Pensieve Header: An attempt on an explicit formula for the Alexander blobs vertex.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2011-08\\w-Computations"];
<< "AlexanderBlobs-Program.m"
```

```
(b[r[1, 3], #] & /@ Table[r[j, 0], {j, 3}]) /. Diag[hs_, ars___] :> hs Diag[1, ars]
```

$$\{0,\ 0,\ -\text{Diag}[1,\ \text{ar}[3,\ 0]]\ \text{h}[1] + \text{Diag}[1,\ \text{ar}[1,\ 0]]\ \text{h}[3]\}$$

```
B[n_, expr_] := Module[
   {bra},
   Transpose[Table[
     bra = b[expr, r[i, 0]] /. Diag[hs_, ars___] :> hs Diag[1, ars];
     Coefficient[bra, r[#, 0]] & /@ Range[n],
     {i, n}
    ]]
  ];
B[3, r[1, 3]] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & \text{h}[3] \\ 0 & 0 & 0 \\ 0 & 0 & -\text{h}[1] \end{pmatrix}$$

```
B3R[i_, j_] := MatrixExp[B[3, r[i, j]]];
B3R[1, 3] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & \frac{e^{-h[1]}\left(-1+e^{h[1]}\right)h[3]}{h[1]} \\ 0 & 1 & 0 \\ 0 & 0 & e^{-h[1]} \end{pmatrix}$$

```
MatrixForm /@ Simplify[{B3R[1, 2].B3R[1, 3], B3R[1, 3].B3R[1, 2]}]
```

$$\left\{\begin{pmatrix} 1 & \frac{\left(1-e^{-h[1]}\right)h[2]}{h[1]} & \frac{\left(1-e^{-h[1]}\right)h[3]}{h[1]} \\ 0 & e^{-h[1]} & 0 \\ 0 & 0 & e^{-h[1]} \end{pmatrix}, \begin{pmatrix} 1 & \frac{\left(1-e^{-h[1]}\right)h[2]}{h[1]} & \frac{\left(1-e^{-h[1]}\right)h[3]}{h[1]} \\ 0 & e^{-h[1]} & 0 \\ 0 & 0 & e^{-h[1]} \end{pmatrix}\right\}$$

```
MatrixForm /@ Simplify[{B3R[1, 2].B3R[1, 3].B3R[2, 3], B3R[2, 3].B3R[1, 3].B3R[1, 2]}]
```

$$\left\{\begin{pmatrix} 1 & \frac{\left(1-e^{-h[1]}\right)h[2]}{h[1]} & \frac{\left(1-e^{-h[1]}\right)h[3]}{h[1]} \\ 0 & e^{-h[1]} & \frac{e^{-h[1]-h[2]}\left(-1+e^{h[2]}\right)h[3]}{h[2]} \\ 0 & 0 & e^{-h[1]-h[2]} \end{pmatrix}, \begin{pmatrix} 1 & \frac{\left(1-e^{-h[1]}\right)h[2]}{h[1]} & \frac{\left(1-e^{-h[1]}\right)h[3]}{h[1]} \\ 0 & e^{-h[1]} & \frac{e^{-h[1]-h[2]}\left(-1+e^{h[2]}\right)h[3]}{h[2]} \\ 0 & 0 & e^{-h[1]-h[2]} \end{pmatrix}\right\}$$

```
v = Diag[f12, ar[1, 2]] + Diag[f21, ar[2, 1]];
B[3, v] // MatrixForm
```

$$\begin{pmatrix} -\text{f21}\,\text{h}[2] & \text{f12}\,\text{h}[2] & 0 \\ \text{f21}\,\text{h}[1] & -\text{f12}\,\text{h}[1] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
MatrixExp[a {{-1, 1}, {1, -1}}] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}e^{-2a}\left(1+e^{2a}\right) & \frac{1}{2}e^{-2a}\left(-1+e^{2a}\right) \\ \frac{1}{2}e^{-2a}\left(-1+e^{2a}\right) & \frac{1}{2}e^{-2a}\left(1+e^{2a}\right) \end{pmatrix}$$

