expand}[\mathcal{S}], \operatorname{vs}] /. \\ (\operatorname{ps}_{-} \rightarrow \operatorname{c}_{-}) : \Rightarrow \operatorname{Factor}[c] (\operatorname{Times} \mathbb{Q} \operatorname{vs}^{\operatorname{ps}})]];$

- Integration using Picard iteration.
- \$ π = Identity; (* Hacks in pink *)

③Unprotect[Integrate]; (* Core in yellow *)

 ω_{-} . $\mathbb{E}[L_{-}] d(vs_{-}List) :=$

Module [{n, L0, Q,
$$\triangle$$
, G, Z0, Z, λ , DZ, DDZ, FZ,
a, b},
n = Length@vs; L0 = L /. $\epsilon \rightarrow 0$;
Q = Table [($-\partial_{vs[a],vs[b]}$ L0) /. Thread [$vs \rightarrow 0$] /
(p | x) __ $\rightarrow 0$, {a, n}, {b, n}];
If[(\triangle = Det[0]) == 0, Return@"Degenerate 0!"

 $Z = ZO = CF@\frac{\pi}{L} + vs.Q.vs/2]; G = Inverse[Q];$

FixedPoint (DZ = Table[∂vZ, {v, vs}];

 $DDZ = Table[\partial_u DZ, \{u, vs\}];$ $FZ = Sum[G[[a, b]] (DDZ[[a, b]] + DZ[[a]] \times DZ[[b]]),$ $\{a, n\}, \{b, n\}] / 2;$ $Z = CF[Z0 + \left[\lambda f \pi FZ \right] d[\lambda] \right] \&, Z];$

PowerExpand@Factor $\left[\omega \Delta^{-1/2}\right] \times$

$$\mathbb{E}\left[\mathsf{CF}\left[\frac{\mathbf{Z}}{\mathbf{A}} \rightarrow \mathbf{1}\right], \mathsf{Thread}\left[\frac{\mathbf{vs}}{\mathbf{vs}} \rightarrow \mathbf{0}\right]\right];$$

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Protect[Integrate];
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 $\sqrt{\mu}$

$$\stackrel{(:)}{=} \int \mathbb{E}\left[-\frac{\mu x^{2}}{2 \mu x^{2}}\right] d\{x\}$$

$$\stackrel{\Box}{=} \frac{\mathbb{E}\left[-\frac{\xi^{2}}{2 \mu}\right]}{\sqrt{\mu}}$$

$$\stackrel{(:)}{=} \mathbf{FofG} = \int \mathbb{E}\left[-\mu (x-a)^{2}/2 + i \xi x\right] d\{x\}$$

$$\stackrel{\Box}{=} \mathbb{E}\left[\frac{i (2a\mu + i \xi) \xi}{2\mu}\right]$$





So we've tested and nearly proven the Fourier inversion formula!

$$\stackrel{()}{=} L = -\frac{1}{2} \{x_1, x_2\} \cdot {\binom{a \ b}{b \ c}} \cdot \{x_1, x_2\} + \{\xi_1, \xi_2\} \cdot \{x_1, x_2\};$$

$$\frac{Z12 = \int \mathbb{E} [L] \ d\{x_1, x_2\}}{\frac{E \left[\frac{c \ \xi_1^2}{2 \ (-b^2 + a \ c)} + \frac{b \ \xi_1 \ \xi_2}{b^2 - a \ c} + \frac{a \ \xi_2^2}{2 \ (-b^2 + a \ c)}\right]}{\sqrt{-b^2 + a \ c}}$$

$$\stackrel{(:)}{=} \left\{ \mathbf{Z1} = \int \mathbb{E} [L] d\{\mathbf{x}_{1}\}, \ \mathbf{Z12} = \int \mathbf{Z1} d\{\mathbf{x}_{2}\} \right\} \\ \frac{\Box}{\left\{ \frac{\mathbb{E} \left[-\frac{(-b^{2}+a\,c)\,\mathbf{x}_{2}^{2}}{2\,a} - \frac{b\,\mathbf{x}_{2}\,\xi_{1}}{a} + \frac{\xi_{1}^{2}}{2\,a} + \mathbf{x}_{2}\,\xi_{2} \right]}{\sqrt{a}}, \ \mathsf{True} \right\}$$



