Pensieve Header: The playground - a minimalistic Lie quotient of ${\mathcal A}^w$, with a BCH product law.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\w-Computations"];
<< "Playground Program.m"
```

```
{ar[1, 2] ** ar[1, 3], ar[1, 3] ** ar[1, 2]}
```

$\{ar[1, 2] + ar[1, 3], ar[1, 2] + ar[1, 3]\}$

```
ToDegree[4, R[1, 3] ** R[2, 3]] // PCollect
```

$ar[2, 3] \left( h + \frac{1}{2} h^2 h[1] + \frac{1}{12} h^3 h[1]^2 - \frac{1}{12} h^3 h[1] h[2] - \frac{1}{24} h^4 h[1]^2 h[2] \right) +$

$ar[1, 3] \left( h - \frac{1}{2} h^2 h[2] - \frac{1}{12} h^3 h[1] h[2] + \frac{1}{12} h^3 h[2]^2 + \frac{1}{24} h^4 h[1] h[2]^2 \right)$

```
ToDegree[4, R[1, 2] ** R[1, 3] ** R[2, 3]] // PCollect
```

$h \, ar[1, 2] + h \, ar[1, 3] + h \, ar[2, 3]$

```
ToDegree[4, R[1, 2] ** R[1, 3] ** R[2, 3] - R[2, 3] ** R[1, 3] ** R[1, 2]]
```

$0$

```
R[1, 2] ~ Up ~ {{1, 2}, 3}
```

$h \, ar[1, 3] + h \, ar[2, 3]$

$$fb = -\frac{1}{4} + h\left(\frac{x}{32} + \frac{5y}{96}\right) + \frac{h^3\left(-57x^3 - 169x^2y - 187xy^2 - 99y^3\right)}{138\,240} +$$

$$\frac{1}{69\,672\,960} h^5\left(555x^5 + 2558x^4y + 5000x^3y^2 + 5078x^2y^3 + 2669xy^4 + 780y^5\right) +$$

$$\frac{1}{200\,658\,124\,800} h^7\left(-31\,989x^7 - 205\,357x^6y - 592\,097x^5y^2 -\right.$$
$$\left. 964\,985x^4y^3 - 949\,967x^3y^4 - 565\,943x^2y^5 - 186\,139xy^6 - 34\,275y^7\right);$$