```
(B3V = MatrixExp[B[3, v]]) // MatrixForm
```

$$\begin{pmatrix} \frac{e^{-f12\,h[1]-f21\,h[2]}\left(e^{f12\,h[1]+f21\,h[2]}\,f12\,h[1]+f21\,h[2]\right)}{f12\,h[1]+f21\,h[2]} & \frac{e^{-f12\,h[1]-f21\,h[2]}\left(-1+e^{f12\,h[1]+f21\,h[2]}\right)f12\,h[2]}{f12\,h[1]+f21\,h[2]} & 0 \\ \frac{e^{-f12\,h[1]-f21\,h[2]}\left(-1+e^{f12\,h[1]+f21\,h[2]}\right)f21\,h[1]}{f12\,h[1]+f21\,h[2]} & \frac{e^{-f12\,h[1]-f21\,h[2]}\left(f12\,h[1]+e^{f12\,h[1]+f21\,h[2]}\,f21\,h[2]\right)}{f12\,h[1]+f21\,h[2]} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
Simplify[B3V.MatrixExp[B[3, r[1, 3] + r[2, 3]]]] // MatrixForm
```

$$\begin{pmatrix} \frac{f12\,h[1]+e^{-f12\,h[1]-f21\,h[2]}\,f21\,h[2]}{f12\,h[1]+f21\,h[2]} & \frac{\left(1-e^{-f12\,h[1]-f21\,h[2]}\right)f12\,h[2]}{f12\,h[1]+f21\,h[2]} & \frac{e^{-(1+f12)\,h[1]-(1+f21)\,h[2]}\left(-1+e^{h[1]+h[2]}\right)\left((-f12+f21)\,h[2]+e\right.}{(h[1]+h[2])\,(f12\,h[1]+f2} \\ \frac{\left(1-e^{-f12\,h[1]-f21\,h[2]}\right)f21\,h[1]}{f12\,h[1]+f21\,h[2]} & \frac{e^{-f12\,h[1]-f21\,h[2]}\,f12\,h[1]+f21\,h[2]}{f12\,h[1]+f21\,h[2]} & \frac{e^{-(1+f12)\,h[1]-(1+f21)\,h[2]}\left(-1+e^{h[1]+h[2]}\right)\left(f12\,h[1]+f21\left(\left(-1+e^{f12\,h\right.\right.\right.}}{(h[1]+h[2])\,(f12\,h[1]+f2} \\ 0 & 0 & e^{-h[1]-h[2]} \end{pmatrix}$$

```
Simplify[B3R[1, 3].B3R[2, 3].B3V] // MatrixForm
```

$$\begin{pmatrix} \frac{f12\,h[1]+e^{-f12\,h[1]-f21\,h[2]}\,f21\,h[2]}{f12\,h[1]+f21\,h[2]} & \frac{\left(1-e^{-f12\,h[1]-f21\,h[2]}\right)f12\,h[2]}{f12\,h[1]+f21\,h[2]} & \frac{e^{-h[1]-h[2]}\left(-1+e^{h[1]}\right)h[3]}{h[1]} \\ \frac{\left(1-e^{-f12\,h[1]-f21\,h[2]}\right)f21\,h[1]}{f12\,h[1]+f21\,h[2]} & \frac{e^{-f12\,h[1]-f21\,h[2]}\,f12\,h[1]+f21\,h[2]}{f12\,h[1]+f21\,h[2]} & \frac{\left(1-e^{-h[2]}\right)h[3]}{h[2]} \\ 0 & 0 & e^{-h[1]-h[2]} \end{pmatrix}$$

```
Simplify[B3V.MatrixExp[B[3, r[1, 3] + r[2, 3]]] == B3R[1, 3].B3R[2, 3].B3V] // MatrixForm
```