A226260 Numerators of mass formula for connected vacuum graphs on 2n nodes for a phi^3 field theory. 1, 5, 5, 1105, 565, 82825, 19675, 1282031525, 80727925, 1683480621875, 13209845125, 2239646759308375, 19739117098375, 6320791709083309375, 32468078556378125, 38362676768845045751875, 281365778455032973125, 2824650747089425586152484375, 776632157034116712734375 (list; graph; refs; listen; history; text: internal format)

The Right-Handed Trefoil.

©K = Mirror@Knot[3, 1]; Features[K]

$$\Box \text{Features}[7, C_4[-1] X_{1,5}[1] X_{3,7}[1] X_{6,2}[1]]$$

$$\odot \mathcal{L}[X_{i_{-},j_{-}}[S_{-}]] := T^{S/2} \mathbb{E}[$$

$$x_i (p_{i+1} - p_i) + x_j (p_{j+1} - p_j) + (T^S - 1) x_i (p_{i+1} - p_{j+1}) + (\varepsilon S/2) \times (X_i (p_i - p_j) ((T^S - 1) x_i p_j + 2 (1 - x_j p_j)) - 1))$$

$$\mathcal{L}[C_{i_{-}}[\varphi_{-}]] := T^{\varphi/2} \mathbb{E} \Big[x_i (p_{i+1} - p_i) + \varepsilon \varphi \left(\frac{1}{2} - x_i p_i\right) \Big]$$

$$\mathcal{L}[K_{-}] := CF[\mathcal{L}/@ \text{Features}[K][2]]$$

$$vs[K_{-}] := Join @@ Table[\{p_i, x_i\}, \{i, \text{Features}[K][1]]\}]$$

Joseph Fourier

© {vs[K], ⊥[K]}

$$\begin{array}{l} \square \left\{ \left\{ p_{1}, x_{1}, p_{2}, x_{2}, p_{3}, x_{3}, p_{4}, x_{4}, p_{5}, x_{5}, p_{6}, x_{6}, p_{7}, x_{7} \right\}, \\ T \mathbb{E} \left[-2 \in -p_{1} x_{1} + \epsilon p_{1} x_{1} + T p_{2} x_{1} - \epsilon p_{5} x_{1} + (1 - T) p_{6} x_{1} + \\ \frac{1}{2} (-1 + T) \in p_{1} p_{5} x_{1}^{2} + \frac{1}{2} (1 - T) \in p_{5}^{2} x_{1}^{2} - p_{2} x_{2} + p_{3} x_{2} - p_{3} x_{3} + \\ e p_{3} x_{3} + T p_{4} x_{3} - \epsilon p_{7} x_{3} + (1 - T) p_{8} x_{3} + \frac{1}{2} (-1 + T) \epsilon p_{3} p_{7} x_{3}^{2} + \\ \frac{1}{2} (1 - T) \epsilon p_{7}^{2} x_{3}^{2} - p_{4} x_{4} + \epsilon p_{4} x_{4} + p_{5} x_{4} - p_{5} x_{5} + p_{6} x_{5} - \\ e p_{1} p_{5} x_{1} x_{5} + \epsilon p_{5}^{2} x_{1} x_{5} - \epsilon p_{2} x_{6} + (1 - T) p_{3} x_{6} - p_{6} x_{6} + \\ e p_{6} x_{6} + T p_{7} x_{6} + \epsilon p_{2}^{2} x_{2} x_{6} - \epsilon p_{2} p_{6} x_{2} x_{6} + \frac{1}{2} (1 - T) \epsilon p_{7}^{2} x_{6}^{2} + \\ \frac{1}{2} (-1 + T) \epsilon p_{2} p_{6} x_{6}^{2} - p_{7} x_{7} + p_{8} x_{7} - \epsilon p_{3} p_{7} x_{3} x_{7} + \epsilon p_{7}^{2} x_{3} x_{7} \right] \right\} \\ \stackrel{(\bigcirc}{} \$\pi = Normal \left[\# + 0 \left[\epsilon \right]^{2} \right] \$; \int \mathcal{L} [K] dl vs [K] \\ \stackrel{\square}{=} \underbrace{i T \mathbb{E} \left[-\frac{(-1+T)^{2} (1+T^{2}) \epsilon}{(1-T+T^{2})^{2}} \right]} \end{array}$$

