

Pensieve Header: The U(I2D) program, extending U(I2D)-PolyTime.nb

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I2DRules = {
  u[lft___, 1, rgt___] :=> u[lft, rgt],
  u[lft___, s_n·, s_m·, rgt___] :=> u[lft, sn+m, rgt],
  u[lft___, yn·, xm·, rgt___] :=> Sum[(-n)k Binomial[m, k] u[lft, x-k+m, yn, rgt],
    {k, 0, m}],
  u[lft___, pyn·, pxm·, rgt___] :=> u[lft, pxm, pyn, rgt],
  u[lft___, xn·, pxm·, rgt___] :=> u[lft, pxm, xn, rgt],
  u[lft___, xn·, pym·, rgt___] :=> Sum[(-m)k Binomial[n, k] u[lft, pym, x-k+n, rgt],
    {k, 0, n}],
  u[lft___, yn·, pxm·, rgt___] :=> u[lft, pxm, yn, rgt],
  u[lft___, yn·, pym·, rgt___] :=>
    Sum[Binomial[n, k] Binomial[m, k] k! u[lft, pxk, pym-k, yn-k, rgt],
      {k, 0, Min[m, n]}],
};

If[Head[$DegreeStack] != List, $DegreeStack = {Infinity}];
$ModDegree = First[$DegreeStack];
SetAttributes[ModDegree, HoldRest];
ModDegree[m_, expr_] := Module[{res},
  PrependTo[$DegreeStack, $ModDegree = m];
  res = expr;
  $DegreeStack = Rest[$DegreeStack];
  $ModDegree = First[$DegreeStack];
  res
];
Deg[t_T] := Plus @@ Take[Plus @@ t, 2];

Outer[u[#1, #2] - u[#2, #1] &, {px, py, x, y}, {px, py, x, y}] // . I2DRules // Expand //
MatrixForm
(
  (
    0 0 0 0
    0 0 u[py] -u[px]
    0 -u[py] 0 u[y]
    0 u[px] -u[y] 0
  )
)

u[y^3, x, x, py^2, px^4] // . I2DRules // Expand
54 u[px^6, y] + 25 u[px^4, py^2, y^3] + 96 u[px^5, py, y^2] - 36 u[px^6, x, y] + 6 u[px^6, x^2, y] -
10 u[px^4, py^2, x, y^3] + u[px^4, py^2, x^2, y^3] - 48 u[px^5, py, x, y^2] + 6 u[px^5, py, x^2, y^2]

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Format[t_T, StandardForm] := Module[
  {k, out},
  out = {};
  Do[
    If[t[[k]] === {0, 0, 0, 0},
      AppendTo[out, 1],
      (* Else *)
      If[t[[k, 1]] == 1, AppendTo[out, "ξ"];
      If[t[[k, 1]] > 1, AppendTo[out, SuperscriptBox["ξ", t[[k, 1]]]];
      If[t[[k, 2]] == 1, AppendTo[out, "η"];
      If[t[[k, 2]] > 1, AppendTo[out, SuperscriptBox["η", t[[k, 2]]]];
      If[t[[k, 3]] == 1, AppendTo[out, "x"];
      If[t[[k, 3]] > 1, AppendTo[out, SuperscriptBox["x", t[[k, 3]]]];
      If[t[[k, 4]] == 1, AppendTo[out, "y"];
      If[t[[k, 4]] > 1, AppendTo[out, SuperscriptBox["y", t[[k, 4]]]];
    ];
    If[k < Length[t], AppendTo[out, "@"],
      {k, Length[t]}
    ];
  ];
  DisplayForm[RowBox[out]]
]

Unprotect[NonCommutativeMultiply];
0 ** _ = 0;
_ ** 0 = 0;
(c_?NumberQ * a_) ** b_ := Expand[c * (a ** b)];
a_ ** (c_?NumberQ * b_) := Expand[c * (a ** b)];
a_Plus ** b_ := (# ** b) & /@ a;
a_ ** b_Plus := (a ** #) & /@ b;
T[{m1_, m2_, m3_, m4_}] ** T[{n1_, n2_, n3_, n4_}] /;
  m1 + m2 + n1 + n2 ≥ $ModDegree := 0;
T[{m1_, m2_, m3_, m4_}] ** T[{n1_, n2_, n3_, n4_}] := Expand[
  u[px^m1, py^m2, x^m3, y^m4, px^n1, py^n2, x^n3, y^n4] // . I2DRules
] /. ut_u => T[Exponent[Times @@ ut, #] & /@ {px, py, x, y}];
T[t1_] ** T[t2_] /; Deg[T[t1]] + Deg[T[t2]] ≥ $ModDegree := 0;
T[t1_] ** T[t2_] := Module[
  {cc, tt},
  Distribute[
    OT @@ MapThread[T[#1] ** T[#2] &, {{t1}, {t2}}]
  ] /. ot_OT => (
    cc = 1; tt = T[];
    Replace[ot, c_. * t_T => (cc *= c; AppendTo[tt, t];), {1}];
    cc * Flatten[tt]
  )
]