$$\left\{\left\{0,\ 0,\ \left(-\frac{e^{-h[1]-h[2]}\left(-1+e^{h[1]}\right)}{h[1]}+\left(e^{-(1+f12)\,h[1]-(1+f21)\,h[2]}\left(-1+e^{h[1]+h[2]}\right)\left((-f12+f21)\,h[2]+\right.\right.\right.\right.$$
$$\left.\left.\left.\left.e^{f12\,h[1]+f21\,h[2]}\,f12\,(h[1]+h[2])\right)\right)\Big/\left((h[1]+h[2])\,(f12\,h[1]+f21\,h[2])\right)\right)h[3]\right\},$$
$$\left\{0,\ 0,\ \left(-\frac{1-e^{-h[2]}}{h[2]}+\left(e^{-(1+f12)\,h[1]-(1+f21)\,h[2]}\left(-1+e^{h[1]+h[2]}\right)\right.\right.\right.$$
$$\left.\left.\left(f12\,h[1]+f21\left(\left(-1+e^{f12\,h[1]+f21\,h[2]}\right)h[1]+e^{f12\,h[1]+f21\,h[2]}\,h[2]\right)\right)\right)\Big/$$
$$\left.\left.\left((h[1]+h[2])\,(f12\,h[1]+f21\,h[2])\right)\right)h[3]\right\},$$
$$\{0,\ 0,\ 0\}\Big\} == \{\{0,\ 0,\ 0\},\ \{0,\ 0,\ 0\},\ \{0,\ 0,$$
$$0\}\}$$

```
Solve[
 {(e^(-(1+f12) h[1]-(1+f21) h[2]) (-1+e^(h[1]+h[2])) ((-f12+f21) h[2]+e^(f12 h[1]+f21 h[2]) f12 (h[1]+h[2])))
      h[3])/((h[1]+h[2]) (f12 h[1]+f21 h[2])) ==
  (e^(-h[1]-h[2]) (-1+e^h[1]) h[3])/h[1], f21==0}, {f12, f21}]
```

Solve::ifun : Inverse functions are being used by Solve, so

some solutions may not be found; use Reduce for complete solution information. >>

$$\left\{\left\{f12\to\frac{\text{Log}\left[\frac{e^{-h[1]-f21\,h[2]}\left(-1+e^{h[1]+h[2]}\right)h[2]}{\left(-1+e^{h[2]}\right)(h[1]+h[2])}\right]}{h[1]},\ f21\to 0\right\}\right\}$$

```
Simplify[Log[E^(-h[1]-f21 h[2]) (-1+E^(h[1]+h[2])) h[2]/((-1+E^h[2]) (h[1]+h[2]))]/h[1] /. {f21 → 0, h[1] → x, h[2] → y}]
```

$$\frac{\text{Log}\left[\frac{e^{-x}\,(-1+e^{x+y})\,y}{(-1+e^{y})\,(x+y)}\right]}{x}$$

```
(Log[E^(-h[1]-f21 h[2]) (-1+E^(h[1]+h[2])) h[2]/((-1+E^h[2]) (h[1]+h[2]))]/h[1] /. {f21 → 0, h[1] → h x, h[2] → h y}) + O[h]^8
```

$$-\frac{1}{2}+\frac{1}{24}\,(x+2y)\,h+\frac{\left(-x^3-4x^2y-6xy^2-4y^3\right)h^3}{2880}+\frac{\left(x^5+6x^4y+15x^3y^2+20x^2y^3+15xy^4+6y^5\right)h^5}{181\,440}+\frac{1}{9\,676\,800}\left(-x^7-8x^6y-28x^5y^2-56x^4y^3-70x^3y^4-56x^2y^5-28xy^6-8y^7\right)h^7+O[h]^8$$

A result from Recycling/AlexanderBlobs-Results-110819.nb:

```
((-1/2 Diag[1, ar[1, 2]] + 1/24 Diag[h[1], ar[1, 2]] - Diag[h[1]^3, ar[1, 2]]/2880 +
    Diag[h[1]^5, ar[1, 2]]/181440 - Diag[h[1]^7, ar[1, 2]]/9676800 + 1/12 Diag[h[2], ar[1, 2]] -
    1/720 Diag[h[1]^2 h[2], ar[1, 2]] + Diag[h[1]^4 h[2], ar[1, 2]]/30240 -
    Diag[h[1]^6 h[2], ar[1, 2]]/1209600 - 1/480 Diag[h[1] h[2]^2, ar[1, 2]] +
    Diag[h[1]^3 h[2]^2, ar[1, 2]]/12096 - Diag[h[1]^5 h[2]^2, ar[1, 2]]/345600 -
    1/720 Diag[h[2]^3, ar[1, 2]] + Diag[h[1]^2 h[2]^3, ar[1, 2]]/9072 -
    Diag[h[1]^4 h[2]^3, ar[1, 2]]/172800 + Diag[h[1] h[2]^4, ar[1, 2]]/12096 -
    Diag[h[1]^3 h[2]^4, ar[1, 2]]/138240 + Diag[h[2]^5, ar[1, 2]]/30240 - Diag[h[1]^2 h[2]^5, ar[1, 2]]/172800 -
    Diag[h[1] h[2]^6, ar[1, 2]]/345600 - Diag[h[2]^7, ar[1, 2]]/1209600) /.
  Diag[hs_, ar[1, 2]] :> hs /. {h[1] → h x, h[2] → h y}) + O[h]^8
```

