Pensieve header: $50 offered for solutions.

Date: Tue, 23 Aug 2011 11:28:18 -0400 (EDT)
From: Dror Bar-Natan <drorbn@math.toronto.edu>
To: LazyKnots
Subject: Invitation / warning.

Dear All,

Within an hour I will post on my pensieve a couple of equations (for unknowns that are functions of two variables) and offer $50 for the first closed-form solution thereof. Within a short time later some hints will be posted, and perhaps some simplifications. If I will deem the simplifications major, I reserve the right to lower the sum offered. If offers for simplifications will be made by you, I may partially pay out the bounty. If no progress will be made for a while, I may raise the prize.

I will send a further notification with a URL when the equations/hints are posted. Shall anybody be interested, I will gladly describe what the equations are about at a LazyKnots meeting today at 4:30PM.

Best,

Dror.

Date: Thu, 25 Aug 2011 10:10:42 -0400 (EDT)
From: Dror Bar-Natan <drorbn@math.toronto.edu>
To: LazyKnots
Subject: Partial credits.

Dear All,

Some very useful ways to earn partial credit:

1. ($5) Prove that indeed a solution exists and is unique.

2. ($10) Prove that indeed f_21(x,y)=1/2+f_12(y,x).

3. ($30) Find a solution which is a composition of elementary functions (+, -, *, /, log, exp) and one single-variable “special” function. That special function may be anything provided it is a function of a single variable, though you can get a $5 extra credit if you can identify that special function in terms of known ones, perhaps in terms of the Lambert W function of, say, http://en.wikipedia.org/wiki/Lambert_W_function.

Best,

Dror.

The Equations.

Ok, here we go. For unknown functions $f_{12}(x, y)$ and $f_{21}(x, y)$, solve the equations:

$$\left(e^{-x(1+f_{12})-y(1+f_{21})}\left(\left(-y+e^{x f_{12}+y f_{21}}(x+y)\right) f_{12}+y f_{21}\right)\right) / ((x+y)(x f_{12}+y f_{21})) == \frac{e^{-x-y}(-1+e^{x})}{(-1+e^{x+y})\, x}$$

and

$$\left(e^{-x-x f_{12}-y f_{21}}\left(x y (f_{12}-f_{21})+e^{x f_{12}+y f_{21}}\, y (x+y) f_{21}+e^{\frac{x+y}{2}}\, x (x f_{12}+y f_{21})\right)\right) / ((x+y)(x f_{12}+y f_{21})) == \frac{e^{-x g_{12}-y g_{21}}\, x g_{12}+y g_{21}}{x g_{12}+y g_{21}}$$

where in the second equation, g_{12} stands for $f_{21}(y, x)$ and likewise g_{21} stands for $f_{12}(y, x)$. I have good reasons to believe (see the "perturbative hint" below) that a solution exists and is unique.

An Analytical Hint

One solution of only the first equation is $f_{21} = 0$ and $f_{12} = \frac{\operatorname{Log}\left[\frac{e^{-x}(-1+e^{x+y})\, y}{(-1+e^{y})(x+y)}\right]}{x}$.

A Perturbative Hint

Up to degree 10 there exists a unique power-series solution; it is given by:

$$f_{12} = -\frac{1}{4}+\left(\frac{x}{32}+\frac{5 y}{96}\right) h+\frac{\left(-57 x^3-169 x^2 y-187 x y^2-99 y^3\right) h^3}{138240}+$$
$$\frac{1}{69672960}\left(555 x^5+2558 x^4 y+5000 x^3 y^2+5078 x^2 y^3+2669 x y^4+780 y^5\right) h^5+$$
$$\frac{1}{200658124800}\left(-31989 x^7-205357 x^6 y-592097 x^5 y^2-964985 x^4 y^3-949967 x^3 y^4-565943 x^2 y^5-186139 x y^6-34275 y^7\right) h^7+$$
$$\frac{1}{1589212348416000}\left(5104911 x^9+42750073 x^8 y+165124408 x^7 y^2+376113312 x^6 y^3+552802446 x^5 y^4+542218746 x^4 y^5+352835712 x^3 y^6+146191048 x^2 y^7+33851563 x y^8+4104741 y^9\right) h^9+O[h]^{11}$$