T[{2, 1, 3, 4}] ** T[{1, 1, 0, 1}]
- (ξ³ η² y⁵) + 3 (ξ³ η² x y⁵) - 3 (ξ³ η² x² y⁵) + ξ³ η² x³ y⁵ + 4 (ξ⁴ η x³ y⁴)

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T[{2, 1, 3, 4}, {0, 0, 0, 1}] ** T[{1, 1, 0, 1}, {0, 0, 1, 0}]

$$\xi^3 \eta^2 y^5 \otimes y - \xi^3 \eta^2 y^5 \otimes x y - 3 (\xi^3 \eta^2 x y^5 \otimes y) + 3 (\xi^3 \eta^2 x y^5 \otimes x y) + 3 (\xi^3 \eta^2 x^2 y^5 \otimes y) -$$


$$3 (\xi^3 \eta^2 x^2 y^5 \otimes x y) - \xi^3 \eta^2 x^3 y^5 \otimes y + \xi^3 \eta^2 x^3 y^5 \otimes x y - 4 (\xi^4 \eta x^3 y^4 \otimes y) + 4 (\xi^4 \eta x^3 y^4 \otimes x y)$$

TPower[expr_, p_Integer] /; p > 0 := NonCommutativeMultiply @@ Table[expr, {p}];
TExp[n_Integer, expr_] := Module[
  {total, term, k},
  k = 0;
  total = term = T@@Table[{0, 0, 0, 0}, {n}];
  While[term != 0,
    ++k;
    total += (term = Expand[term ** expr / k])
  ];
  total
];
TExp[expr_] := Module[
  {n},
  {n} = Cases[expr, t_T :-> Length[t], Infinity, 1];
  TExp[n, expr]
];
ModDegree[7, TExp[1, T[{0, 1, 1, 0}]]]

$$1 + \eta x - \frac{\eta^2 x}{2} + \frac{1}{2} (\eta^2 x^2) + \frac{\eta^3 x}{3} - \frac{1}{2} (\eta^3 x^2) + \frac{1}{6} (\eta^3 x^3) - \frac{\eta^4 x}{4} + \frac{11}{24} (\eta^4 x^2) -$$


$$\frac{1}{4} (\eta^4 x^3) + \frac{1}{24} (\eta^4 x^4) + \frac{\eta^5 x}{5} - \frac{5}{12} (\eta^5 x^2) + \frac{7}{24} (\eta^5 x^3) - \frac{1}{12} (\eta^5 x^4) + \frac{1}{120} (\eta^5 x^5) -$$


$$\frac{\eta^6 x}{6} + \frac{137}{360} (\eta^6 x^2) - \frac{5}{16} (\eta^6 x^3) + \frac{17}{144} (\eta^6 x^4) - \frac{1}{48} (\eta^6 x^5) + \frac{1}{720} (\eta^6 x^6)$$

PutOn[s_Integer, ind_List, T[{0, 0, 0, 0} ...]] := T @@ Table[{0, 0, 0, 0}, {s}];
PutOn[s_Integer, ind_List, t_T] := Module[
  {k, indk, j, base, rest},
  base = T @@ Table[{0, 0, 0, 0}, {s}];
  {{k, j}} = Position[t, p_Integer /; p > 0, {2}, 1];
  indk = Replace[ind[[k]], i_Integer :-> {i}];
  rest = t;
  --rest[[k, j]];
  rest = PutOn[s, ind, rest];
  Sum[
    ReplacePart[base, {indk[[i]], j} -> 1] ** rest,
    {i, Length[indk]}
  ]
];
PutOn[s_Integer, ind_List, expr_] := Expand[expr /. t_T :-> PutOn[s, ind, t]];
PutOn[3, {{1, 3}}, T[{0, 1, 1, 0}]]

$$1 \otimes 1 \otimes \eta x + x \otimes 1 \otimes \eta + \eta \otimes 1 \otimes x + \eta x \otimes 1 \otimes 1$$


