

Pensieve Header: Some symbolic computations in U(I2D), with conventions following the Chicago ax+b handout of <http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/>

**The 2D Lie Algebra.** Let  $\mathfrak{g} = \text{lie}(x^1, x^2)/[x^1, x^2] = x^2$ , let  $\mathfrak{g}^* = \langle \phi_1, \phi_2 \rangle$  with  $\phi_i(x^j) = \delta_i^j$ , let  $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$  so  $[\phi_i, \phi_j] = [\phi_1, x^i] = 0$  while  $[x^1, \phi_2] = -\phi_2$  and  $[x^2, \phi_2] = \phi_1$ . Let  $r = Id = \phi_1 \otimes x^1 + \phi_2 \otimes x^2 \in \mathfrak{g}^* \otimes \mathfrak{g} \subset I\mathfrak{g} \otimes I\mathfrak{g}$ . Let  $\mathcal{U} = \{\text{words in } I\mathfrak{g}\}/ab - ba = [a, b]$ , degree-completed with respect to  $\deg \phi_i = 1$  and  $\deg x^i = 0$  (so  $\mathcal{U} \equiv$  (power series is 4 variables)). Let  $R = \exp(r) \in \mathcal{U} \otimes \mathcal{U}$ .

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kcount = 0;
rules = {
  u[lft___, 1, rgt___] => u[lft, rgt],
  s_Symbol^n_Integer /; n > 1 => Sequence @@ Table[s, {n}],
  u[lft___, y, x, rgt___] => u[lft, x, y, rgt] - u[lft, y, rgt],
  u[lft___, y^n_, x^m_, rgt___] /; !NumberQ[n] || !NumberQ[m] =>
  ( ( Sum[(-n)^tk Binomial[m, tk] u[lft, x^-tk+m, y^n, rgt] ] /. tk -> k[+kcount] ),
  u[lft___, py, px, rgt___] => u[lft, px, py, rgt],
  u[lft___, py^n_, px^m_, rgt___] /; !NumberQ[n] || !NumberQ[m] =>
  u[lft, px^m, py^n, rgt],
  u[lft___, x, px, rgt___] => u[lft, px, x, rgt],
  u[lft___, x^n_, px^m_, rgt___] /; !NumberQ[n] || !NumberQ[m] =>
  u[lft, px^m, x^n, rgt],
  u[lft___, y, px, rgt___] => u[lft, px, y, rgt],
  u[lft___, y^n_, px^m_, rgt___] /; !NumberQ[n] || !NumberQ[m] =>
  u[lft, px^m, y^n, rgt],
  u[lft___, x, py, rgt___] => u[lft, py, x, rgt] - u[lft, py, rgt],
  u[lft___, x^n_, py^m_, rgt___] /; !NumberQ[n] || !NumberQ[m] =>
  ( ( Sum[(-m)^tk Binomial[n, tk] u[lft, py^m, x^-tk+n, rgt] ] /. tk -> k[+kcount] ),
  u[lft___, y, py, rgt___] => u[lft, py, y, rgt] + u[lft, px, rgt],
  u[lft___, y^n_, py^m_, rgt___] /; !NumberQ[n] || !NumberQ[m] =>
  ( ( Sum[Binomial[n, tk] Binomial[m, tk] tk! u[lft, px^tk, py^m-tk, y^n-tk, rgt] ] /.
  tk -> k[+kcount] )
};

u[y^5, x^3] //. rules // Expand
-125 u[y, y, y, y, y] + 75 u[x, y, y, y, y] -
15 u[x, x, y, y, y] + u[x, x, x, y, y, y, y]

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**u[y^n, x^m] // rules // Expand**

$$\sum_{k[1]=0}^m (-n)^{k[1]} \text{Binomial}[m, k[1]] u[x^{m-k[1]}, y^n]$$

**(u[y^n, x^m] // rules // Expand) /. {n -> 5, m -> 3}**

$$-125 u[1, y^5] + 75 u[x, y^5] - 15 u[x^2, y^5] + u[x^3, y^5]$$

**u[x^4, py^6] // rules // Expand**

$$1296 u[py, py, py, py, py, py] - 864 u[py, py, py, py, py, py, x] + 216 u[py, py, py, py, py, py, x, x] - 24 u[py, py, py, py, py, py, x, x, x] + u[py, py, py, py, py, py, x, x, x, x]$$

**u[x^n, py^m] // rules // Expand**

$$\sum_{k[3]=0}^n (-m)^{k[3]} \text{Binomial}[n, k[3]] u[py^m, x^{n-k[3]}]$$

**(u[x^n, py^m] // rules // Expand) /. {n -> 4, m -> 6}**

$$1296 u[py^6, 1] - 864 u[py^6, x] + 216 u[py^6, x^2] - 24 u[py^6, x^3] + u[py^6, x^4]$$

**u[y^4, py^6] // rules // Expand**

$$360 u[px, px, px, px, py, py] + 480 u[px, px, px, py, py, py, y] + 180 u[px, px, py, py, py, py, y, y] + 24 u[px, py, py, py, py, py, y, y, y] + u[py, py, py, py, py, py, y, y, y, y]$$

**(u[y^n, py^m] // rules // Expand) /. {n -> 4, m -> 6}**

$$u[1, py^6, y^4] + 24 u[px, py^5, y^3] + 180 u[px^2, py^4, y^2] + 480 u[px^3, py^3, y] + 360 u[px^4, py^2, 1]$$