$$f_{21} = \frac{1}{4}+\left(\frac{5 x}{96}+\frac{y}{32}\right) h+\frac{\left(-99 x^3-187 x^2 y-169 x y^2-57 y^3\right) h^3}{138240}+$$
$$\frac{1}{69672960}\left(780 x^5+2669 x^4 y+5078 x^3 y^2+5000 x^2 y^3+2558 x y^4+555 y^5\right) h^5+$$
$$\frac{1}{200658124800}\left(-34275 x^7-186139 x^6 y-565943 x^5 y^2-949967 x^4 y^3-964985 x^3 y^4-592097 x^2 y^5-205357 x y^6-31989 y^7\right) h^7+$$
$$\frac{1}{1589212348416000}\left(4104741 x^9+33851563 x^8 y+146191048 x^7 y^2+352835712 x^6 y^3+542218746 x^5 y^4+552802446 x^4 y^5+376113312 x^3 y^6+165124408 x^2 y^7+42750073 x y^8+5104911 y^9\right) h^9+O[h]^{11}$$

(Here the variable "h" is only used to organize the power series by degree. Feel free to substitute $h = 1$.)

It seems that $f_{12}(x, y) = -\frac{1}{2} + f_{21}(y, x)$ and that both functions are odd except for their constant term.

Scratch Work.

```
Simplify[f12 /.
  Solve[(e^(-x (1+f12)-y (1+f21)) ((-y + e^(x f12+y f21) (x + y)) f12 + y f21)) / ((x + y) (x f12 + y f21)) ==
    (e^(-x-y) (-1 + e^x))/((-1 + e^(x+y)) x) /. f21 -> 0, f12]
]
```

Solve::ifun : Inverse functions are being used by Solve, so
some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{-\frac{x+y+\mathrm{Log}\left[\frac{e^{-y}\,(-1+e^{y})\,(x+y)}{(-1+e^{x+y})\,y}\right]}{x}\right\}$$

```
Simplify[f21 /.
  Solve[(e^(-x (1+f12)-y (1+f21)) ((-y + e^(x f12+y f21) (x + y)) f12 + y f21)) / ((x + y) (x f12 + y f21)) ==
    (e^(-x-y) (-1 + e^x))/((-1 + e^(x+y)) x) /. f12 -> 0, f21]
]
```

Solve::ifun : Inverse functions are being used by Solve, so
some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{-\frac{x+y+\mathrm{Log}\left[\frac{e^{-x-y}\,(-1+e^{x})\,(x+y)}{(-1+e^{x+y})\,x}\right]}{y}\right\}$$

```
Simplify[(# / (-1 + e^(x+y))) & /@
  ((e^(-(1+f12) h[1]-(1+f21) h[2]) (-1 + e^(h[1]+h[2])) ((-f12 + f21) h[2] + e^(f12 h[1]+f21 h[2]) f12 (h[1] + h[2]))
      h[3]) / ((h[1] + h[2]) (f12 h[1] + f21 h[2])) == (e^(-h[1]-h[2]) (-1 + e^(h[1])) h[3])/h[1] /.
    {h[1] -> x, h[2] -> y, h[3] -> 1, f12 -> f12, f21 -> f21})
]
```

$$\left(e^{-x\,(1+f_{12})-y\,(1+f_{21})}\left(\left(-y+e^{x f_{12}+y f_{21}}\,(x+y)\right) f_{12}+y f_{21}\right)\right) / \left((x+y)\,(x f_{12}+y f_{21})\right) = \frac{e^{-x-y}\,(-1+e^{x})}{(-1+e^{x+y})\,x}$$

```
Simplify[(e^(-(1+f12) x-f21 y) ((f12 - f21) x y + e^(f12 x+f21 y) f21 y (x + y) + e^((x+y)/2) x (f12 x + f21 y))) /
    ((x + y) (f12 x + f21 y)) == (e^(-g12 x-g21 y) g12 x + g21 y)/(g12 x + g21 y) /.
  {f12 → f₁₂, f21 → f₂₁, g12 → g₁₂, g21 → g₂₁}
]
```