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OrderedPartitions[n_, 1] := {{n}};
OrderedPartitions[n_Integer, p_Integer] /; p > 1 :=
  OrderedPartitions[n, p] = Join @@ Table[
    Append[#, k] & /@ OrderedPartitions[n - k, p - 1],
    {k, 0, n}
  ];
Basis[n_, d_] := Flatten[Outer[
  Function[{p1, p2},
    T @@ MapThread[
      Join,
      {Partition[p1, 2], Partition[p2, 2]}
    ]
  ],
  OrderedPartitions[d, 2 n],
  OrderedPartitions[d, 2 n + 1],
  1
]]

OrderedPartitions[5, 3]
{{5, 0, 0}, {4, 1, 0}, {3, 2, 0}, {2, 3, 0}, {1, 4, 0}, {0, 5, 0}, {4, 0, 1},
 {3, 1, 1}, {2, 2, 1}, {1, 3, 1}, {0, 4, 1}, {3, 0, 2}, {2, 1, 2}, {1, 2, 2},
 {0, 3, 2}, {2, 0, 3}, {1, 1, 3}, {0, 2, 3}, {1, 0, 4}, {0, 1, 4}, {0, 0, 5}}

Partition[Range[9], 2]
{{1, 2}, {3, 4}, {5, 6}, {7, 8}}

Basis[2, 2]
{ $\xi^2 x^2 \otimes 1$ ,  $\xi^2 xy \otimes 1$ ,  $\xi^2 y^2 \otimes 1$ ,  $\xi^2 x \otimes x$ ,  $\xi^2 y \otimes x$ ,  $\xi^2 x \otimes x^2$ ,  $\xi^2 x \otimes y$ ,  $\xi^2 y \otimes y$ ,  $\xi^2 x \otimes xy$ ,  $\xi^2 \otimes y^2$ ,
 $\xi^2 x \otimes 1$ ,  $\xi^2 y \otimes 1$ ,  $\xi^2 \otimes x$ ,  $\xi^2 \otimes y$ ,  $\xi^2 \otimes 1$ ,  $\xi \eta x^2 \otimes 1$ ,  $\xi \eta xy \otimes 1$ ,  $\xi \eta y^2 \otimes 1$ ,  $\xi \eta x \otimes x$ ,  $\xi \eta y \otimes x$ ,
 $\xi \eta \otimes x^2$ ,  $\xi \eta x \otimes y$ ,  $\xi \eta y \otimes y$ ,  $\xi \eta \otimes xy$ ,  $\xi \eta \otimes y^2$ ,  $\xi \eta x \otimes 1$ ,  $\xi \eta y \otimes 1$ ,  $\xi \eta \otimes x$ ,  $\xi \eta \otimes y$ ,  $\xi \eta \otimes 1$ ,
 $\eta^2 x^2 \otimes 1$ ,  $\eta^2 xy \otimes 1$ ,  $\eta^2 y^2 \otimes 1$ ,  $\eta^2 x \otimes x$ ,  $\eta^2 y \otimes x$ ,  $\eta^2 x \otimes x^2$ ,  $\eta^2 x \otimes y$ ,  $\eta^2 y \otimes y$ ,  $\eta^2 \otimes xy$ ,  $\eta^2 \otimes y^2$ ,
 $\eta^2 x \otimes 1$ ,  $\eta^2 y \otimes 1$ ,  $\eta^2 \otimes x$ ,  $\eta^2 \otimes y$ ,  $\eta^2 \otimes 1$ ,  $\xi x^2 \otimes \xi$ ,  $\xi xy \otimes \xi$ ,  $\xi y^2 \otimes \xi$ ,  $\xi x \otimes \xi x$ ,  $\xi y \otimes \xi x$ ,
 $\xi \otimes \xi x^2$ ,  $\xi x \otimes \xi y$ ,  $\xi y \otimes \xi y$ ,  $\xi \otimes \xi xy$ ,  $\xi \otimes \xi y^2$ ,  $\xi x \otimes \xi$ ,  $\xi y \otimes \xi$ ,  $\xi \otimes \xi x$ ,  $\xi \otimes \xi y$ ,  $\xi \otimes \xi$ ,