$$1 - T + T^2$$

A faster program to compute ρ_1 , and more stories about it, are at [BV2].

Invariance Under Reidemeister 3.



<u>□</u>False

Invariance Under Reidemeister 3, Take 2.

$$\hat{\mathbb{C}} lhs = \int (\mathcal{L} / @ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1])) d \{x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\}; rhs = \int (\mathcal{L} / @ (X_{j,k}[1] X_{i,k+1}[1] X_{i+1,j+1}[1])) d \{x_i, x_j, x_k, x_{i+1}, p_{i+1}, p_{j+1}, p_{k+1}, x_{j+1}, x_{k+1}\}; lhs === rhs$$

□Degenerate Q!

Invariance Under Reidemeister 3, Take 3.

 $(::) lhs = \int (\mathbb{E} [i \pi_i p_i + i \pi_j p_j + i \pi_k p_k] \times \mathcal{L} / @ (X_{i,j}[1] X_{i+1,k}[1] X_{j+1,k+1}[1]))$ $d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$ $rhs = \left(\left(\mathbb{E} \left[i \pi_{i} p_{i} + i \pi_{j} p_{j} + i \pi_{k} p_{k} \right] \times \mathcal{L} / \mathcal{Q} \left(X_{j,k} [1] X_{i,k+1} [1] X_{i+1,j+1} [1] \right) \right) \right)$ $d\{p_i, p_j, p_k, x_i, x_j, x_k, p_{i+1}, p_{j+1}, p_{k+1}, x_{i+1}, x_{j+1}, x_{k+1}\};$ lhs == rhs True) **1hs** <u>−</u>T^{3/2} ℝ $-\frac{3\,\varepsilon}{2}\,+\,\, \mathrm{i}\,\, T^2\,\, p_{2+1}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, T\,\, p_{2+j}\,\, \pi_1\,+\,\, \mathrm{i}\,\, T^2\, \varepsilon\,\, p_{2+j}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, p_{2+k}\,\, \pi_1\,+\, \mathrm{i}\,\, T^2\, \varepsilon\,\, p_{2+j}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, p_{2+k}\,\, \pi_1\,+\, \mathrm{i}\,\, T^2\, \varepsilon\,\, p_{2+j}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, p_{2+k}\,\, \pi_1\,+\, \mathrm{i}\,\, T^2\,\, \varepsilon\,\, p_{2+j}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, p_{2+k}\,\, \pi_1\,+\, \mathrm{i}\,\, T^2\,\, \varepsilon\,\, p_{2+j}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, p_{2+k}\,\, \pi_1\,+\, \mathrm{i}\,\, T^2\,\, \varepsilon\,\, p_{2+j}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, p_{2+k}\,\, \pi_1\,+\, \mathrm{i}\,\, T^2\,\, \varepsilon\,\, p_{2+j}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, p_{2+k}\,\, \pi_1\,+\, \mathrm{i}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, p_{2+k}\,\, \pi_1\,+\, \mathrm{i}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, p_{2+k}\,\, \pi_1\,+\, \mathrm{i}\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, \pi_1\,-\,\, \mathrm{i}\,\, (\,-\,1\,+\,T\,)\,\, \pi_1\,-\,\, \mathrm{i}\,\, \pi$ $i \ T \in p_{2+k} \ \pi_1 - \frac{1}{2} \ (-1 + T) \ T^3 \in p_{2+i} \ p_{2+j} \ \pi_1^2 + \frac{1}{2} \ (-1 + T) \ T^3 \in p_{2+j}^2 \ \pi_1^2 - \frac{1}{2} \ (-1 + T) \ T^3 \in p_{2+j}^2 \ \pi_1^2 - \frac{1}{2} \ (-1 + T) \ T^3 \in p_{2+j}^2 \ \pi_1^2 - \frac{1}{2} \ (-1 + T) \ T^3 \in p_{2+j}^2 \ \pi_1^2 - \frac{1}{2} \ (-1 + T) \ T^3 \in p_{2+j}^2 \ \pi_1^2 - \frac{1}{2} \ (-1 + T) \ T^3 \in p_{2+j}^2 \ \pi_1^2 - \frac{1}{2} \ \pi_1^2 \ \pi_1^2 \ \pi_1^2 + \frac{1}{2} \ \pi_1^2 \ \pi_1^2$ $\frac{1}{2} \ (-1+T) \ T^2 \in p_{2+i} \ p_{2+k} \ \pi_i^2 + \frac{1}{2} \ (-1+T) \ ^2 \ T \in p_{2+j} \ p_{2+k} \ \pi_i^2 +$ $(-1 + T) \ T^{2} \in p_{2+i} \ p_{2+k} \ \pi_{i} \ \pi_{j} + \ (-1 + T)^{2} \ T \in p_{2+j} \ p_{2+k} \ \pi_{i} \ \pi_{j} +$ $(-1+T) \ T \in p_{2+k}^2 \ \pi_1 \ \pi_j - \frac{1}{2} \ (-1+T) \ T \in p_{2+j} \ p_{2+k} \ \pi_j^2 + \frac{1}{2} \ (-1+T) \ T \in p_{2+k}^2 \ \pi_j^2 + \frac{1}{2} \ (-1+T) \ T \in p_{2+k}^2 \ \pi_j^2 + \frac{1}{2} \ \pi_j^2 \ \pi_j^2 + \frac{1}{2} \ \pi_j^2 \ \pi_j^2 + \frac{1}{2} \ \pi_j^2 \$ $\texttt{i} \ p_{2+k} \ \pi_k \ - \ 2 \ \texttt{i} \ \in \ p_{2+k} \ \pi_k \ + \ T^2 \ \in \ p_{2+i} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \in \ p_{2+j} \ p_{2+k} \ \pi_i \ \pi_k \ - \ (-1 \ + \ T) \ T \ \pi_k \ - \ \pi_k \ \pi_$ $\mathsf{T} \in \mathsf{p}_{2+k}^2 \, \pi_{\mathbf{i}} \, \pi_k + \mathsf{T} \in \mathsf{p}_{2+\mathbf{j}} \, \mathsf{p}_{2+k} \, \pi_{\mathbf{j}} \, \pi_k - \mathsf{T} \in \mathsf{p}_{2+k}^2 \, \pi_{\mathbf{j}} \, \pi_k \, \Big|$

Invariance under the other Reidemeister moves is proven in a similar way. See IType.nb at $\omega\epsilon\beta/ap$.

There's more! To get sl_2 invariants mod ϵ^3 , add the following to $L(X_{ij}^+)$, $L(X_{ij}^-)$, and $L(C_i^{\varphi})$, respectively (and see More.nb at ω - $\epsilon\beta/ap$ for the verifications):

$$\mathbb{S}^{\text{lster}} \odot \epsilon^2 \mathbf{r}_2[\mathbf{1}, \mathbf{i}, \mathbf{j}]$$

$$\begin{array}{c} \hline \begin{array}{c} \hline \\ 12 \end{array} \stackrel{1}{\leftarrow} \end{tabular} \begin{array}{c} e^2 & \left(-6 \end{tabular}_{i} + 6 \end{tabular}_{j} \end{tabular}_{i} - 3 \end{tabular}_{i} - 1 + 3 \end{tabular}_{j} \end{tabular}_{i} \end{tabular}_{i} \\ & 3 \end{tabular}_{i} \end{tabular}_{i} - 1 + 3 \end{tabular}_{j} \end{tabular}_{i} \end{tabular}_{i} \end{tabular}_{i} \end{tabular}_{i} \end{tabular}_{j} \end{tabular}_{i} \end{tabu$$

