10 Equations in a Noodle Soup

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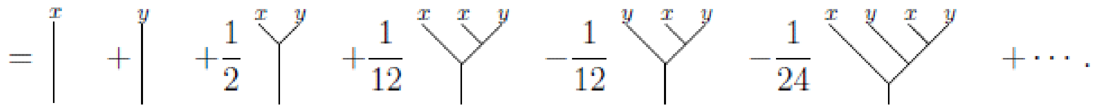
http://www.math.toronto.edu/~drorbn/papers/Aarhus/II/AarhusII.pdf, pages 20-21:

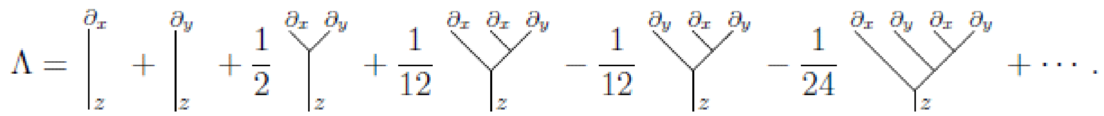
5.3. **An explicit formula for the map $\overrightarrow{m}^{xy}_z$ on uni-trivalent diagrams.** Let x and y be two elements in a free associative (but not-commutative) completed algebra. The Baker-Campbel-Hausdorf (BCH) formula (see e.g. [Ja]) measures the failure of the identity $e^{x+y} = e^x e^y$ to hold, in terms of Lie elements, or, what is the same, in terms of trees modulo the IHX and AS relation. The first few terms in the BCH formula are:

(9)
$$\log e^x e^y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] - \frac{1}{12}[y,[x,y]] - \frac{1}{24}[x,[y,[x,y]]] + \cdots$$



The proposition below states that as an operation on uni-trivalent diagrams, the map $\overrightarrow{m}^{xy}_z : \mathcal{A}(\uparrow_x\uparrow_y\uparrow_E) \to \mathcal{A}\uparrow_z\uparrow_E)$ of Definition 3.5 is given by gluing the disjoint-union exponential of the trees in the BCH formula (9). Precisely, let Λ be the sum of trees in the BCH formula, only with ∂_x replacing x, with ∂_y replacing y, and with a z marked on each root:

(10)



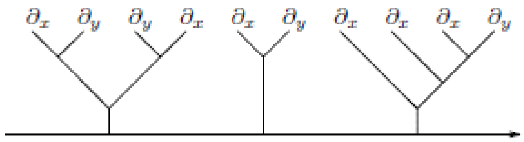
Proposition 5.4. *For any $C \in \mathcal{B}(\{x,y\}\cup E)$,*

(11)
$$\sigma_n \overrightarrow{m}^{xy}_z \chi_{n+1} C = \langle \exp_{\cup} \Lambda, C \rangle_{x,y},$$

where χ_{n+1} denotes the standard isomorphism $\mathcal{B}(\{x,y\}\cup E) \to \mathcal{A}(\uparrow_x\uparrow_y\uparrow_E)$ (whose inverse is σ_{n+1}) and σ_n denotes the standard isomorphism $\mathcal{A}(\uparrow_z\uparrow_E) \to \mathcal{B}(\{z\}\cup E)$ (whose inverse is χ_n).

A noteworthy special case of this proposition is the case where $n=1$ and C is a disjoint union $C_x \cup C_y$ of a uni-trivalent diagram C_x whose legs are labeled only by x and a uni-trivalent diagram C_y whose legs are labeled only by y. In this case $\overrightarrow{m}^{xy}_z$ is (up to leg labelings) the product $\times_A : \mathcal{B}\otimes\mathcal{B} \to \mathcal{B}$ that $\mathcal{B}$ inherits from $\mathcal{A}$, and equation (11) becomes a specific formula for this product in terms of gluing forests. The existence of such a formula is immediate from the definition of $\sigma : \mathcal{A} \to \mathcal{B}$, and this existence was used in several places before (see e.g. [B-N2]), but we are not aware of a previous place where this formula was written explicitly. A similar formula is the "wheeling formula" of [BGRT].