$$\left(e^{-x-x f_{12}-y f_{21}}\left(x\,y\,(f_{12}-f_{21})+e^{x f_{12}+y f_{21}}\,y\,(x+y)\,f_{21}+e^{\frac{x+y}{2}}\,x\,(x f_{12}+y f_{21})\right)\right)\Big/\left((x+y)\,(x f_{12}+y f_{21})\right)==\frac{e^{-x g_{12}-y g_{21}}\,x\,g_{12}+y\,g_{21}}{x\,g_{12}+y\,g_{21}}$$

$$\text{pf12} = -\frac{1}{4}+\left(\frac{x}{32}+\frac{5y}{96}\right)h+\frac{\left(-57x^3-169x^2y-187xy^2-99y^3\right)h^3}{138240}+\frac{1}{69672960}\left(555x^5+2558x^4y+5000x^3y^2+5078x^2y^3+2669xy^4+780y^5\right)h^5+\frac{1}{200658124800}\left(-31989x^7-205357x^6y-592097x^5y^2-964985x^4y^3-949967x^3y^4-565943x^2y^5-186139xy^6-34275y^7\right)h^7+\frac{1}{1589212348416000}\left(5104911x^9+42750073x^8y+165124408x^7y^2+376113312x^6y^3+552802446x^5y^4+542218746x^4y^5+352835712x^3y^6+146191048x^2y^7+33851563xy^8+4104741y^9\right)h^9+O[h]^{11};$$

$$\text{pf21} = \frac{1}{4}+\left(\frac{5x}{96}+\frac{y}{32}\right)h+\frac{\left(-99x^3-187x^2y-169xy^2-57y^3\right)h^3}{138240}+\frac{1}{69672960}\left(780x^5+2669x^4y+5078x^3y^2+5000x^2y^3+2558xy^4+555y^5\right)h^5+\frac{1}{200658124800}\left(-34275x^7-186139x^6y-565943x^5y^2-949967x^4y^3-964985x^3y^4-592097x^2y^5-205357xy^6-31989y^7\right)h^7+\frac{1}{1589212348416000}\left(4104741x^9+33851563x^8y+146191048x^7y^2+352835712x^6y^3+542218746x^5y^4+552802446x^4y^5+376113312x^3y^6+165124408x^2y^7+42750073xy^8+5104911y^9\right)h^9+O[h]^{11};$$

```
pf12 + (pf21 /. {x → -y, y → -x})
```

$O[h]^{11}$

```
Simplify[pf12 - (-1 / 2 + pf21 /. {x → y, y → x})]
```

$O[h]^{11}$

$$\left(\left(e^{-x-y-x f_{12}-y f_{21}}\left(\left(-y+e^{x f_{12}+y f_{21}}(x+y)\right)f_{12}+y f_{21}\right)\right)\Big/\left((x+y)(x f_{12}+y f_{21})\right)==\frac{e^{-x-y}\,(-1+e^{x})}{(-1+e^{x+y})\,x}\right)\;/.\;y f_{21}\to -x f_{12}+g$$

$$\frac{e^{-g-x-y}\left(g-x f_{12}+\left(-y+e^{g}(x+y)\right)f_{12}\right)}{g\,(x+y)}==\frac{e^{-x-y}\,(-1+e^{x})}{(-1+e^{x+y})\,x}$$

```
x pf12 + y pf21 // Simplify
```

