Semi-Symmetrized Calculus

September-24-11
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$$\mathcal{D}(S) \cong R \oplus M_{S\times S}(R) \qquad R = \mathbb{Q}[[c_i]]_{i\in S}$$

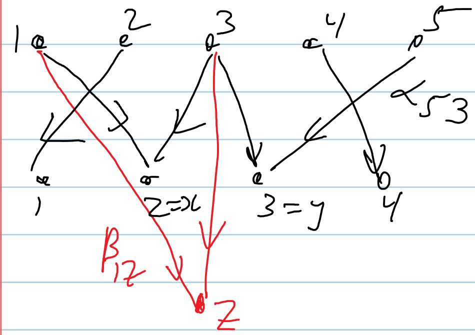
$$= \{ w + \sum \alpha_{ij} t_i h_j : w, \alpha_{ij} \in R \}$$

$$\begin{tikzcd}
M \\
\mathcal{D}_0(S) \arrow[u, phantom, "\cap"] \arrow[r, "x^{ss}"] \arrow[d, "m^{xy}_z"] & \mathcal{A}_0(S) \arrow[d, "m^{xy}_z"] \\
\mathcal{D}_0(S) \arrow[r, "x^{ss}"] & \mathcal{A}_0(S')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}_0(S) \arrow[r, "x^{ss}"] \arrow[d, "c^x_y"] & \mathcal{A}_0(S) \arrow[d, "c^x_y"] \\
\mathcal{D}_0(S) \arrow[r, "x^{ss}"] & \mathcal{A}_0(S)
\end{tikzcd}$$

Question. Determine the $\mathcal{D}$-side m^{xy}_z & c^x_y.

[They are non-linear?]

Multiplication.

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HMultiply[x_, y_, z_][mix_] := Module[
  {ξ, η, j},
  ξ = D[mix, h[x]] /. t[s_] :> c[s];
  η = D[mix, h[y]] /. t[s_] :> c[s];
  j[ζ_] := If[ζ === 0, 1, (e^ζ - 1)/ζ];
  (mix /. {h[x] -> 0, h[y] -> 0}) +
   j[ξ] h[z] D[mix, h[x]] / j[ξ + η] + e^ξ j[η] h[z] D[mix, h[y]] / j[ξ + η]
]
```

A 2D B-Picture, IV.nb

$$\xi = \sum_i c_i \alpha_{ix} \qquad \eta = \sum c_j \alpha_{jy} \qquad j(\zeta) = \frac{e^\zeta - 1}{\zeta}$$

$$\beta_{iz} = \frac{1}{j(\xi+\eta)} \left(j(\xi)\alpha_{ix} + e^{\xi} j(\eta) \alpha_{iy} \right)$$

Associativity: (for x, y, z, with duals $\check{x}, \check{y}, \check{z}$)

$$\beta^i_{(xy)} = \frac{1}{j(\check{x}+\check{y})} \left(j(\check{x})\alpha^i_x + e^{\check{x}} j(\check{y}) \alpha^i_y \right) \qquad (*)$$

$$\check{(xy)} = c_i \beta^i_{(xy)} = \frac{c_i}{j(\check x+\check y)}\left(j(\check x)\alpha^i_x + e^{\check x} j(\check y)\alpha^i_y\right)$$

$$= \frac{1}{j(\check x+\check y)}\left((e^{\check x}-1) + e^{\check x}(e^{\check y}-1)\right)$$

$$= \frac{1}{j(\check x+\check y)}\left(e^{\check x+\check y}-1\right) = \check x+\check y$$

So $j(\check{(xy)}) = j(\check x+\check y)$ and

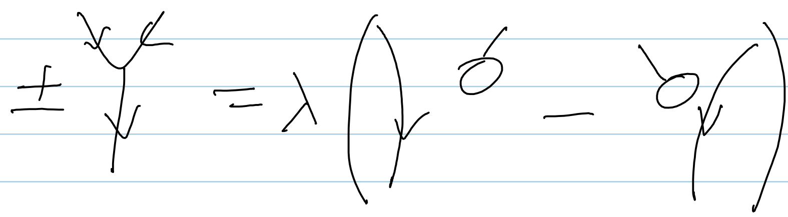
$$\beta^i_{(xy)z} = \frac{1}{j(\check x+\check y+\check z)}\left(\frac{j(\check x+\check y)}{j(\check x+\check y)}\left(j(\check x)\alpha^i_x + e^{\check x} j(\check y)\alpha^i_y\right) + e^{\check x+\check y} j(\check z)\alpha^i_z\right)$$

$$= \frac{1}{j(\check x+\check y+\check z)}\left(j(\check x)\alpha^i_x + e^{\check x} j(\check y)\alpha^i_y + e^{\check x}e^{\check y} j(\check z)\alpha^i_z\right)$$

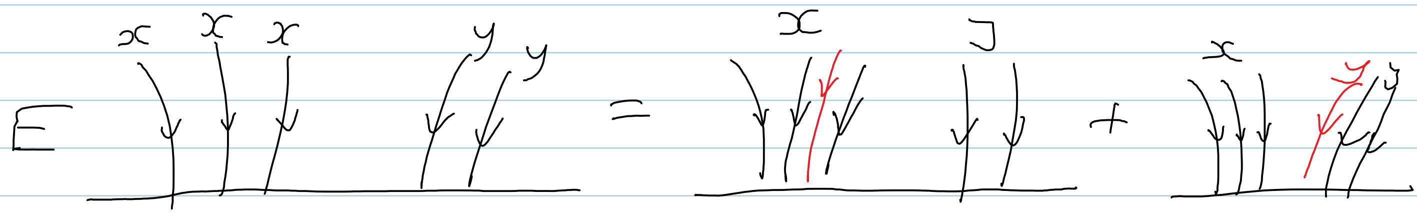
It works, but I could use some intuition!

A more symmetrical form of (*) is:

$$j(\check x+\check y)\,\beta^i_{(xy)} = j(\check x)\alpha^i_x + e^{\check x} j(\check y)\alpha^i_y$$

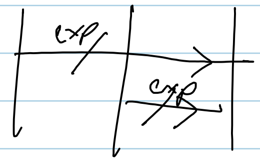


Perhaps I should use λ to get an extra grading?



If there are no coefficients, there can be no "extraction costs"! Yet the result is non-trivial.

$\Longrightarrow$ I should find the glow way of computing



See a similar attempt at 2008-08/BCH modulo [[L,L],[L,L]]

$$G(\exp L) = e^{-L} E e^{L} = J(\operatorname{ad} L) E(L) \quad \text{where } J(x) = \frac{1-e^{-x}}{x}$$