Wheeled Calculus

December-20-11
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$$\mathcal{D}_0(S) \cong R \oplus M_{S\times S}(R) \qquad R = \mathbb{Q}[\![c_i]\!]_{i\in S}$$

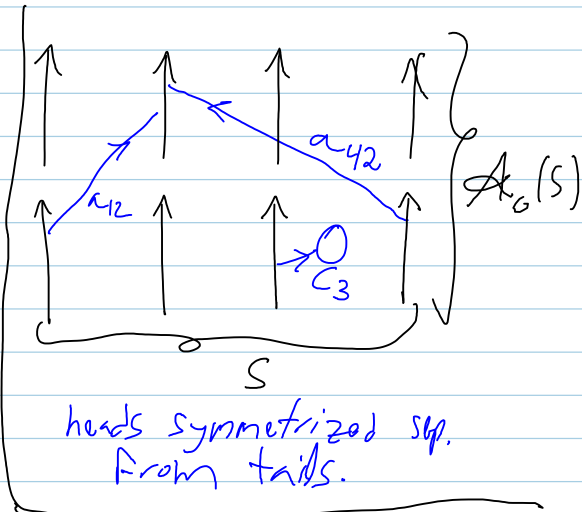
$$= \{ w + \sum \alpha_{ij} t_i h_j : w, \alpha_{ij} \in R \}$$

with $\chi^{wss}: \mathcal{D}_0(S) \to \mathcal{A}_0(S)$ "wheeled semi-symmetrization"

mapping $w + \sum \alpha_{ij} t_i h_j$ to

$$\chi^{ss}\left(\exp\left(w + \sum_{i,j} J_j^{-1} \alpha_{ij}\, a_{ij}\right)\right)$$

where $J_j := \sum c_i \alpha_{ij} \big/ \log(1 + \sum_i c_i \alpha_{ij})$



heads symmetrized sep. from tails.

That is, w/ $\tilde{\alpha}_{ij} = J_j^{-1} \alpha_{ij}$, $\sum c_i \tilde{\alpha}_{ij} = \log(1 + \sum c_i J_j \tilde{\alpha}_{ij})$

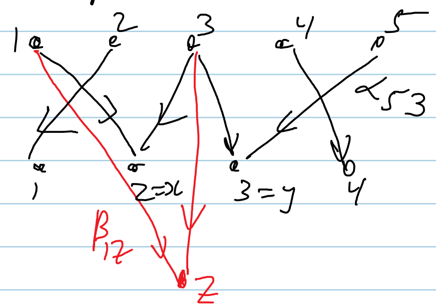
so w/ $\eta_j = \sum c_i \tilde{\alpha}_{ij}$, $\quad J_j = \dfrac{e^{\eta_j} - 1}{\eta_j}$

worth noting: $e^{\eta_j} = 1 + \sum c_i \alpha_{ij}$

$$\begin{array}{ccc} \mu \in \mathcal{D}_0(S) & \xrightarrow{\chi^{wss}} & \mathcal{A}_0(S) \\ \downarrow m^{xy}_z & & \downarrow m^{xy}_z \\ \mathcal{D}_0(S) & \xrightarrow{\chi^{wss}} & \mathcal{A}_0(S') \end{array} \qquad \begin{array}{ccc} \mathcal{D}_0(S) & \xrightarrow{\chi^{wss}} & \mathcal{A}_0(S) \\ \downarrow c^x_y & & \downarrow c^x_y \\ \mathcal{D}_0(S) & \xrightarrow{\chi^{wss}} & \mathcal{A}_0(S) \end{array}$$

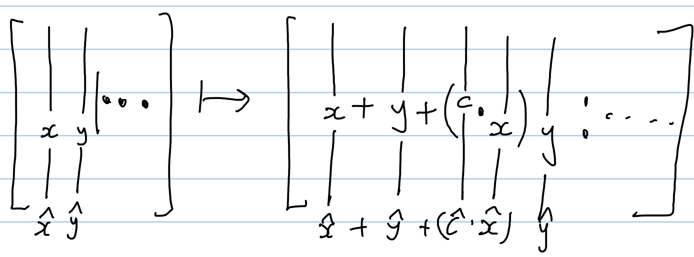
Question. Determine the $\mathcal{D}$-side m^{xy}_z & c^x_y.

