Setting

Def A marked sutured manifold is a sutured manifold M w/ labelled + arrows on R_+ (red), − arrows on R_- (blue), endpoints on sutures.

Balanced: $\chi(R_+) = \chi(R_-)$ [NB Different from bordered sutured]

Combed vector field $\vec{v}$ w/ $\vec{v}$ pointing out on R_+, in on R_-, from R_+ to R_- on sutures.

Framed ~~transverse~~ indep. vect field $\vec{v}_x, \vec{v}_y, \vec{v}_z$

$\vec{v}_z$ a combing
+ arrows tangent to $\vec{v}_x$ } at endpoints
− arrows tangent to $\vec{v}_x$

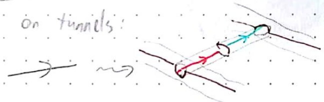
Relevant in ~~sutured~~ setting $S^2 \neq 1$

Equivalence • cut tunnels • cut on bigons] a little unclear in framed setting

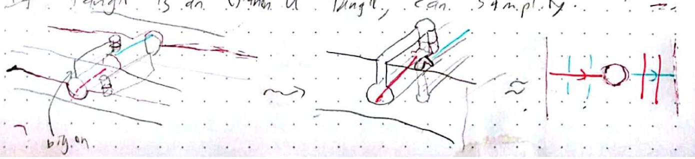
~~Framed 3-manifolds are kind of complicated. Eg have (at least) 3 different Chern classes, coming from 3 different 2-plane fields c_{xy}, c_{yz}, c_{zx}. May be a constraint like $c_{xy} + c_{yz} + c_{zx} = 0$. Also have more info, like combing.~~

Nope: for combing v_x, $c_1(v_x^\perp) = 0$, since that's a non-zero section.

Embedding virtual tangle:
Given tangle T on Σ
$M = \Sigma \times I \setminus T$
Sutures on $\partial\Sigma \times \{z\}$ and on tunnels:

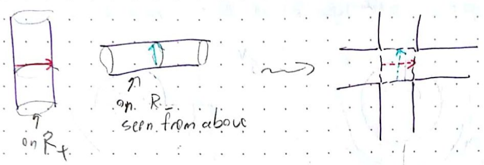


If tangle is an O then U tangle, can simplify:



This is hard to draw...
Will be harder to get the framings right.

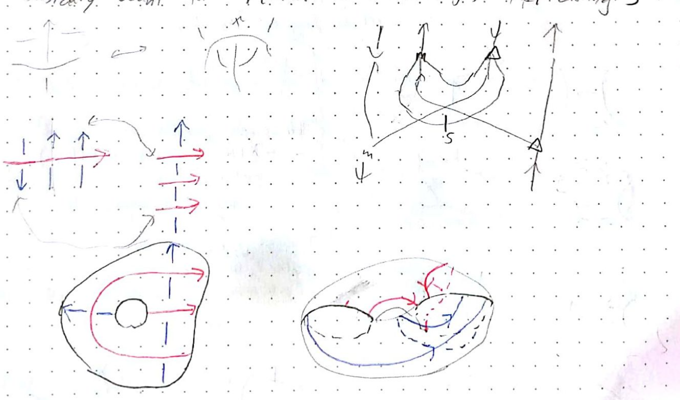
Gluings. Attach like so:



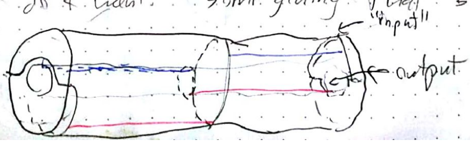
Basic objects. Drawn earlier

Drinfeld Double

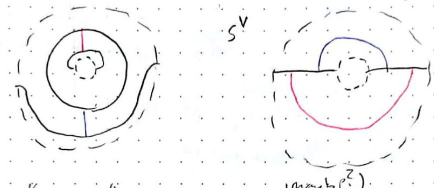
Basically want to reverse a tube − just like doing S



What do I want? Some gluing that switches "red"/"blue"?

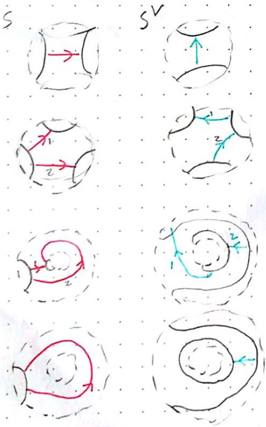


Need a "duality" operation.
Given a marked subsurface S, find another surface $S^\vee$ that gives the effect of gluing with S. $S^{\vee\vee} = -S$ ~ reverse all arrows
In this example, need both + & − markings



This would give the result...

What's the rule? Which surfaces are fully parametrized?
- probably: spine of the surface & sutures touch each boundary?



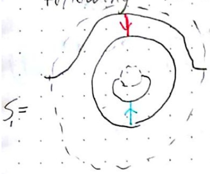
What happens to χ?
- should switch $\chi(R_+)$, $\chi(R_-)$

How many arcs do you need?
- $-\chi(R_+)$ and $-\chi(R_-)$ probably (counting ∂ appropriately)

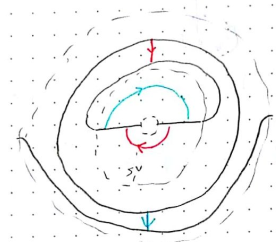
I guess the answer is clear: replace each arc by a dual arc (presumably getting another spine)

orientations seem to be aiming out wrong.

Q: For representing tangles on tunnels, put following sutures:

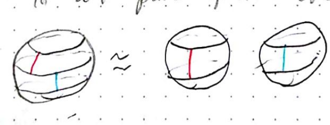


"Dualizing" operation (for Drinfeld double) is $S_1^\vee \cup S_2$ filled by solid torus



Dotted circles bounding circles (as $S_2 \times I$).
⤷ this matches! yay.

Q. Do you get every sutured marked 3-manifold?
- What's a more standard representation? Heegaard ~~surface~~ diagram?
Take $\Sigma \times I$, attach α and β handles (equal #s to be balanced)
can you arrange for marking to be disjoint from attaching circles? Probably.
So then: Represent (markings) ∪ (α circles) ∪ (β circles) as a virtual OU tangle
Attach a gadget to α-β pairs (picked arbitrarily)
Gadget:



So, yes.