```
Vb = PCollect[
  (fb /. {x → h[2], y → h[1]}) h * ar[2, 1] +
  (1 / 2 + fb /. {x → h[1], y → h[2]}) h * ar[1, 2]
  + 0 h (ar[1, 1] + ar[2, 2])
  + 1 / 96 h^2 (h[1] ar[2, 2] + h[2] ar[1, 1]) + 0 h^2 (h[1] ar[1, 1] + h[2] ar[2, 2])
 ]
```

$$\frac{1}{96} h^2 \, ar[2, 2] \, h[1] + \frac{1}{96} h^2 \, ar[1, 1] \, h[2] +$$

$$ar[1, 2]\left(\frac{h}{4} + \frac{1}{32} h^2 h[1] - \frac{19 h^4 h[1]^3}{46\,080} + \frac{37 h^6 h[1]^5}{4\,644\,864} - \frac{10\,663 h^8 h[1]^7}{66\,886\,041\,600} + \frac{5}{96} h^2 h[2] -\right.$$
$$\frac{169 h^4 h[1]^2 h[2]}{138\,240} + \frac{1279 h^6 h[1]^4 h[2]}{34\,836\,480} - \frac{205\,357 h^8 h[1]^6 h[2]}{200\,658\,124\,800} - \frac{187 h^4 h[1] h[2]^2}{138\,240} +$$
$$\frac{125 h^6 h[1]^3 h[2]^2}{1\,741\,824} - \frac{592\,097 h^8 h[1]^5 h[2]^2}{200\,658\,124\,800} - \frac{11 h^4 h[2]^3}{15\,360} + \frac{2539 h^6 h[1]^2 h[2]^3}{34\,836\,480} -$$
$$\frac{27\,571 h^8 h[1]^4 h[2]^3}{5\,733\,089\,280} + \frac{2669 h^6 h[1] h[2]^4}{69\,672\,960} - \frac{949\,967 h^8 h[1]^3 h[2]^4}{200\,658\,124\,800} +$$
$$\left.\frac{13 h^6 h[2]^5}{1\,161\,216} - \frac{80\,849 h^8 h[1]^2 h[2]^5}{28\,665\,446\,400} - \frac{186\,139 h^8 h[1] h[2]^6}{200\,658\,124\,800} - \frac{457 h^8 h[2]^7}{2\,675\,441\,664}\right) +$$

$$ar[2, 1]\left(-\frac{h}{4} + \frac{5}{96} h^2 h[1] - \frac{11 h^4 h[1]^3}{15\,360} + \frac{13 h^6 h[1]^5}{1\,161\,216} - \frac{457 h^8 h[1]^7}{2\,675\,441\,664} + \frac{1}{32} h^2 h[2] -\right.$$
$$\frac{187 h^4 h[1]^2 h[2]}{138\,240} + \frac{2669 h^6 h[1]^4 h[2]}{69\,672\,960} - \frac{186\,139 h^8 h[1]^6 h[2]}{200\,658\,124\,800} - \frac{169 h^4 h[1] h[2]^2}{138\,240} +$$
$$\frac{2539 h^6 h[1]^3 h[2]^2}{34\,836\,480} - \frac{80\,849 h^8 h[1]^5 h[2]^2}{28\,665\,446\,400} - \frac{19 h^4 h[2]^3}{46\,080} + \frac{125 h^6 h[1]^2 h[2]^3}{1\,741\,824} -$$
$$\frac{949\,967 h^8 h[1]^4 h[2]^3}{200\,658\,124\,800} + \frac{1279 h^6 h[1] h[2]^4}{34\,836\,480} - \frac{27\,571 h^8 h[1]^3 h[2]^4}{5\,733\,089\,280} +$$
$$\left.\frac{37 h^6 h[2]^5}{4\,644\,864} - \frac{592\,097 h^8 h[1]^2 h[2]^5}{200\,658\,124\,800} - \frac{205\,357 h^8 h[1] h[2]^6}{200\,658\,124\,800} - \frac{10\,663 h^8 h[2]^7}{66\,886\,041\,600}\right)$$

```
ToDegree[2, Vb]
```

$$\frac{1}{96} h^2 \, ar[2, 2] \, h[1] + \frac{1}{96} h^2 \, ar[1, 1] \, h[2] +$$

$$ar[2, 1]\left(-\frac{h}{4} + \frac{5}{96} h^2 h[1] + \frac{1}{32} h^2 h[2]\right) + ar[1, 2]\left(\frac{h}{4} + \frac{1}{32} h^2 h[1] + \frac{5}{96} h^2 h[2]\right)$$

```
ToDegree[4, Vb ** (R[1, 3] + R[2, 3])]
```

$$\frac{1}{4} h\, ar[1, 2] + h\, ar[1, 3] - \frac{1}{4} h\, ar[2, 1] + h\, ar[2, 3] + \frac{1}{32} h^2\, ar[1, 2]\, h[1] +$$

$$\frac{5}{96} h^2\, ar[2, 1]\, h[1] + \frac{1}{96} h^2\, ar[2, 2]\, h[1] - \frac{1}{4} h^2\, ar[2, 3]\, h[1] + \frac{1}{16} h^3\, ar[2, 3]\, h[1]^2 -$$

$$\frac{19\, h^4\, ar[1, 2]\, h[1]^3}{46\,080} - \frac{11\, h^4\, ar[2, 1]\, h[1]^3}{15\,360} - \frac{7\, h^4\, ar[2, 3]\, h[1]^3}{1152} + \frac{1}{96} h^2\, ar[1, 1]\, h[2] +$$

$$\frac{5}{96} h^2\, ar[1, 2]\, h[2] + \frac{1}{4} h^2\, ar[1, 3]\, h[2] + \frac{1}{32} h^2\, ar[2, 1]\, h[2] - \frac{1}{16} h^3\, ar[1, 3]\, h[1]\, h[2] +$$

$$\frac{1}{48} h^3\, ar[2, 3]\, h[1]\, h[2] - \frac{169\, h^4\, ar[1, 2]\, h[1]^2\, h[2]}{138\,240} + \frac{7\, h^4\, ar[1, 3]\, h[1]^2\, h[2]}{1152} -$$

$$\frac{187\, h^4\, ar[2, 1]\, h[1]^2\, h[2]}{138\,240} + \frac{1}{192} h^4\, ar[2, 3]\, h[1]^2\, h[2] - \frac{1}{48} h^3\, ar[1, 3]\, h[2]^2 -$$

$$\frac{187\, h^4\, ar[1, 2]\, h[1]\, h[2]^2}{138\,240} - \frac{1}{192} h^4\, ar[1, 3]\, h[1]\, h[2]^2 - \frac{169\, h^4\, ar[2, 1]\, h[1]\, h[2]^2}{138\,240} +$$

$$\frac{1}{128} h^4\, ar[2, 3]\, h[1]\, h[2]^2 - \frac{11\, h^4\, ar[1, 2]\, h[2]^3}{15\,360} - \frac{1}{128} h^4\, ar[1, 3]\, h[2]^3 - \frac{19\, h^4\, ar[2, 1]\, h[2]^3}{46\,080}$$