Multiplication.



$$\beta_{1z} = \alpha_{1x} + (1 + \sum c_j \alpha_{jx}) \alpha_{1y}$$

$$= \alpha_{1x} + \alpha_{1y} + (\sum c_j \alpha_{jx}) \alpha_{1y}$$



Associativity:

$$[\hat{x}\hat{y}\hat{z} : B] \mapsto [\hat{x} + \hat{y} + (\hat{c}\cdot\hat{x})\hat{y} : \hat{z} : B] \mapsto [\cancel{\hat{x}}^{1} + \cancel{\hat{y}}^{2} + \cancel{\hat{z}}^{3} + \cancel{(\hat{c}\cdot\hat{x})\hat{y}}^{5} + (\cancel{\hat{c}\cdot\hat{x}} + \cancel{\hat{c}\cdot\hat{y}}^{4} + (\hat{c}\cdot\hat{x})(\hat{c}\cdot\hat{y}))\hat{z}^{z} : B]$$

It works, but I could

$$\left[+(\hat{C}\cdot\hat{x}+\hat{C}\cdot\hat{y}+(\hat{C}\hat{x})(\hat{C}\hat{y}))\hat{z} : B\right]$$

It works, but I could use some intuition!

$$\left[\hat{x} : \hat{y}+\hat{z}+(\hat{C}\cdot\hat{y})\hat{z} : B\right] \mapsto \left[\begin{array}{l}\hat{x}+\hat{y}+\hat{z}+(\hat{C}\cdot\hat{y})\hat{z} \\ +(\hat{C}\cdot\hat{x})(\hat{y}+\hat{z}+(\hat{C}\cdot\hat{y})\hat{z})\end{array} : B\right]$$

Compact Notation.

$$(\hat{x} : \hat{y} : B) \longrightarrow (\hat{x}+\hat{y}(1+\bar{C}\hat{x}) : B)$$

Comultiplication & Antipode.

$$(V : B) \xrightarrow{\Delta} (V : V : B)$$

Satisfies

$$\begin{array}{ccc} (V : W : B) & \xrightarrow{\Delta\otimes\Delta} & (V : V : W : W : B) \\ \downarrow m & & \downarrow m\otimes m \\ & \longrightarrow & \checkmark \end{array}$$

$$(\hat{x} : B) \xrightarrow{S} (-V/1+\bar{C}\hat{x})$$

To do. 1. R. 2. Factorization.
3. Conjugation, no self terms $-C^x_y$.
4. "Feedback" conjugation.

1. R. $R_{12} = \mu\, a_{12}$, So that $\mu \cdot \frac{\log(1+C_1\mu)}{C_1\mu} = 1$.

hence $1+C_1\mu = e^{C_1}$, $\mu = \frac{e^{C_1}-1}{C_1}$

2. Factorization.

$x|_X = z|_X \quad x|_Y = 0$

$y|_X = 0 \quad (1+C\cdot x)y|_Y = z|_Y$

$\Rightarrow x = z/.\left(\begin{array}{c}Y\text{ tails} \\ \to 0\end{array}\right)$

$y = \frac{1}{1+C\cdot x}\cdot z/.\left(\begin{array}{c}X\text{ tails} \\ \to 0\end{array}\right)$