$$-\frac{1}{2}+\frac{1}{24}\,(x+2y)\,h+\frac{\left(-x^3-4x^2y-6xy^2-4y^3\right)h^3}{2880}+\frac{\left(x^5+6x^4y+15x^3y^2+20x^2y^3+15xy^4+6y^5\right)h^5}{181\,440}+\frac{1}{9\,676\,800}\left(-x^7-8x^6y-28x^5y^2-56x^4y^3-70x^3y^4-56x^2y^5-28xy^6-8y^7\right)h^7+O[h]^8$$

```
MatrixForm @ Simplify[
  MatrixExp[B[2, r[1, 2]]].MatrixExp[B[2, v]].
    MatrixExp[B[2, -1 / 2 (r[1, 2] + r[2, 1])]] /. {h[1] → x, h[2] → y}
 ]
```

$$\left(\begin{array}{cc} \frac{e^{-(1+f12)\,x-f21\,y}\left(\left(-f12+f21\right)x\,y-e^{f12\,x+f21\,y}\,f21\,y\,(x+y)+e^{\frac{x+y}{2}}\,y\,(f12\,x+f21\,y)+e^{x+f12\,x+f21\,y}\,(x+y)\,(f12\,x+f21\,y)\right)}{(x+y)\,(f12\,x+f21\,y)} & \frac{e^{-(1+f12)\,x-f21\,y}\,y\left(\left(-f12+f21\right.\right.}{} \\ \frac{e^{-(1+f12)\,x-f21\,y}\,x\left(-\left(-1+e^{\frac{x+y}{2}}\right)f12\,x+f21\left(\left(-1+e^{f12\,x+f21\,y}\right)x+\left(-e^{\frac{x+y}{2}}+e^{f12\,x+f21\,y}\right)y\right)\right)}{(x+y)\,(f12\,x+f21\,y)} & \frac{e^{-(1+f12)\,x-}}{} \end{array}\right)$$

```
Simplify[MatrixExp[B[2, v /. {f12 → g12, f21 → g21}]] /. {h[1] → x, h[2] → y}] //
 MatrixForm
```

$$\left(\begin{array}{cc} \frac{g12\,x+e^{-g12\,x-g21\,y}\,g21\,y}{g12\,x+g21\,y} & \frac{\left(1-e^{-g12\,x-g21\,y}\right)g12\,y}{g12\,x+g21\,y} \\ \frac{\left(1-e^{-g12\,x-g21\,y}\right)g21\,x}{g12\,x+g21\,y} & \frac{e^{-g12\,x-g21\,y}\,g12\,x+g21\,y}{g12\,x+g21\,y} \end{array}\right)$$

A result from AlexanderBlobs-Results.nb:

$$\begin{aligned}
\text{pertres} = \Bigg(& -\frac{1}{4}\,\text{Diag}[1, \text{ar}[1, 2]] + \frac{1}{4}\,\text{Diag}[1, \text{ar}[2, 1]] + \frac{1}{32}\,\text{Diag}[h[1], \text{ar}[1, 2]] + \\
& \frac{5}{96}\,\text{Diag}[h[1], \text{ar}[2, 1]] - \frac{19\,\text{Diag}\left[h[1]^3, \text{ar}[1, 2]\right]}{46\,080} - \frac{11\,\text{Diag}\left[h[1]^3, \text{ar}[2, 1]\right]}{15\,360} + \\
& \frac{37\,\text{Diag}\left[h[1]^5, \text{ar}[1, 2]\right]}{4\,644\,864} + \frac{13\,\text{Diag}\left[h[1]^5, \text{ar}[2, 1]\right]}{1\,161\,216} - \frac{10\,663\,\text{Diag}\left[h[1]^7, \text{ar}[1, 2]\right]}{66\,886\,041\,600} - \\
& \frac{457\,\text{Diag}\left[h[1]^7, \text{ar}[2, 1]\right]}{2\,675\,441\,664} + \frac{243\,091\,\text{Diag}\left[h[1]^9, \text{ar}[1, 2]\right]}{75\,676\,778\,496\,000} + \\
& \frac{1\,368\,247\,\text{Diag}\left[h[1]^9, \text{ar}[2, 1]\right]}{529\,737\,449\,472\,000} + \frac{5}{96}\,\text{Diag}[h[2], \text{ar}[1, 2]] + \frac{1}{32}\,\text{Diag}[h[2], \text{ar}[2, 1]] - \\
& \frac{169\,\text{Diag}\left[h[1]^2\,h[2], \text{ar}[1, 2]\right]}{138\,240} - \frac{187\,\text{Diag}\left[h[1]^2\,h[2], \text{ar}[2, 1]\right]}{138\,240} + \\
& \frac{1279\,\text{Diag}\left[h[1]^4\,h[2], \text{ar}[1, 2]\right]}{34\,836\,480} + \frac{2669\,\text{Diag}\left[h[1]^4\,h[2], \text{ar}[2, 1]\right]}{69\,672\,960} - \\
& \frac{205\,357\,\text{Diag}\left[h[1]^6\,h[2], \text{ar}[1, 2]\right]}{200\,658\,124\,800} - \frac{186\,139\,\text{Diag}\left[h[1]^6\,h[2], \text{ar}[2, 1]\right]}{200\,658\,124\,800} + \\
& \frac{42\,750\,073\,\text{Diag}\left[h[1]^8\,h[2], \text{ar}[1, 2]\right]}{1\,589\,212\,348\,416\,000} + \frac{33\,851\,563\,\text{Diag}\left[h[1]^8\,h[2], \text{ar}[2, 1]\right]}{1\,589\,212\,348\,416\,000} - \\
& \frac{187\,\text{Diag}\left[h[1]\,h[2]^2, \text{ar}[1, 2]\right]}{138\,240} - \frac{169\,\text{Diag}\left[h[1]\,h[2]^2, \text{ar}[2, 1]\right]}{138\,240} + \\
& \frac{125\,\text{Diag}\left[h[1]^3\,h[2]^2, \text{ar}[1, 2]\right]}{1\,741\,824} + \frac{2539\,\text{Diag}\left[h[1]^3\,h[2]^2, \text{ar}[2, 1]\right]}{34\,836\,480} - \\
& \frac{592\,097\,\text{Diag}\left[h[1]^5\,h[2]^2, \text{ar}[1, 2]\right]}{200\,658\,124\,800} - \frac{80\,849\,\text{Diag}\left[h[1]^5\,h[2]^2, \text{ar}[2, 1]\right]}{28\,665\,446\,400} + \\
& \frac{20\,640\,551\,\text{Diag}\left[h[1]^7\,h[2]^2, \text{ar}[1, 2]\right]}{198\,651\,543\,552\,000} + \frac{18\,273\,881\,\text{Diag}\left[h[1]^7\,h[2]^2, \text{ar}[2, 1]\right]}{198\,651\,543\,552\,000} -
\end{aligned}$$