```
ToDegree[4, {R4Eqn[Vb], TwistEqn[Vb], TwistMinusEqn[Vb]}]
```

$\{0, 0, 0\}$

```
ToDegree[2, Phi[Vb]]
```

$$\frac{1}{24} h^2\, ar[2, 3]\, h[1] - \frac{1}{24} h^2\, ar[3, 2]\, h[1] - \frac{1}{24} h^2\, ar[1, 3]\, h[2] +$$

$$\frac{1}{24} h^2\, ar[3, 1]\, h[2] + \frac{1}{24} h^2\, ar[1, 2]\, h[3] - \frac{1}{24} h^2\, ar[2, 1]\, h[3]$$

**Phi[4, Vb]**

$\text{ar}[3, 3]\left(\dfrac{h^4\,h[1]^2\,h[2]}{1536} + \dfrac{h^4\,h[1]\,h[2]^2}{1152} + \dfrac{h^4\,h[1]\,h[2]\,h[3]}{2304}\right) +$

$\text{ar}[2, 3]\left(\dfrac{1}{24}\,h^2\,h[1] - \dfrac{31\,h^4\,h[1]^3}{46\,080} - \dfrac{23\,h^4\,h[1]^2\,h[2]}{46\,080} - \dfrac{13\,h^4\,h[1]\,h[2]^2}{46\,080} + \dfrac{19\,h^4\,h[1]^2\,h[3]}{138\,240} - \right.$

$\left.\dfrac{11\,h^4\,h[1]\,h[2]\,h[3]}{69\,120} - \dfrac{23\,h^4\,h[1]\,h[3]^2}{138\,240}\right) + \text{ar}[2, 2]\left(-\dfrac{h^4\,h[1]^2\,h[3]}{4608} + \dfrac{h^4\,h[1]\,h[3]^2}{4608}\right) +$

$\text{ar}[3, 2]\left(-\dfrac{1}{24}\,h^2\,h[1] + \dfrac{11\,h^4\,h[1]^3}{15\,360} + \dfrac{89\,h^4\,h[1]^2\,h[2]}{46\,080} + \dfrac{11\,h^4\,h[1]\,h[2]^2}{5120} + \right.$

$\left.\dfrac{67\,h^4\,h[1]^2\,h[3]}{138\,240} + \dfrac{157\,h^4\,h[1]\,h[2]\,h[3]}{69\,120} + \dfrac{169\,h^4\,h[1]\,h[3]^2}{138\,240}\right) +$

$\text{ar}[3, 1]\left(\dfrac{1}{24}\,h^2\,h[2] - \dfrac{h^4\,h[1]^2\,h[2]}{3456} - \dfrac{7\,h^4\,h[1]\,h[2]^2}{8640} - \dfrac{7\,h^4\,h[2]^3}{5760} - \right.$

$\left.\dfrac{h^4\,h[1]\,h[2]\,h[3]}{5760} - \dfrac{7\,h^4\,h[2]^2\,h[3]}{4320} - \dfrac{19\,h^4\,h[2]\,h[3]^2}{17\,280}\right) +$

$\text{ar}[1, 1]\left(-\dfrac{h^4\,h[1]\,h[2]\,h[3]}{2304} - \dfrac{h^4\,h[2]^2\,h[3]}{1152} - \dfrac{h^4\,h[2]\,h[3]^2}{1536}\right) +$

$\text{ar}[1, 3]\left(-\dfrac{1}{24}\,h^2\,h[2] + \dfrac{19\,h^4\,h[1]^2\,h[2]}{17\,280} + \dfrac{7\,h^4\,h[1]\,h[2]^2}{4320} + \right.$

$\left.\dfrac{7\,h^4\,h[2]^3}{5760} + \dfrac{h^4\,h[1]\,h[2]\,h[3]}{5760} + \dfrac{7\,h^4\,h[2]^2\,h[3]}{8640} + \dfrac{h^4\,h[2]\,h[3]^2}{3456}\right) +$

$\text{ar}[1, 2]\left(\dfrac{1}{24}\,h^2\,h[3] - \dfrac{169\,h^4\,h[1]^2\,h[3]}{138\,240} - \dfrac{157\,h^4\,h[1]\,h[2]\,h[3]}{69\,120} - \right.$

$\left.\dfrac{11\,h^4\,h[2]^2\,h[3]}{5120} - \dfrac{67\,h^4\,h[1]\,h[3]^2}{138\,240} - \dfrac{89\,h^4\,h[2]\,h[3]^2}{46\,080} - \dfrac{11\,h^4\,h[3]^3}{15\,360}\right) +$

$\text{ar}[2, 1]\left(-\dfrac{1}{24}\,h^2\,h[3] + \dfrac{23\,h^4\,h[1]^2\,h[3]}{138\,240} + \dfrac{11\,h^4\,h[1]\,h[2]\,h[3]}{69\,120} + \right.$

$\left.\dfrac{13\,h^4\,h[2]^2\,h[3]}{46\,080} - \dfrac{19\,h^4\,h[1]\,h[3]^2}{138\,240} + \dfrac{23\,h^4\,h[2]\,h[3]^2}{46\,080} + \dfrac{31\,h^4\,h[3]^3}{46\,080}\right)$

**ToDegree[2, #[Phi[Vb]] & /@ {Pentagon, Hexagon, uUnitarity, HexagonMinus}]**

{0, 0, 0, 0}

**ToDegree[4, #[Phi[Vb]] & /@ {Pentagon, Hexagon, uUnitarity, HexagonMinus}]**

{0, 0, 0, 0}