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HFactorize[z_, x_, ytails_List → y_][mix_] := Module[
  {ξ, ξ0, α, α0, β, β0},
  ξ0 = D[mix, h[z]];
  α0 = ξ0 /. ((t[#] → 0) & /@ ytails);
  β0 = ξ0 - α0;
  {ξ, α, β} = {ξ0, α0, β0} /. t[s_] :> c[s];
  Expand[mix - ξ0 h[z] + Log[(α E^ξ + β)/ξ]/α α0 h[x] - Log[(α + β E^-ξ)/ξ]/β β0 h[y]]
]
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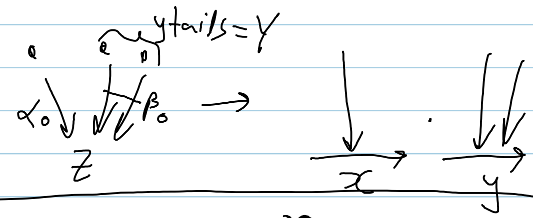
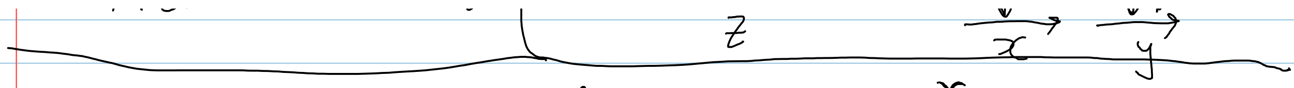
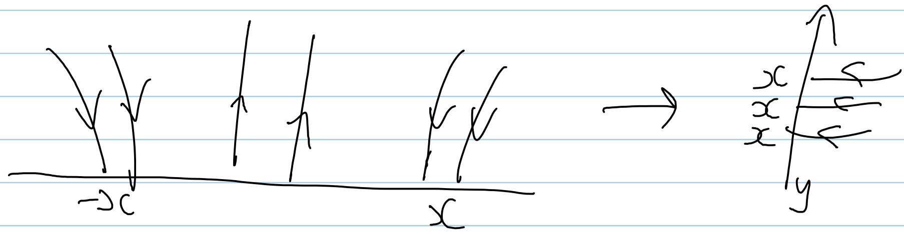
From Semi-Symmetrized 2D Calculus.nb

The unwheeled case



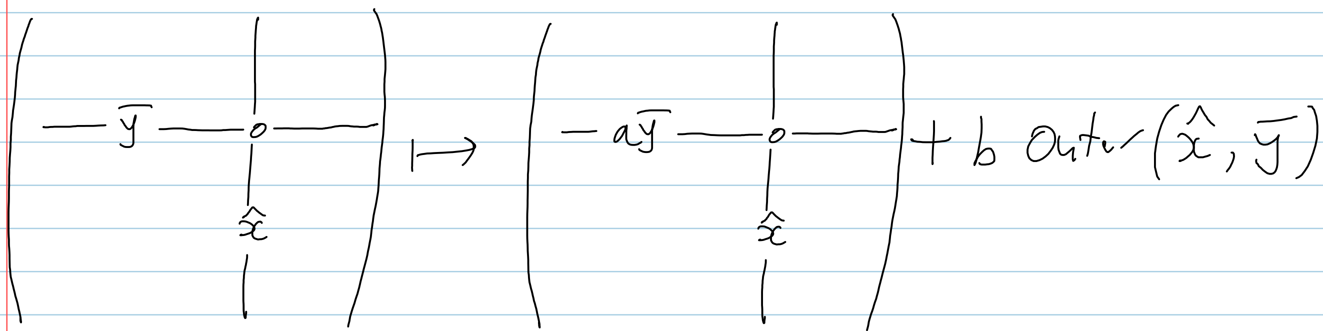


3. Conjugation, no self terms — C^x_y.



1. Every y-tail term gets multiplied by $\sim e^{\hat{c}\cdot\hat{x}}$ (perhaps $\sim 1+\hat{c}\hat{x}$)

2. or has its y replaced by $\sim\hat{x}$.