**u[y^6, py^4] // rules // Expand**

$$360 u[px, px, px, px, y, y] + 480 u[px, px, px, py, y, y, y] + 180 u[px, px, py, py, y, y, y, y] + 24 u[px, py, py, py, y, y, y, y, y] + u[py, py, py, py, y, y, y, y, y, y]$$

**(u[y^n, py^m] // rules // Expand) /. {n -> 6, m -> 4}**

$$u[1, py^4, y^6] + 24 u[px, py^3, y^5] + 180 u[px^2, py^2, y^4] + 480 u[px^3, py, y^3] + 360 u[px^4, 1, y^2]$$

**u[y^5, x^6, py^3, px^4] //. rules // Expand**

937500 u[px, px, px, px, px, px, px, px, y, y] - 1125000 u[px, px, px, px, px, px, px, px, x, y, y] +  
 2799360 u[px, px, px, px, px, px, px, py, y, y, y] +  
 562500 u[px, px, px, px, px, px, px, x, x, y, y] -  
 2799360 u[px, px, px, px, px, px, px, py, x, y, y, y] +  
 1764735 u[px, px, px, px, px, px, py, py, y, y, y, y] -  
 150000 u[px, px, px, px, px, px, px, x, x, x, y, y] +  
 1166400 u[px, px, px, px, px, px, px, py, x, x, y, y, y] -  
 1512630 u[px, px, px, px, px, px, py, py, x, y, y, y, y] +  
 262144 u[px, px, px, px, py, py, py, y, y, y, y, y] +  
 22500 u[px, px, px, px, px, px, px, x, x, x, x, y, y] -  
 259200 u[px, px, px, px, px, px, py, x, x, x, y, y, y] +  
 540225 u[px, px, px, px, px, py, py, x, x, y, y, y, y] -  
 196608 u[px, px, px, px, py, py, py, x, y, y, y, y, y] -  
 1800 u[px, px, px, px, px, px, px, x, x, x, x, x, y, y] +  
 32400 u[px, px, px, px, px, px, py, x, x, x, x, x, y, y, y] -  
 102900 u[px, px, px, px, px, py, py, x, x, x, y, y, y, y] +  
 61440 u[px, px, px, px, py, py, py, x, x, y, y, y, y, y] +  
 60 u[px, px, px, px, px, px, px, x, x, x, x, x, x, y, y] -  
 2160 u[px, px, px, px, px, px, py, x, x, x, x, x, x, y, y, y] +  
 11025 u[px, px, px, px, px, py, py, x, x, x, x, x, y, y, y, y] -  
 10240 u[px, px, px, px, py, py, py, x, x, x, y, y, y, y, y] +  
 60 u[px, px, px, px, px, px, py, x, x, x, x, x, x, y, y, y] -  
 630 u[px, px, px, px, px, py, py, x, x, x, x, x, x, y, y, y, y] +  
 960 u[px, px, px, px, py, py, py, x, x, x, x, x, y, y, y, y, y] +  
 15 u[px, px, px, px, px, py, py, x, x, x, x, x, x, x, y, y, y, y] -  
 48 u[px, px, px, px, py, py, py, x, x, x, x, x, x, y, y, y, y, y] +  
 u[px, px, px, px, py, py, py, x, x, x, x, x, x, x, y, y, y, y, y]

**(u[y^n1, x^n2, px^n3] //. rules // Expand)**

$$\sum_{k[7]=0}^{n2} (-n1)^{k[7]} \text{Binomial}[n2, k[7]] u[px^{n3}, x^{n2-k[7]}, y^{n1}]$$

**(u[y^n1, x^n2, y^n3] //. rules // Expand)**

$$\sum_{k[8]=0}^{n2} (-n1)^{k[8]} \text{Binomial}[n2, k[8]] u[x^{n2-k[8]}, y^{n1}, y^{n3}]$$

**(u[x^n2, py^n3] //. rules // Expand)**

$$\sum_{k[9]=0}^{n2} (-n3)^{k[9]} \text{Binomial}[n2, k[9]] u[py^{n3}, x^{n2-k[9]}]$$

**(u[y^n1, x^n2, py^3] /. rules // Expand)**

$$\sum_{k[12]=0}^{n2} (-n1)^{k[12]} \text{Binomial}[n2, k[12]] u[x^{n2-k[12]}, y^{n1}, py^3]$$

$$\sum_{k[12]=0}^{n2} (-n1)^{k[12]} \text{Binomial}[n2, k[12]] u[x^{n2-k[12]}, y^{n1}, py^3] /. rules$$

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u[xn2-k[12], yn1, py3] /. rules

$$\sum_{k[15]=0}^{\text{Min}[3, n1]} \text{Binomial}[3, k[15]] \text{Binomial}[n1, k[15]] k[15]! u[x^{n2-k[12]}, px^{k[15]}, py^{3-k[15]}, y^{n1-k[15]}]$$


$$\sum_{k[12]=0}^{n2} (-n1)^{k[12]} \text{Binomial}[n2, k[12]] /. rules$$

(1 - n1)n2

$$\sum_{k[12]=0}^{n2} (-n1)^{k[12]} \text{Binomial}[n2, k[12]]$$

(1 - n1)n2

$$\sum_{k[12]=0}^{n2} (-n1)^{k[12]} \text{Bin}[n2, k[12]] u[x^{n2-k[12]}, y^{n1}, py^3] /. rules$$

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kcount
11
(u[yn1, xn2, pyn3] //. rules // Expand)
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(u[yn1, xn2, pyn3, pxn4] //. rules // Expand)

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