$$\frac{11\,\text{Diag}[h[2]^3, ar[1, 2]]}{15\,360} - \frac{19\,\text{Diag}[h[2]^3, ar[2, 1]]}{46\,080} +$$
$$\frac{2539\,\text{Diag}[h[1]^2\,h[2]^3, ar[1, 2]]}{34\,836\,480} + \frac{125\,\text{Diag}[h[1]^2\,h[2]^3, ar[2, 1]]}{1\,741\,824} -$$
$$\frac{27\,571\,\text{Diag}[h[1]^4\,h[2]^3, ar[1, 2]]}{5\,733\,089\,280} - \frac{949\,967\,\text{Diag}[h[1]^4\,h[2]^3, ar[2, 1]]}{200\,658\,124\,800} +$$
$$\frac{1\,305\,949\,\text{Diag}[h[1]^6\,h[2]^3, ar[1, 2]]}{5\,518\,098\,432\,000} + \frac{306\,281\,\text{Diag}[h[1]^6\,h[2]^3, ar[2, 1]]}{1\,379\,524\,608\,000} +$$
$$\frac{2669\,\text{Diag}[h[1]\,h[2]^4, ar[1, 2]]}{69\,672\,960} + \frac{1279\,\text{Diag}[h[1]\,h[2]^4, ar[2, 1]]}{34\,836\,480} -$$
$$\frac{949\,967\,\text{Diag}[h[1]^3\,h[2]^4, ar[1, 2]]}{200\,658\,124\,800} - \frac{27\,571\,\text{Diag}[h[1]^3\,h[2]^4, ar[2, 1]]}{5\,733\,089\,280} +$$
$$\frac{4\,387\,321\,\text{Diag}[h[1]^5\,h[2]^4, ar[1, 2]]}{12\,612\,796\,416\,000} + \frac{90\,369\,791\,\text{Diag}[h[1]^5\,h[2]^4, ar[2, 1]]}{264\,868\,724\,736\,000} +$$
$$\frac{13\,\text{Diag}[h[2]^5, ar[1, 2]]}{1\,161\,216} + \frac{37\,\text{Diag}[h[2]^5, ar[2, 1]]}{4\,644\,864} -$$
$$\frac{80\,849\,\text{Diag}[h[1]^2\,h[2]^5, ar[1, 2]]}{28\,665\,446\,400} - \frac{592\,097\,\text{Diag}[h[1]^2\,h[2]^5, ar[2, 1]]}{200\,658\,124\,800} +$$
$$\frac{90\,369\,791\,\text{Diag}[h[1]^4\,h[2]^5, ar[1, 2]]}{264\,868\,724\,736\,000} + \frac{4\,387\,321\,\text{Diag}[h[1]^4\,h[2]^5, ar[2, 1]]}{12\,612\,796\,416\,000} -$$
$$\frac{186\,139\,\text{Diag}[h[1]\,h[2]^6, ar[1, 2]]}{200\,658\,124\,800} - \frac{205\,357\,\text{Diag}[h[1]\,h[2]^6, ar[2, 1]]}{200\,658\,124\,800} +$$
$$\frac{306\,281\,\text{Diag}[h[1]^3\,h[2]^6, ar[1, 2]]}{1\,379\,524\,608\,000} + \frac{1\,305\,949\,\text{Diag}[h[1]^3\,h[2]^6, ar[2, 1]]}{5\,518\,098\,432\,000} -$$
$$\frac{457\,\text{Diag}[h[2]^7, ar[1, 2]]}{2\,675\,441\,664} - \frac{10\,663\,\text{Diag}[h[2]^7, ar[2, 1]]}{66\,886\,041\,600} +$$
$$\frac{18\,273\,881\,\text{Diag}[h[1]^2\,h[2]^7, ar[1, 2]]}{198\,651\,543\,552\,000} + \frac{20\,640\,551\,\text{Diag}[h[1]^2\,h[2]^7, ar[2, 1]]}{198\,651\,543\,552\,000} +$$
$$\frac{33\,851\,563\,\text{Diag}[h[1]\,h[2]^8, ar[1, 2]]}{1\,589\,212\,348\,416\,000} + \frac{42\,750\,073\,\text{Diag}[h[1]\,h[2]^8, ar[2, 1]]}{1\,589\,212\,348\,416\,000} +$$
$$\frac{1\,368\,247\,\text{Diag}[h[2]^9, ar[1, 2]]}{529\,737\,449\,472\,000} + \frac{243\,091\,\text{Diag}[h[2]^9, ar[2, 1]]}{75\,676\,778\,496\,000}\Bigg);$$

```
pf12 = (pertres /. {Diag[hs_, ar[1, 2]] :> hs, Diag[hs_, ar[2, 1]] :> 0} /.
    {h[1] → h x, h[2] → h y}) + O[h]^11
```