```
S1[Phi_] := {
    Phi ~ Up ~ {1, 3, {2, 4}},
    (-Phi) ~ Up ~ {3, 2, 4},
    RR[3, 2],
    Phi ~ Up ~ {2, 3, 4},
    (-Phi) ~ Up ~ {1, 2, {3, 4}}
  };
ToDegree[1, S1[Phi[Vb]]]
```

$$\left\{0,\ 0,\ \frac{1}{2}\, h\, ar[2,\ 3] + \frac{1}{2}\, h\, ar[3,\ 2],\ 0,\ 0\right\}$$

```
S2[Phi_] := PCollect[NonCommutativeMultiply @@ S1[Phi]];
ToDegree[2, S2[Phi[Vb]]]
```

$$ar[4,\ 3]\ \left(-\frac{1}{24}\, h^2\, h[1] - \frac{1}{12}\, h^2\, h[2]\right) + ar[3,\ 4]\ \left(\frac{1}{24}\, h^2\, h[1] + \frac{1}{12}\, h^2\, h[2]\right) +$$

$$ar[2,\ 4]\ \left(-\frac{1}{24}\, h^2\, h[1] - \frac{1}{12}\, h^2\, h[3]\right) + ar[1,\ 4]\ \left(\frac{1}{24}\, h^2\, h[2] - \frac{1}{24}\, h^2\, h[3]\right) +$$

$$ar[4,\ 1]\ \left(-\frac{1}{24}\, h^2\, h[2] + \frac{1}{24}\, h^2\, h[3]\right) + ar[4,\ 2]\ \left(\frac{1}{24}\, h^2\, h[1] + \frac{1}{12}\, h^2\, h[3]\right) +$$

$$ar[3,\ 2]\ \left(\frac{h}{2} + \frac{1}{12}\, h^2\, h[1] - \frac{1}{12}\, h^2\, h[4]\right) + ar[3,\ 1]\ \left(-\frac{1}{12}\, h^2\, h[2] - \frac{1}{24}\, h^2\, h[4]\right) +$$

$$ar[1,\ 2]\ \left(-\frac{1}{12}\, h^2\, h[3] - \frac{1}{24}\, h^2\, h[4]\right) + ar[1,\ 3]\ \left(\frac{1}{12}\, h^2\, h[2] + \frac{1}{24}\, h^2\, h[4]\right) +$$

$$ar[2,\ 1]\ \left(\frac{1}{12}\, h^2\, h[3] + \frac{1}{24}\, h^2\, h[4]\right) + ar[2,\ 3]\ \left(\frac{h}{2} - \frac{1}{12}\, h^2\, h[1] + \frac{1}{12}\, h^2\, h[4]\right)$$