 $\odot \epsilon^2 r_2[-1, i, j]$

$$\begin{array}{c} \hline \\ \hline \\ \hline \\ 12 \ T^2 \end{array} \stackrel{?}{\in} \stackrel{?}{=} \left(-6 \ T^2 \ p_i \ x_i + 6 \ T^2 \ p_j \ x_i + \\ & 3 \ (-3 + T) \ T \ p_i \ p_j \ x_i^2 - 3 \ (-3 + T) \ T \ p_j^2 \ x_i^2 - 4 \ (-1 + T) \ T \ p_i^2 \ p_j \ x_i^3 + \\ & 2 \ (-1 + T) \ (1 + 5 \ T) \ p_i \ p_j^2 \ x_i^3 - 2 \ (-1 + T) \ (1 + 3 \ T) \ p_j^3 \ x_i^3 + \\ & 18 \ T^2 \ p_i \ p_j \ x_i \ x_j - 18 \ T^2 \ p_j^2 \ x_i \ x_j - 6 \ T^2 \ p_i^2 \ p_j \ x_i^2 \ x_j + 6 \ T \ (1 + 2 \ T) \ p_i \ p_j^2 \ x_i^2 \ x_j - \\ & 6 \ T \ (1 + T) \ p_j^3 \ x_i^2 \ x_j - 6 \ T^2 \ p_i \ p_j^2 \ x_i \ x_j^2 + 6 \ T^2 \ p_j^3 \ x_i \ x_j^2 \right)$$

 $\stackrel{\frown}{\odot} \boldsymbol{\epsilon}^{2} \boldsymbol{\gamma}_{2} [\boldsymbol{\varphi}, \mathbf{i}]$ $\stackrel{\Box}{=} -\frac{1}{2} \boldsymbol{\epsilon}^{2} \boldsymbol{\varphi}^{2} \boldsymbol{p}_{i} \boldsymbol{x}_{i}$

Even more! • The sl_2 formulas mod ϵ^4 are in the last page of the handout of [BN3].

• Using [GPV] we can show that every finite type invariant is I-Type.

- Probably, $\langle \text{Reshetikhin-Turaev} \rangle \subset \langle \text{I-Type} \rangle$ efficiently.
- Possibly, $\langle Rozansky Polynomials \rangle \subset \langle I-Type \rangle$ efficiently.
- Knot signatures are I-Type, at least mod 8.
- We already have some work on sl_3 , and it leads to the strongest genuinely-computable knot invariant presently known.

The $sl_3^{/\epsilon^2}$ **Example** (continues Schaveling [Sch]). Here we have two formal variables T_1 and T_2 , we set $T_3 := T_1T_2$, we integrate over 6 variables for each edge: p_{1i} , p_{2i} , p_{3i} , x_{1i} , x_{2i} , and x_{3i} .

$$\begin{split} d \{ vs_{i}, vs_{j}, vs_{k}, vs_{i^{+}}, vs_{j^{+}}, vs_{k^{+}} \} \\ & \square T_{1}^{3} T_{2}^{3} \mathbb{E} \Big[\frac{3 \in}{2} + \ll 138 \gg \Big] \\ & \textcircled{i} rhs = \int \mathcal{F}[i, j, k] \times \mathcal{L} / @ (X_{j,k}[1] X_{i,k^{+}}[1] X_{i^{+},j^{+}}[1]) \\ & d \{ vs_{i}, vs_{j}, vs_{k}, vs_{i^{+}}, vs_{j^{+}}, vs_{k^{+}} \}; \\ & lhs = rhs \end{split}$$

<u>□</u>True

A faster program, in which the Feynman diagrams are "pre-computed" (see theta.nb at $\omega\epsilon\beta/ap$):