$$-\frac{1}{4}+\left(\frac{x}{32}+\frac{5\,y}{96}\right)h+\frac{\left(-57\,x^3-169\,x^2\,y-187\,x\,y^2-99\,y^3\right)h^3}{138\,240}+$$
$$\frac{\left(555\,x^5+2558\,x^4\,y+5000\,x^3\,y^2+5078\,x^2\,y^3+2669\,x\,y^4+780\,y^5\right)h^5}{69\,672\,960}+$$
$$\frac{1}{200\,658\,124\,800}\left(-31\,989\,x^7-205\,357\,x^6\,y-592\,097\,x^5\,y^2-\right.$$
$$\left.964\,985\,x^4\,y^3-949\,967\,x^3\,y^4-565\,943\,x^2\,y^5-186\,139\,x\,y^6-34\,275\,y^7\right)h^7+$$
$$\left(\left(5\,104\,911\,x^9+42\,750\,073\,x^8\,y+165\,124\,408\,x^7\,y^2+376\,113\,312\,x^6\,y^3+\right.\right.$$
$$\left.\left.552\,802\,446\,x^5\,y^4+542\,218\,746\,x^4\,y^5+352\,835\,712\,x^3\,y^6+146\,191\,048\,x^2\,y^7+\right.\right.$$
$$\left.\left.33\,851\,563\,x\,y^8+4\,104\,741\,y^9\right)h^9\right)/1\,589\,212\,348\,416\,000+O[h]^{11}$$

```
pf21 = (pertres /. {Diag[hs_, ar[1, 2]] :> 0, Diag[hs_, ar[2, 1]] :> hs} /.
    {h[1] -> h x, h[2] -> h y}) + O[h]^11
```

$$\frac{1}{4}+\left(\frac{5\,x}{96}+\frac{y}{32}\right)h+\frac{\left(-99\,x^3-187\,x^2\,y-169\,x\,y^2-57\,y^3\right)h^3}{138\,240}+$$
$$\frac{\left(780\,x^5+2669\,x^4\,y+5078\,x^3\,y^2+5000\,x^2\,y^3+2558\,x\,y^4+555\,y^5\right)h^5}{69\,672\,960}+$$
$$\frac{1}{200\,658\,124\,800}\left(-34\,275\,x^7-186\,139\,x^6\,y-565\,943\,x^5\,y^2-\right.$$
$$\left.949\,967\,x^4\,y^3-964\,985\,x^3\,y^4-592\,097\,x^2\,y^5-205\,357\,x\,y^6-31\,989\,y^7\right)h^7+$$
$$\left(\left(4\,104\,741\,x^9+33\,851\,563\,x^8\,y+146\,191\,048\,x^7\,y^2+352\,835\,712\,x^6\,y^3+\right.\right.$$
$$\left.\left.542\,218\,746\,x^5\,y^4+552\,802\,446\,x^4\,y^5+376\,113\,312\,x^3\,y^6+165\,124\,408\,x^2\,y^7+\right.\right.$$
$$\left.\left.42\,750\,073\,x\,y^8+5\,104\,911\,y^9\right)h^9\right)/1\,589\,212\,348\,416\,000+O[h]^{11}$$

```
Simplify[
  (e^(-(1+f12) h[1]-(1+f21) h[2]) (-1+e^(h[1]+h[2])) ((-f12+f21) h[2]+e^(f12 h[1]+f21 h[2]) f12 (h[1]+h[2]))
      h[3])/((h[1]+h[2]) (f12 h[1]+f21 h[2])) == (e^(-h[1]-h[2]) (-1+e^h[1]) h[3])/h[1] /.
   {h[1] -> h x, h[2] -> h y} /. {f12 -> pf12, f21 -> pf21}]
```

$$h[3]\,O[h]^{11}==0$$

```
Simplify[((e^(-(1+f12) x-f21 y) ((f12-f21) x y+e^(f12 x+f21 y) f21 y (x+y)+e^((x+y)/2) x (f12 x+f21 y)))/
      ((x+y) (f12 x+f21 y)) == (e^(-g12 x-g21 y) g12 x+g21 y)/(g12 x+g21 y)) /. {x -> h x, y -> h y} /. {
   f12 -> pf12, f21 -> pf21, g12 -> (pf21 /. {x -> y, y -> x}),
   g21 -> (pf12 /. {x -> y, y -> x})
  }]
```

True