Over then Under Tangles

Thanks for inviting me to ICTS Bangalore!

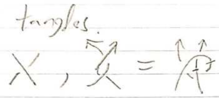
Abstract. Brilliant wrong ideas should not be buried and forgotten. Instead, they should be mined for the gold that lies underneath the layer of wrong. In my talk I will explain how "over then under tangles" lead to an easy classification of knots, and under the surface, also to some valid mathematics: …

Main Fheorem. Every tangle is an Over then Under (OU) tangle.
Pfoof. … □

Impromptu (conversation-starter) lecture on OU tangles ~~and~~ the Drinfel'd Double Formula, and quantization of Lie bialgebras.

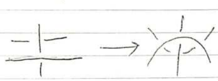
Define OU tangles.

Examples:



Thm Every tangle is an OU tangle.

proof



Corollary Knot theory is trivial!

Too good to be true? What about ?

Too good to be ignored? Perhaps true in some quotient/completion/image?

1. Quotient-completion: complete to virtual knots, divide by OC, get my paper "Knotted balloons & hoops". There may be more…
2. Completion like in $\mathbb{Q} \hookrightarrow \mathbb{A}$: See Dylan's talk.
3. Completion like in $\mathbb{Q}[x] \hookrightarrow \mathbb{Q}[[x]]$ or $K \to \varprojlim K/I^n$:

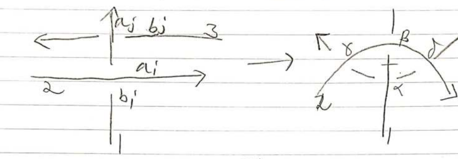
$\boxed{\text{I wish I knew}}$

4. represent!

A "bales" formula.
$R \in \mathbb{A}\otimes\mathbb{B} = \sum a_i \otimes b_i$
need $b \cdot a := \sum a_i' b_i'$

(Need a rule making $\mathbb{A}\otimes\mathbb{B}$ into an algebra)

Requirement on $\mathbb{A}, \mathbb{B}$, $R = \sum a_i \otimes b_i \in \mathbb{A}\otimes\mathbb{B}$



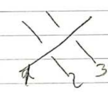
Fod's Gold
"FG relation"

1. $\mathbb{A}$ & $\mathbb{B}$ must be f.d. algebras. | $\mathbb{A}^* \to \mathbb{B}$ onto, $\mathbb{B}^* \to \mathbb{A}$ onto.

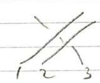
2. The b_i's must span all of $\mathbb{B}$, The a_i's must span all of $\mathbb{A}$ $\Rightarrow$ R is of max rank, $\dim\mathbb{A} = \dim\mathbb{B}$, $\exists$ pairing $P = \langle , \rangle \in \mathbb{B}^*\otimes\mathbb{A}^*$ "dual" to R:

$\sum a_i \langle b_i, a\rangle = a \qquad \sum \langle b, a_i\rangle b_i = b$

3.



$\to \sum a_i a_j \otimes b_i \otimes b_j$ $\quad b \mapsto \sum \langle b, a_i a_j\rangle b_i \otimes b_j$ a co-product on $\mathbb{B}$ s.t. $(1\otimes\Delta)R = R^{12}R^{13}$



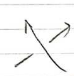
$\to \sum a_i \otimes a_j \otimes b_j b_i$ $\quad a \mapsto \sum a_i \otimes a_j \langle b_j b_i, a\rangle$ a co-product on $\mathbb{A}$ s.t. $(\Delta\otimes 1)R = R^{23}R^{13}$

$\rightsquigarrow$ $\mathbb{A}, \mathbb{B}$ are (dual) bi-algebras.

4.