 $\eta x^2 \otimes \xi$ ,  $\eta xy \otimes \xi$ ,  $\eta y^2 \otimes \xi$ ,  $\eta x \otimes \xi x$ ,  $\eta y \otimes \xi x$ ,  $\eta \otimes \xi x^2$ ,  $\eta x \otimes \xi y$ ,  $\eta y \otimes \xi y$ ,  $\eta \otimes \xi xy$ ,  $\eta \otimes \xi y^2$ ,
 $\eta x \otimes \xi$ ,  $\eta y \otimes \xi$ ,  $\eta \otimes \xi x$ ,  $\eta \otimes \xi y$ ,  $\eta \otimes \xi$ ,  $x^2 \otimes \xi^2$ ,  $xy \otimes \xi^2$ ,  $y^2 \otimes \xi^2$ ,  $x \otimes \xi^2 x$ ,  $y \otimes \xi^2 x$ ,  $1 \otimes \xi^2 x^2$ ,
 $x \otimes \xi^2 y$ ,  $y \otimes \xi^2 y$ ,  $1 \otimes \xi^2 xy$ ,  $1 \otimes \xi^2 y^2$ ,  $x \otimes \xi^2$ ,  $y \otimes \xi^2$ ,  $1 \otimes \xi^2 x$ ,  $1 \otimes \xi^2 y$ ,  $1 \otimes \xi^2$ ,  $\xi x^2 \otimes \eta$ ,
 $\xi xy \otimes \eta$ ,  $\xi y^2 \otimes \eta$ ,  $\xi x \otimes \eta x$ ,  $\xi y \otimes \eta x$ ,  $\xi \otimes \eta x^2$ ,  $\xi x \otimes \eta y$ ,  $\xi y \otimes \eta y$ ,  $\xi \otimes \eta xy$ ,  $\xi \otimes \eta y^2$ ,  $\xi x \otimes \eta$ ,
 $\xi y \otimes \eta$ ,  $\xi \otimes \eta x$ ,  $\xi \otimes \eta y$ ,  $\xi \otimes \eta$ ,  $\eta x^2 \otimes \eta$ ,  $\eta xy \otimes \eta$ ,  $\eta y^2 \otimes \eta$ ,  $\eta x \otimes \eta x$ ,  $\eta y \otimes \eta x$ ,  $\eta \otimes \eta x^2$ ,
 $\eta x \otimes \eta y$ ,  $\eta y \otimes \eta y$ ,  $\eta \otimes \eta xy$ ,  $\eta \otimes \eta y^2$ ,  $\eta x \otimes \eta$ ,  $\eta y \otimes \eta$ ,  $\eta \otimes \eta x$ ,  $\eta \otimes \eta y$ ,  $\eta \otimes \eta$ ,  $x^2 \otimes \xi \eta$ ,
 $xy \otimes \xi \eta$ ,  $y^2 \otimes \xi \eta$ ,  $x \otimes \xi \eta x$ ,  $y \otimes \xi \eta x$ ,  $1 \otimes \xi \eta x^2$ ,  $x \otimes \xi \eta y$ ,  $y \otimes \xi \eta y$ ,  $1 \otimes \xi \eta xy$ ,  $1 \otimes \xi \eta y^2$ ,
 $x \otimes \xi \eta$ ,  $y \otimes \xi \eta$ ,  $1 \otimes \xi \eta x$ ,  $1 \otimes \xi \eta y$ ,  $1 \otimes \xi \eta$ ,  $x^2 \otimes \eta^2$ ,  $xy \otimes \eta^2$ ,  $y^2 \otimes \eta^2$ ,  $x \otimes \eta^2 x$ ,  $y \otimes \eta^2 x$ ,
 $1 \otimes \eta^2 x^2$ ,  $x \otimes \eta^2 y$ ,  $y \otimes \eta^2 y$ ,  $1 \otimes \eta^2 xy$ ,  $1 \otimes \eta^2 y^2$ ,  $x \otimes \eta^2$ ,  $y \otimes \eta^2$ ,  $1 \otimes \eta^2 x$ ,  $1 \otimes \eta^2 y$ ,  $1 \otimes \eta^2$ }

Basis[2, 5] // Length
7056

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