$$\frac{1}{4}(-x+y)+\frac{1}{96}\left(3x^2+10xy+3y^2\right)h+\frac{\left(-57x^4-268x^3y-374x^2y^2-268xy^3-57y^4\right)h^3}{138\,240}+$$
$$\frac{1}{69\,672\,960}\left(555x^6+3338x^5y+7669x^4y^2+10\,156x^3y^3+7669x^2y^4+3338xy^5+555y^6\right)h^5+$$
$$\frac{1}{200\,658\,124\,800}\left(-31\,989x^8-239\,632x^7y-778\,236x^6y^2-1\,530\,928x^5y^3-1\,899\,934x^4y^4-1\,530\,928x^3y^5-778\,236x^2y^6-239\,632xy^7-31\,989y^8\right)h^7+$$
$$\left(\left(5\,104\,911x^{10}+46\,854\,814x^9y+198\,975\,971x^8y^2+522\,304\,360x^7y^3+905\,638\,158x^6y^4+1\,084\,437\,492x^5y^5+905\,638\,158x^4y^6+522\,304\,360x^3y^7+198\,975\,971x^2y^8+46\,854\,814xy^9+5\,104\,911y^{10}\right)h^9\right)/1\,589\,212\,348\,416\,000+O[h]^{11}$$

```
(x pf12 + y pf21 /. {x → a + b, y → a − b}) // Simplify
```

$$-\frac{b}{2}+\frac{1}{24}\left(4a^2-b^2\right)h+\frac{\left(-64a^4+4a^2b^2+3b^4\right)h^3}{8640}+\frac{\left(520a^6+66a^4b^2-25a^2b^4-6b^6\right)h^5}{1\,088\,640}+$$
$$\frac{1}{783\,820\,800}\left(-27\,584a^8-3920a^6b^2-1224a^4b^4+658a^2b^6+81b^8\right)h^7+$$
$$\left(\left(4\,338\,080a^{10}+430\,456a^8b^2+342\,720a^6b^4+38\,702a^4b^6-41\,807a^2b^8-3240b^{10}\right)h^9\right)/1\,551\,965\,184\,000+O[h]^{11}$$

Simplify[

$$\mathbf{\left(\left(\left(e^{-x-x f_{12}-y f_{21}}\left(x y\,(f_{12}-f_{21})+e^{x f_{12}+y f_{21}}\,y\,(x+y)\,f_{21}+e^{\frac{x+y}{2}}\,x\,(x f_{12}+y f_{21})\right)\right)\Big/\left((x+y)\,(x f_{12}+y f_{21})\right)\right)\Big/\left(\frac{e^{-x g_{12}-y g_{21}}\,x\,g_{12}+y\,g_{21}}{x\,g_{12}+y\,g_{21}}\right)\right)\ /.\ \{g_{12}\to 1/2+f_{12},\ g_{21}\to -1/2+f_{21}\}}$$

]

$$\left(e^{-x-x f_{12}-y f_{21}}\left(x\left(\frac{1}{2}+f_{12}\right)+y\left(-\frac{1}{2}+f_{21}\right)\right)\left(x y\,(f_{12}-f_{21})+e^{x f_{12}+y f_{21}}\,y\,(x+y)\,f_{21}+e^{\frac{x+y}{2}}\,x\,(x f_{12}+y f_{21})\right)\right)\Big/\left((x+y)\left(e^{\frac{1}{2}(-x+y-2x f_{12}-2y f_{21})}\,x\left(\frac{1}{2}+f_{12}\right)+y\left(-\frac{1}{2}+f_{21}\right)\right)(x f_{12}+y f_{21})\right)$$

$$\mathbf{Simplify\left[\left(e^{-x-x f_{12}-y f_{21}}\left(x\left(\frac{1}{2}+f_{12}\right)+y\left(-\frac{1}{2}+f_{21}\right)\right)\left(x y\,(f_{12}-f_{21})+e^{x f_{12}+y f_{21}}\,y\,(x+y)\,f_{21}+e^{\frac{x+y}{2}}\,x\,(x f_{12}+y f_{21})\right)\right)\Big/\left((x+y)\left(e^{\frac{1}{2}(-x+y-2x f_{12}-2y f_{21})}\,x\left(\frac{1}{2}+f_{12}\right)+y\left(-\frac{1}{2}+f_{21}\right)\right)(x f_{12}+y f_{21})\right)\ /.\ \{x\to h\,x,\ y\to h\,y,\ f_{12}\to pf12,\ f_{21}\to pf21\}\right]}$$

$$1+O[h]^{11}$$