```
S3[V_] := PCollect /@ {
    (-V) ~ Up ~ {{1, 3}, {2, 4}},
    (-V) ~ Up ~ {1, 3},
    (-V) ~ Up ~ {2, 4},
    R[3, 2],
    V,
    V ~ Up ~ {3, 4},
    V ~ Up ~ {{1, 2}, {3, 4}}
   };
ToDegree[1, S3[Vb]]
```

$$\left\{-\frac{1}{4}\, h\, ar[1,\ 2] - \frac{1}{4}\, h\, ar[1,\ 4] + \frac{1}{4}\, h\, ar[2,\ 1] + \frac{1}{4}\, h\, ar[2,\ 3] - \frac{1}{4}\, h\, ar[3,\ 2] -\right.$$

$$\frac{1}{4}\, h\, ar[3,\ 4] + \frac{1}{4}\, h\, ar[4,\ 1] + \frac{1}{4}\, h\, ar[4,\ 3],\ -\frac{1}{4}\, h\, ar[1,\ 3] + \frac{1}{4}\, h\, ar[3,\ 1],$$

$$-\frac{1}{4}\, h\, ar[2,\ 4] + \frac{1}{4}\, h\, ar[4,\ 2],\ h\, ar[3,\ 2],\ \frac{1}{4}\, h\, ar[1,\ 2] - \frac{1}{4}\, h\, ar[2,\ 1],$$

$$\frac{1}{4}\, h\, ar[3,\ 4] - \frac{1}{4}\, h\, ar[4,\ 3],\ -\frac{1}{4}\, h\, ar[1,\ 3] + \frac{1}{4}\, h\, ar[1,\ 4] + \frac{1}{4}\, h\, ar[2,\ 3] +$$

$$\left.\frac{1}{4}\, h\, ar[2,\ 4] - \frac{1}{4}\, h\, ar[3,\ 1] - \frac{1}{4}\, h\, ar[3,\ 2] - \frac{1}{4}\, h\, ar[4,\ 1] - \frac{1}{4}\, h\, ar[4,\ 2]\right\}$$

```
S4[V_] := PCollect[NonCommutativeMultiply @@ S3[V]];
ToDegree[2, S4[Vb]]
```

$$\text{ar}[4, 3] \left(-\frac{1}{24} h^2 h[1] - \frac{1}{12} h^2 h[2]\right) + \text{ar}[3, 4] \left(\frac{1}{24} h^2 h[1] + \frac{1}{12} h^2 h[2]\right) +$$

$$\text{ar}[2, 4] \left(-\frac{1}{24} h^2 h[1] - \frac{1}{12} h^2 h[3]\right) + \text{ar}[1, 4] \left(\frac{1}{24} h^2 h[2] - \frac{1}{24} h^2 h[3]\right) +$$

$$\text{ar}[4, 1] \left(-\frac{1}{24} h^2 h[2] + \frac{1}{24} h^2 h[3]\right) + \text{ar}[4, 2] \left(\frac{1}{24} h^2 h[1] + \frac{1}{12} h^2 h[3]\right) +$$

$$\text{ar}[3, 2] \left(\frac{h}{2} + \frac{1}{12} h^2 h[1] - \frac{1}{12} h^2 h[4]\right) + \text{ar}[3, 1] \left(-\frac{1}{12} h^2 h[2] - \frac{1}{24} h^2 h[4]\right) +$$

$$\text{ar}[1, 2] \left(-\frac{1}{12} h^2 h[3] - \frac{1}{24} h^2 h[4]\right) + \text{ar}[1, 3] \left(\frac{1}{12} h^2 h[2] + \frac{1}{24} h^2 h[4]\right) +$$

$$\text{ar}[2, 1] \left(\frac{1}{12} h^2 h[3] + \frac{1}{24} h^2 h[4]\right) + \text{ar}[2, 3] \left(\frac{h}{2} - \frac{1}{12} h^2 h[1] + \frac{1}{12} h^2 h[4]\right)$$

