

Pensieve Header: Poly-time computations in U(I2D), with conventions following the Chicago ax+b handout of <http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/>

The 2D Lie Algebra. Let $\mathfrak{g} = \text{lie}(x^1, x^2)/[x^1, x^2] = x^2$, let $\mathfrak{g}^* = \langle \phi_1, \phi_2 \rangle$ with $\phi_i(x^j) = \delta_i^j$, let $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$ so $[\phi_i, \phi_j] = [\phi_1, x^i] = 0$ while $[x^1, \phi_2] = -\phi_2$ and $[x^2, \phi_2] = \phi_1$. Let $r = Id = \phi_1 \otimes x^1 + \phi_2 \otimes x^2 \in \mathfrak{g}^* \otimes \mathfrak{g} \subset I\mathfrak{g} \otimes I\mathfrak{g}$. Let $\mathcal{U} = \{\text{words in } I\mathfrak{g}\}/ab - ba = [a, b]$, degree-completed with respect to $\deg \phi_i = 1$ and $\deg x^i = 0$ (so $\mathcal{U} \equiv$ (power series in 4 variables)). Let $R = \exp(r) \in \mathcal{U} \otimes \mathcal{U}$.

```
rules = {
  u[lft___, 1, rgt___] => u[lft, rgt],
  u[lft___, s^n_., s^m_., rgt___] => u[lft, s^(n+m), rgt],
  u[lft___, y^n_., x^m_., rgt___] =>
    Sum[k=0, m] (-n)^k Binomial[m, k] u[lft, x^-k+m, y^n, rgt],
  u[lft___, py^n_., px^m_., rgt___] => u[lft, px^m, py^n, rgt],
  u[lft___, x^n_., px^m_., rgt___] => u[lft, px^m, x^n, rgt],
  u[lft___, x^n_., py^m_., rgt___] =>
    Sum[k=0, n] (-m)^k Binomial[n, k] u[lft, py^m, x^-k+n, rgt],
  u[lft___, y^n_., px^m_., rgt___] => u[lft, px^m, y^n, rgt],
  u[lft___, y^n_., py^m_., rgt___] =>
    Sum[k=0, Min[m,n]] Binomial[n, k] Binomial[m, k] k! u[lft, px^k, py^m-k, y^n-k, rgt]
};
```

```
u[y, x] //. rules // Expand
```

```
-u[y] + u[x, y]
```

```
u[y^5, x^3] //. rules // Expand
```

```
-125 u[y^5] + 75 u[x, y^5] - 15 u[x^2, y^5] + u[x^3, y^5]
```

```
u[x^4, py^6] //. rules // Expand
```

```
1296 u[py^6] - 864 u[py^6, x] + 216 u[py^6, x^2] - 24 u[py^6, x^3] + u[py^6, x^4]
```

```
u[y^4, py^6] //. rules // Expand
```

```
360 u[px^4, py^2] + u[py^6, y^4] + 24 u[px, py^5, y^3] + 180 u[px^2, py^4, y^2] + 480 u[px^3, py^3, y]
```

```
u[y^6, py^4] //. rules // Expand
```

```
360 u[px^4, y^2] + u[py^4, y^6] + 24 u[px, py^3, y^5] + 180 u[px^2, py^2, y^4] + 480 u[px^3, py, y^3]
```