 $\bigcirc \mathbf{R}_1[1, i_j] = CF[$ $1/2 - T_3 g_{1ji} g_{2ji} - g_{3ii} + g_{2jj} g_{3ii} + T_1 (T_3 - 1) g_{1ji} g_{3ji} +$ $T_2 (T_3 - 1) g_{2ji} g_{3ji} - T_2 g_{2ji} g_{3jj} +$ $(g_{1jj} g_{2ii} + (T_3 - 1) g_{1jj} g_{2ji} - T_1 g_{1ii} g_{2jj} - g_{1jj} g_{3ii} T_1 (T_3 - 1) g_{1jj} g_{3ji} + T_1 g_{1ii} g_{3jj}) / (T_1 - 1)];$ $\bigcirc \mathbf{R}_1[-1, i_j] = CF[$ $-1/2 - T_1^{-1} g_{1ji} g_{2ii} - (1 - T_1^{-1} - T_2^{-1}) g_{1ji} g_{2ji} - g_{1jj} g_{2ji}$ $g_{1ji} g_{2jj} + g_{3ii} + T_1^{-1} g_{1ji} g_{3ii} - (1 - T_2^{-1}) g_{2ji} g_{3ii} - g_{2jj} g_{3ii} +$ $\left(1 - T_3^{-1}\right) \, g_{1ji} \, g_{3ji} - \left(1 - T_3^{-1}\right) \, g_{2ii} \, g_{3ji} + \left(2 - T_2^{-1}\right) \, \left(1 - T_3^{-1}\right) \, g_{2ji} \, g_{3ji} + \left(1 - T_3^{-1}\right) \, g_{2ji} \, g_{3ji} \, g_{3ji} + \left(1 - T_3^{-1}\right) \, g_{2ji} \, g_{3ji} \, g_{3ji}$ $(1 - T_3^{-1}) g_{2jj} g_{3ji} + g_{1ji} g_{3jj} + g_{2ji} g_{3jj} +$ $(T_1 (1 - T_2^{-1}) g_{1ii} g_{2ji} - g_{1jj} g_{2ii} + T_1 g_{1ii} g_{2jj} + g_{1jj} g_{3ii} T_{2}^{-1}$ (T₃ - 1) $g_{1ii} g_{3ji} - T_{1} g_{1ii} g_{3jj} / (T_{1} - 1)];$ $\odot \Theta[\{1, i0_, j0_\}, \{1, i1_, j1_\}] =$ $-T_1$ (T₃ - 1) $g_{1,j1,i0} g_{2,i1,i0} g_{3,j0,i1} + (T_3 - 1) g_{1,j1,j0} g_{2,i1,i0} g_{3,j0,i1} +$ $\mathsf{T}_1 \ (\mathsf{T}_3 - 1) \ \mathsf{g}_{1,j1,i0} \ \mathsf{g}_{2,j1,i0} \ \mathsf{g}_{3,j0,i1} - \ (\mathsf{T}_3 - 1) \ \mathsf{g}_{1,j1,j0} \ \mathsf{g}_{2,j1,i0} \ \mathsf{g}_{3,j0,i1};$ $\odot \theta[\{1, i0_, j0_\}, \{-1, i1_, j1_\}] =$ $(T_3 - 1) g_{1,j1,i0} g_{2,i1,i0} g_{3,j0,i1} - T_1^{-1} (T_3 - 1) g_{1,j1,j0} g_{2,i1,i0} g_{3,j0,i1} (T_3 - 1) g_{1,j1,i0} g_{2,j1,i0} g_{3,j0,i1} + T_1^{-1} (T_3 - 1) g_{1,j1,j0} g_{2,j1,i0} g_{3,j0,i1};$ $\odot \theta[\{-1, i0_, j0_\}, \{1, i1_, j1_\}] = CF[$ $T_1^{-1} T_2^{-1} (T_3 - 1)$ $(g_{1,j1,i0}, g_{2,i1,i0}, g_{3,j0,i1}, - T_1, g_{1,j1,j0}, g_{2,j1,j0}, g_{3,j0,j1}, - T_1, g_{1,j1,j0}, g_{2,j1,j0}, g_{2,j1,j0}, g_{3,j0,j1}, - T_1, g_{1,j1,j0}, g_{2,j1,j0}, g_{3,j0,j1}, - T_1, g_{1,j1,j0}, g_{2,j1,j0}, g_{2,j1,j0},$ $g_{1,j1,i0} g_{2,j1,i0} g_{3,j0,i1} + T_1 g_{1,j1,j0} g_{2,j1,i0} g_{3,j0,i1}$; $\odot \Theta[\{-1, i0_{, j0_{}}, \{-1, i1_{, j1_{}}\}] = CF[$ $(1 - T_3^{-1}) (-T_1^{-1} g_{1,j1,i0} g_{2,i1,i0} g_{3,j0,i1} + g_{1,j1,j0} g_{2,i1,i0} g_{3,j0,i1} +$ $T_1^{-1} g_{1,j1,i0} g_{2,j1,i0} g_{3,j0,i1} - g_{1,j1,j0} g_{2,j1,i0} g_{3,j0,i1})];$ $\odot \Gamma_1[\varphi_{,k_{}}] = -\varphi / 2 + \varphi g_{3,k,k};$ We call the invariant computed θ : $\stackrel{()}{=} \Theta[K_{]} := Module \left| \{ Cs, \varphi, n, A, s, i, j, k, \Delta, G, \gamma, \alpha, \beta, gEval, c, z \} \right\}$ $\{Cs, \varphi\} = Rot[K]; n = Length[Cs];$ A = IdentityMatrix[2 n + 1]; Cases Cs, $\{s_{, i_{, j_{}}\}$:> $\left(\mathbb{A}[\{i, j\}, \{i+1, j+1\}] + = \begin{pmatrix} -\mathsf{T}^{s} \; \mathsf{T}^{s} - 1 \\ 0 & -1 \end{pmatrix} \right)];$ $\Delta = \mathbf{T}^{(-\text{Total}[\varphi] - \text{Total}[Cs[All,1]])/2} \text{Det}[A]$ G = Inverse[A]; $\mathsf{gEval}[\mathcal{S}_{_}] := \mathsf{Factor}[\mathcal{S} / . \mathsf{g}_{\nu_{_},\alpha_{_},\beta_{_}} \Rightarrow (\mathsf{G}\llbracket \alpha, \beta \rrbracket / . \mathsf{T} \rightarrow \mathsf{T}_{\nu})];$ $z = gEval\left[\sum_{k=1}^{n} \sum_{k=1}^{n} \Theta[Cs[k1], Cs[k2]]\right];$ z += gEval $\left[\sum_{k=1}^{n} \mathbf{R}_{1} @@ Cs [[k]]\right];$ $z += gEval[\sum_{k=1}^{2n} \Gamma_1[\phi[k]], k]];$ { Δ , (Δ /. T \rightarrow T₁) (Δ /. T \rightarrow T₂) (Δ /. T \rightarrow T₃) z} // Factor ;