```
ToDegree[4, S2[Phi[Vb]] - S4[Vb]]  // PCollect
```

0

```
ToDegree[8, Puncture[2, -Vb]]
```

$$-\text{ar}[1, 2] \left(\frac{h}{4} + \frac{1}{32} h^2 h[1] - \frac{19 h^4 h[1]^3}{46080} + \frac{37 h^6 h[1]^5}{4644864} - \frac{10663 h^8 h[1]^7}{66886041600}\right)$$

```
CoefficientList[ToDegree[8, Puncture[2, Vb]] /. {h[1] → 1, ar[1, 2] → 1}, h]
```

$$\left\{0, \frac{1}{4}, \frac{1}{32}, 0, -\frac{19}{46080}, 0, \frac{37}{4644864}, 0, -\frac{10663}{66886041600}\right\}$$

```
Series[Log[(E^(x / 2) - E^(-x / 2)) / x], {x, 0, 8}]
```

$$\frac{x^2}{24} - \frac{x^4}{2880} + \frac{x^6}{181440} - \frac{x^8}{9676800} + O[x]^9$$

```
Series[Log[(E^x - 1) / x], {x, 0, 8}]
```

$$\frac{x}{2} + \frac{x^2}{24} - \frac{x^4}{2880} + \frac{x^6}{181440} - \frac{x^8}{9676800} + O[x]^9$$

$$\left( -\frac{1}{4} + \left( \frac{x}{32} + \frac{5\,y}{96} \right) h + \frac{\left( -57\,x^3 - 169\,x^2\,y - 187\,x\,y^2 - 99\,y^3 \right) h^3}{138\,240} + \right.$$

$$\frac{1}{69\,672\,960} \left( 555\,x^5 + 2558\,x^4\,y + 5000\,x^3\,y^2 + 5078\,x^2\,y^3 + 2669\,x\,y^4 + 780\,y^5 \right) h^5 +$$

$$\frac{1}{200\,658\,124\,800} \left( -31\,989\,x^7 - 205\,357\,x^6\,y - 592\,097\,x^5\,y^2 - \right.$$

$$964\,985\,x^4\,y^3 - 949\,967\,x^3\,y^4 - 565\,943\,x^2\,y^5 - 186\,139\,x\,y^6 - 34\,275\,y^7 \Big) h^7 +$$

$$\frac{1}{1\,589\,212\,348\,416\,000} \left( 5\,104\,911\,x^9 + 42\,750\,073\,x^8\,y + 165\,124\,408\,x^7\,y^2 + \right.$$

$$376\,113\,312\,x^6\,y^3 + 552\,802\,446\,x^5\,y^4 + 542\,218\,746\,x^4\,y^5 + 352\,835\,712\,x^3\,y^6 +$$

$$\left. 146\,191\,048\,x^2\,y^7 + 33\,851\,563\,x\,y^8 + 4\,104\,741\,y^9 \right) h^9 + O[h]^{11} \Bigg) \; /. \; y \to 0$$

$$-\frac{1}{4} + \frac{x\,h}{32} - \frac{19\,x^3\,h^3}{46\,080} + \frac{37\,x^5\,h^5}{4\,644\,864} - \frac{10\,663\,x^7\,h^7}{66\,886\,041\,600} + \frac{243\,091\,x^9\,h^9}{75\,676\,778\,496\,000} + O[h]^{11}$$