Proof of Proposition 5.4. Let $\mathcal{A}^{xy}$ be the space of "planted forests" whose leaves are labeled ∂_x and ∂_y, modulo the usual *STU* (and hence *AS* and *IHX*) relations. A planted forest is simply a forest in which the roots of the trees are "planted" along a directed line:



$\mathcal{A}^{xy}$ is similar in nature to $\mathcal{A}$; in particular, it is an algebra by the juxtaposition product $\times_A$ and it is graded, and hence an exponential $\exp_A$ and a logarithm $\log_A$ can be defined on it using power series.

Let ξ and η be the elements  and  of $\mathcal{A}^{xy}$, respectively. Clearly,

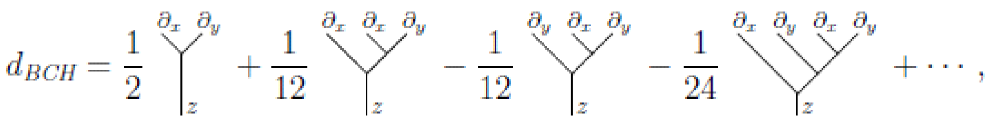
$$\overrightarrow{m}^{xy}_z \chi_{n+1} C = \langle (\exp_A \xi) \times_A (\exp_A \eta), C\rangle_{x,y} = \langle \exp_A \log_A \left((\exp_A \xi) \times_A (\exp_A \eta)\right), C\rangle_{x,y}.$$

But $\log_A((\exp_A \xi) \times_A (\exp_A \eta))$ can be evaluated using the BCH formula (9). The result is $\chi^{xy}_z \Lambda$, where Λ was defined in equation (10) and $\chi^{xy}_z : \mathcal{B}^{xy}_z \to \mathcal{A}^{xy}$ is the natural isomorphism (whose inverse is σ^{xy}_z) of the space $\mathcal{B}^{xy}_z$ of forests with trees as in equation (10) (modulo *AS* and *IHX*) and the space $\mathcal{A}^{xy}$. Therefore $\overrightarrow{m}^{xy}_z \chi_{n+1} C = \langle \exp_A \chi^{xy}_z \Lambda, C\rangle_{x,y}$, and hence

$$\sigma_n \overrightarrow{m}^{xy}_z \chi_{n+1} C = \langle \sigma^{xy}_z \exp_A \chi^{xy}_z \Lambda, C\rangle_{x,y}. \tag{12}$$

The only thing left to note is that Λ is a sum of *trees*, namely forests in which z appears only once. On such forests $\exp_A \circ \chi^{xy}_z = \chi^{xy}_z \circ \exp_\cup$, and we see that equation 12 proves equation (11). □

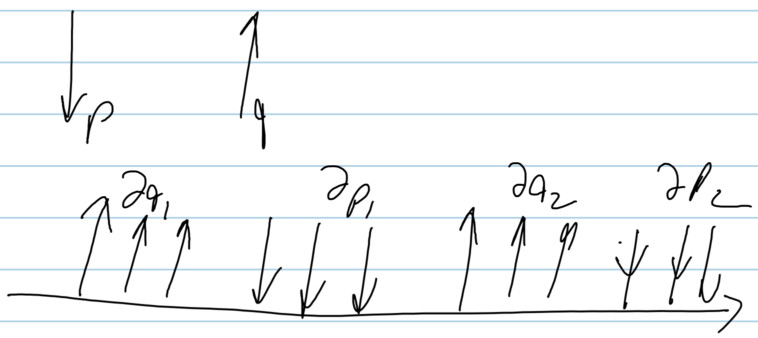
Corollary 5.5. *(Compare with [Å-I, Exercise 1.7]) Let d_{BCH} be Λ with the first two terms removed,*



and let $D_{BCH} = \exp_\cup d_{BCH}$. Then

$$\sigma_n \overrightarrow{m}^{xy}_z \chi_{n+1} C = (D_{BCH} \flat C)/(x, y \to z).$$

Proof. Gluing the exponentials of the struts $|^{\partial_x}_z$ and $|^{\partial_y}_z$ is equivalent to applying the change of variables $(x, y \to z)$. □



$$\log\left(e^{\partial q_1} e^{\partial p_1} e^{\partial q_2} e^{\partial p_2}\right) = \ldots$$