$R^{-1} = \sum a_i' \otimes b_i'$

R is invertible in $\mathbb{A}\otimes\mathbb{B}$



$\to \sum a_i' \otimes b_i'$

$a \mapsto \sum a_i' \otimes \langle b_i', a\rangle$
$\to$ s.t. $(S\otimes 1)R = R^{-1}$ .. likewise for $\mathbb{B}$. an antipode for $\mathbb{A}$ $\to$

The FG relation:

$$\sum b_i \cdot a_j \otimes a_i \otimes b_j = \sum a_\alpha b_\beta \otimes a_\gamma a_\beta \bar{a}_\delta \otimes b_\delta b_\alpha b_\gamma$$

Pair w/ $b\otimes a$ on strands 2,3: $\quad \Delta(b) = b_1\otimes b_2\otimes b_3$, $\Delta(a) = a_1\otimes a_2\otimes a_3$

$$b \cdot a := \sum a_\alpha b_\beta \otimes \langle b_1, a_\gamma\rangle\langle b_2, a_\beta\rangle\langle b_3, \bar{a}_\delta\rangle$$
$$\langle b_\delta, a_1\rangle\langle b_\alpha, a_2\rangle\langle b_\gamma, a_3\rangle$$
$$= \langle b_1, a_3\rangle\langle b_3, \bar{a}_1\rangle a_2 b_2$$

Example $\mathbb{A} = \mathcal{U}(\langle a, x\rangle/[a,x]=x)$ $\quad \mathbb{B} = \mathcal{U}(\langle b, y\rangle/[b,y]=-\epsilon y)$

$\Delta(y, b, a, x) = y_1 + B_1 y_2,\ b_1 + b_2,\ a_1 + a_2,\ x_1 + A_1 x_2$ $\quad$ Pair w/ $\langle b, a\rangle = \hbar^{-1}$, $\langle y, x\rangle = \hbar^{-1}$

$B_0 = e^{-\hbar b}$
$A = e^{-\hbar\epsilon a}$

$S(y\,b\,a\,x) = -B^{-1}y, -b, -a, -A^{-1}x$

$\Rightarrow \langle b^n, a^m\rangle = \hbar^n \delta_{nm} n! \quad \langle y^n, x^m\rangle = \hbar^{-n}\delta_{nm}[n]_q!$ $\quad q = e^{\hbar\epsilon}$

$$\Rightarrow R = \sum \frac{a^n b^n}{n!} \cdot \frac{x^m y^m}{[m]_q!}$$

$ba = ab$

$xy = qyx + (1-AB)/\hbar \ldots$

Setting

Def A marked sutured manifold is a sutured manifold M w/ labelled + arrows on R_+ (red) / − arrows on R_- (blue), endpoints on sutures.

Balanced: $\chi(R_+) = \chi(R_-)$ [Nb Different from bordered sutured]

Combed vector field $\vec{v}$ w/ $\vec{v}$ pointing out on R_+, in on R_-, from R_+ to R_- on sutures.

Framed ~~transverse~~ indep. vect field $\vec{v}_x, \vec{v}_y, \vec{v}_z$

$\vec{v}_z$ a combing.
+ arrows tangent to $\vec{v}_x$ } at endpoints
− arrows tangent to $\vec{v}_x$ }

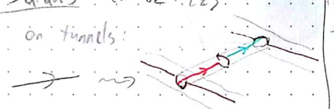
Relevant in ~~sutured~~ setting, $\Sigma \neq 1$

Equivalence • cut tunnels • cut on bigon] a little unclear in framed setting

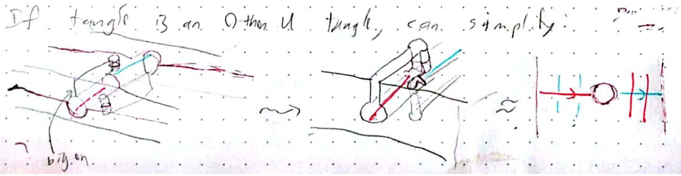
Framed 3-manifolds are kind of complicated.
Eg, have (at least) 3 different Chern classes, coming from 3 different 2-plane fields: c_{xy}, c_{yz}, c_{zx}.
May be a constraint like $c_{xy} + c_{yz} + c_{zx} = 0$.
Also has more info, like ~~framing~~ combing.

Nope: for combing v_x, $c_1(v_x^\perp) = 0$, since it has a non-zero section.