In compact notation: (a is really $a_1+a_2\bar{c}\hat{x}$)

$$\begin{array}{cc} & x \\ y & \begin{pmatrix}\alpha_1 & 0\\ \alpha_2 & \alpha_3\end{pmatrix}\end{array} \xmapsto{C^x_y} \begin{pmatrix} a\alpha_1 & 0\\ \alpha_2 & \alpha_3\end{pmatrix} + b\begin{pmatrix}0 & 0\\ \alpha_3\alpha_1 & 0\end{pmatrix}$$

linear in y? ✓

The action axiom —

$$\begin{array}{cc} & x \quad y \\ z & \begin{pmatrix}\alpha_1 & 0 & 0\\ \alpha_2 & \alpha_3 & \alpha_4\end{pmatrix}\end{array} \xrightarrow{C^x_z} \Big(\ldots \text{ checked in } \Big)$$

Wheeled Semi-Symmetrized 2D Calculus.nb
& 11-12-23 calculator.nb

Sol'n: $a_1=a_2=1$, b is free.
Then b is fixed to $-c_y$ by 4T.

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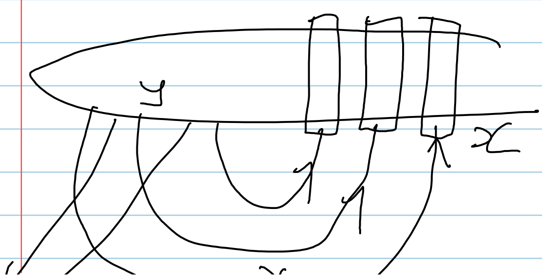
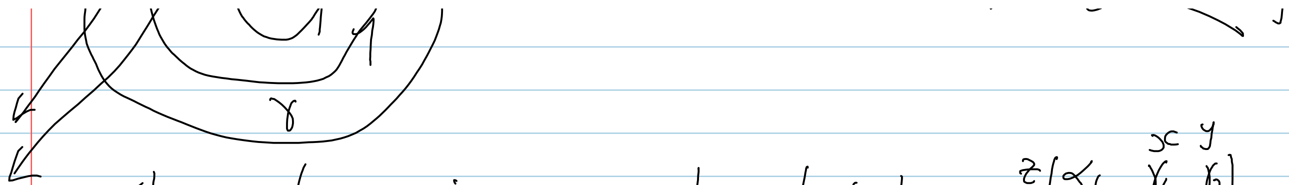
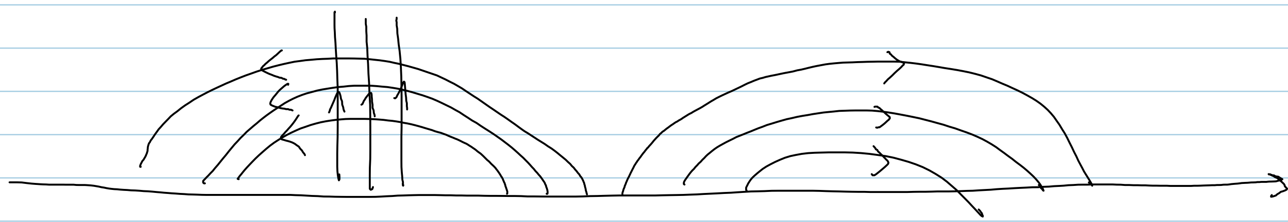
4. "Feedback" conjugation. [the x heads have no tails other than y]



$$\cancel{\begin{array}{cc} & x \\ y & \begin{pmatrix}\alpha_1 & \gamma\\ \alpha_2 & 0\end{pmatrix}\end{array} \xrightarrow{C^x_y} y\begin{pmatrix} a\alpha_1 c_y \gamma & \gamma\\ \alpha_2 & 0\end{pmatrix}}$$



The action axiom may be checked on $\begin{array}{cc} & \begin{array}{ccc} & x & y \end{array} \\ z & \begin{pmatrix} \alpha_1 & \gamma_1 & \gamma_2 \\ \alpha_2 & 0 & 0 \end{pmatrix} \end{array}$



only wheels matter!