Some Knots.

$$\widehat{\odot} \text{ Expand} [\Theta[\text{Knot}[3, 1]]] \\ \overline{\Box} \left\{ -1 + \frac{1}{T} + T, -\frac{1}{T_1^2} - T_1^2 - \frac{1}{T_2^2} - \frac{1}{T_1^2 T_2^2} + \frac{1}{T_1 T_2^2} + \frac{1}{T_1 T_2^2} + \frac{1}{T_1^2 T_2} + \frac{1}{T_1^2 T_2} + \frac{1}{T_2^2 T_2} + \frac{1}{T_2^2 T_2} + \frac{1}{T_2^2 T_2^2} + \frac{1}{T_2^2 T_2^$$



So θ detects knot mutation and separates the Conway knot K11n34 from the Kinoshita-Terasaka knot K11n42!

The48-crossingGompf-Scharlemann-Thompsonknot[GST] is significant because it maybe a counterexample to the slice-ribbon conjecture:



on knot ecause it may



K11n42

K11n34

Gompf Scharlemann Thompson



③ AbsoluteTiming@

PolyPlot[

 $\theta \left[\mathsf{EPD} \left[X_{14,1}, \overline{X}_{2,29}, X_{3,40}, X_{43,4}, \overline{X}_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, \overline{X}_{9,28}, X_{10,41}, X_{42,11}, \overline{X}_{27,12}, X_{30,15}, \overline{X}_{16,61}, \overline{X}_{17,72}, \overline{X}_{18,83}, X_{19,34}, \overline{X}_{89,20}, \overline{X}_{21,92}, \overline{X}_{79,22}, \overline{X}_{68,23}, \overline{X}_{57,24}, \overline{X}_{25,56}, X_{62,31}, X_{73,32}, X_{84,33}, \overline{X}_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, \overline{X}_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, \overline{X}_{63,74}, \overline{X}_{64,85}, \overline{X}_{76,65}, \overline{X}_{87,66}, \overline{X}_{67,94}, \overline{X}_{75,86}, \overline{X}_{88,77}, \overline{X}_{78,93} \right] \left[\llbracket 2 \rrbracket \right]$



Next, a random 250 crossing knot (knot provided by N. Dunfield):



Prior Art. θ is probably equal to the "2-loop polynomial" studied by Ohtsuki at [Oh2] (at much greater difficulty, and with harder computations). θ is

Ohtsuki Garoufalidis Kashaev

related, but probably not equivalent, to the invariant studied by Garoufalidis and Kashaev at [GK].

 θ Sees Topology! Indeed, for a knot *K*, half the T_1 degree (say) of $\theta(K)$ bounds the genus of *K* from below, and this bound is sometimes better (and sometimes worse) than the bound coming from Δ . It is fair to hope that "anything Δ can do θ can do too" (see [BN2]), and in particular, that θ may say something about ribbon and/or slice properties.

The Rolfsen Table of Knots.



Where is it coming from? The most honest answer is "we don't know" (and that's good!). The second most, "undetermined co-efficients for an ansatz that made sense". The ansatz comes from the following principles / earlier work:

Morphisms have generating functions. Indeed, there is an isomorphism

$$\mathcal{G}\colon \operatorname{Hom}(\mathbb{Q}[x_i],\mathbb{Q}[y_j])\to\mathbb{Q}[y_j][\![\xi_i]\!],$$

and by PBW, many relevant spaces are polynomial rings, though only as vector spaces.

Composition is integration. Indeed, if $f \in \text{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j])$ and $g \in \text{Hom}(\mathbb{Q}[y_i], \mathbb{Q}[z_k])$, then

$$\mathcal{G}(g \circ f) = \int e^{-y \cdot \eta} fg \, dy \, d\eta$$

Use universal invariants. These take values in a universal enveloping algebra (perhaps quantized), and thus they are expressible as long compositions of generating functions. See [La, Oh1].

"Solvable approximation" \rightarrow perturbed Gaussians. Let g be a semisimple Lie algebra, let h be its Cartan subalgebra, and let b^u and b^l be its upper and lower Borel subalgebras. Then b^u has a bracket β , and as the dual of b^l it also has a cobracket δ , and in fact, $g \oplus h \equiv \text{Double}(b^u, \beta, \delta)$. Let $g_{\epsilon}^+ := \text{Double}(b^u, \beta, \epsilon\delta)$ (mod ϵ^{d+1} it is solvable for any *d*). Then by [BV3, BN1] (in the case of $g = sl_2$) all the interesting tensors of $\mathcal{U}(g_{\epsilon}^+)$ (quantized or not) are perturbed Gaussian with perturbation parameter ϵ with with understood bounds on the degrees of the perturbations.

The Philosophy Corner. "Universal invariants", valued in universal enveloping algebra (possibly quantized) rather than in representations thereof, are a priori better than the representation theoretic ones. They are compatible with strand doubling (the Hopf coproduct), and as the knot genus and the ribbon property



for knots are expressible in terms of strand doubling, universal invariants stand a chance to say something about these properties. Indeed, they sometimes do! See e.g. [BN2, Oh2, GK, LV, BG]. Representation theoretic invariants don't do that!

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