Embedding virtual tangle:
Given tangle T on Σ
$M = \Sigma \times I \setminus T$
Sutures on $\partial\Sigma \times \{\tfrac{1}{2}\}$ and on tunnels:



If tangle is an O than U tangle, can simplify:



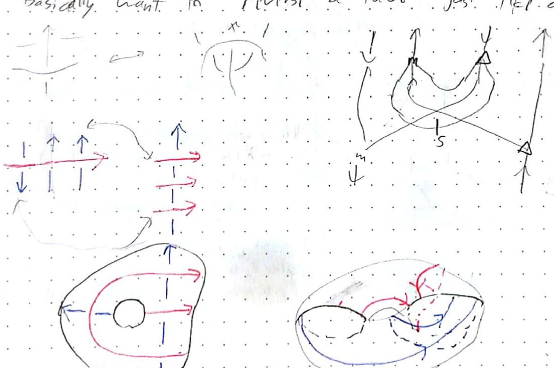
This is hard to draw...
Will be harder to get the framings right.

Gluings Attach like so:

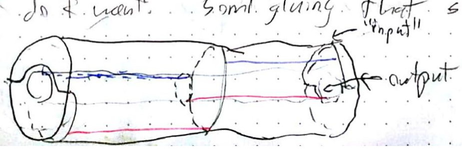
Basic objects Drawn earlier

Drinfeld Double

Basically want to reverse a tube – just like doing S

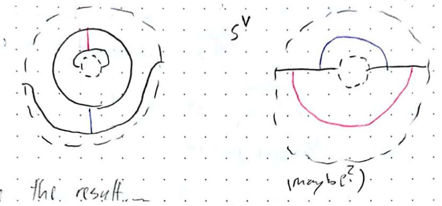


What do I want? Some gluing that switches red/blue



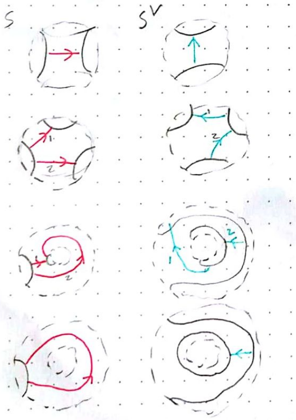
Need a "duality" operation
Given a marked subsurface S, find another surface $S^\vee$ that gives the effect of gluing with S. $S^{\vee\vee} = -S$ ⇝ reverse all arrows
In this example, need both + & − markings

Subsurface S: $S^\vee$



This would give the result...
What's the rule? Which surfaces are fully parametrized?
– probably: spine of the surface & sutures touch each boundary?

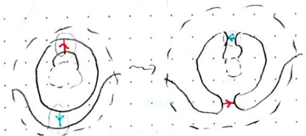
S $S^\vee$



What happens to χ?
– should switch $\chi(R_+)$, $\chi(R_-)$

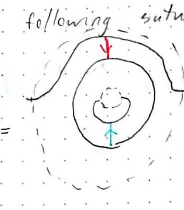
How many arcs do you need?
$-\chi(R_+)$ and $-\chi(R_-)$
(counting ∂ appropriately) probably

I guess the answer is clear:
replace each arc by a dual arc
(presumably getting another spine)



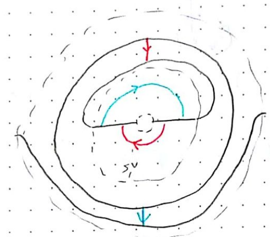
orientations seem to be coming out wrong..

Ques For representing tangles on tunnels, put following sutures:



$S_1 =$

"Dualizing" operation (for Drinfeld double) is $S_1^\vee \cup S_2$ filled by solid torus

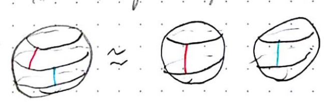


Dotted circles bounding circles (as $S_2 \times I$).
⤷ this matches! yay.

Q. Do you get every sutured marked 3-manifold?
– What's a more standard representation? Heegaard ~~sutured~~ diagram?
Take $\Sigma \times I$, attach α and β handles (equal #s to be balanced)
can you arrange for marking to be disjoint from attaching circles? Probably.

So then Represent (markings) ∪ (α circles) ∪ (β circles) as a virtual O/U tangle

Attach a gadget to α-β pairs (picked arbitrarily)
Greg's gadget:



So, yes.