u[y^5, x^6, py^3, px^4] // . rules // Expand

$$\begin{aligned}
 & 937\,500 u[px^7, y^2] + 262\,144 u[px^4, py^3, y^5] + 1\,764\,735 u[px^5, py^2, y^4] + \\
 & 2\,799\,360 u[px^6, py, y^3] - 1\,125\,000 u[px^7, x, y^2] + 562\,500 u[px^7, x^2, y^2] - \\
 & 150\,000 u[px^7, x^3, y^2] + 22\,500 u[px^7, x^4, y^2] - 1\,800 u[px^7, x^5, y^2] + 60 u[px^7, x^6, y^2] - \\
 & 196\,608 u[px^4, py^3, x, y^5] + 61\,440 u[px^4, py^3, x^2, y^5] - 10\,240 u[px^4, py^3, x^3, y^5] + \\
 & 960 u[px^4, py^3, x^4, y^5] - 48 u[px^4, py^3, x^5, y^5] + u[px^4, py^3, x^6, y^5] - \\
 & 1\,512\,630 u[px^5, py^2, x, y^4] + 540\,225 u[px^5, py^2, x^2, y^4] - 102\,900 u[px^5, py^2, x^3, y^4] + \\
 & 11\,025 u[px^5, py^2, x^4, y^4] - 630 u[px^5, py^2, x^5, y^4] + 15 u[px^5, py^2, x^6, y^4] - \\
 & 2\,799\,360 u[px^6, py, x, y^3] + 1\,166\,400 u[px^6, py, x^2, y^3] - 259\,200 u[px^6, py, x^3, y^3] + \\
 & 32\,400 u[px^6, py, x^4, y^3] - 2\,160 u[px^6, py, x^5, y^3] + 60 u[px^6, py, x^6, y^3]
 \end{aligned}$$

u[y^5, x^6, py^3, px^4, y, x, py, px] // . rules // Expand

$$\begin{aligned}
 & -8\,437\,500 u[px^9, y^2] - 3\,720\,087 u[px^5, py^4, y^6] - 33\,030\,144 u[px^6, py^3, y^5] - \\
 & 79\,413\,075 u[px^7, py^2, y^4] - 55\,987\,200 u[px^8, py, y^3] + 12\,937\,500 u[px^9, x, y^2] - \\
 & 8\,437\,500 u[px^9, x^2, y^2] + 3\,037\,500 u[px^9, x^3, y^2] - 652\,500 u[px^9, x^4, y^2] + \\
 & 83\,700 u[px^9, x^5, y^2] - 5\,940 u[px^9, x^6, y^2] + 180 u[px^9, x^7, y^2] + \\
 & 3\,011\,499 u[px^5, py^4, x, y^6] - 1\,043\,199 u[px^5, py^4, x^2, y^6] + 200\,475 u[px^5, py^4, x^3, y^6] - \\
 & 23\,085 u[px^5, py^4, x^4, y^6] + 15\,93 u[px^5, py^4, x^5, y^6] - 61 u[px^5, py^4, x^6, y^6] + \\
 & u[px^5, py^4, x^7, y^6] + 30\,277\,632 u[px^6, py^3, x, y^5] - 11\,870\,208 u[px^6, py^3, x^2, y^5] + \\
 & 2\,580\,480 u[px^6, py^3, x^3, y^5] - 336\,000 u[px^6, py^3, x^4, y^5] + 26\,208 u[px^6, py^3, x^5, y^5] - \\
 & 11\,134 u[px^6, py^3, x^6, y^5] + 21 u[px^6, py^3, x^7, y^5] + 83\,950\,965 u[px^7, py^2, x, y^4] - \\
 & 37\,923\,795 u[px^7, py^2, x^2, y^4] + 9\,492\,525 u[px^7, py^2, x^3, y^4] - \\
 & 1\,422\,225 u[px^7, py^2, x^4, y^4] + 127\,575 u[px^7, py^2, x^5, y^4] - 6\,345 u[px^7, py^2, x^6, y^4] + \\
 & 135 u[px^7, py^2, x^7, y^4] + 69\,984\,000 u[px^8, py, x, y^3] - 37\,324\,800 u[px^8, py, x^2, y^3] + \\
 & 11\,016\,000 u[px^8, py, x^3, y^3] - 1\,944\,000 u[px^8, py, x^4, y^3] + \\
 & 205\,200 u[px^8, py, x^5, y^3] - 12\,000 u[px^8, py, x^6, y^3] + 300 u[px^8, py, x^7, y^3]
 \end{aligned}$$

Outer[u[#1, #2] - u[#2, #1] &, {px, py, x, y}, {px, py, x, y}] // . rules // Expand // MatrixForm

$$\begin{pmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & u[py] & -u[px] \\
 0 & -u[py] & 0 & u[y] \\
 0 & u[px] & -u[y] & 0
 \end{pmatrix}$$